27. Relativity of the refractive index

Links to the book and to other chapters of the book.

27.1 Introduction

As noted in Chapter 15 on the relativistic refraction of light, the refractive index of an optically isotropic medium (and then the speed of light in that medium) is usually considered as independent of relative motion. The consequences of that assumption were examined there. In this chapter we will analyze the other two possibilities:

1. The refractive index varies with relative motion.
2. The refractive index only makes sense in the proper frame of the medium.

Although the second alternative will be simply considered for reasons of completeness.

27.2 On the invariance of the refractive index

In this discussion we will make use of a solid transparent rod R made of an isotropic medium m, whose proper length is L_o. It is placed horizontally and parallel to the X_o axis of its proper inertial frame RF_o. At both (the left and the right) ends of R two emitting sources S_a and S_b emit photons in opposite directions parallel to X_o (see Figure 27.1). Being n_o the refractive index of m, and being m optically isotropic, light will travel through R at the same constant speed c/n_o in all directions.

At instant $t = 0$ in RF_o, the source S_a emits a photon a^* parallel to X_o, in the direction from left to right; simultaneously S_b emits another photon b^* parallel to X_o, though in this case from right to left. After a time t_o both photons collide at C (the geometrical center of R), where a visible red flash is fired as a consequence of the collision of both

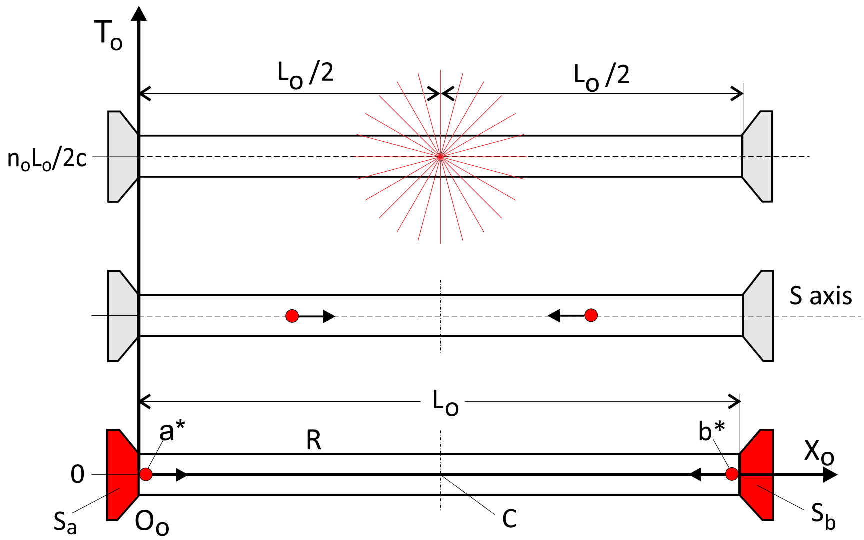

Figure 27.1 – In the proper frame RF_o of R, two photons a^* and b^* begin to move at the same instant $t = 0$ and in opposite directions parallel to X_o. After a time t_o, a^* and b^* collide just at C, the geometrical center of R, where a detector emits a red flash as a consequence of the collision of both photons.

photons. Evidently, we will have:

$$t_o = \frac{(L_o/2)}{c/n_o} \tag{1}$$

$$= \frac{n_o L_o}{2c} \tag{2}$$

RF_v is another inertial reference frame that coincides with RF_o at $t = 0$ and from whose perspective RF_o moves parallel to X_v, from left to right, at a velocity v according to our standard conditions (Figure 27.2). In RF_v the photon a^* begins to move at instant $t = 0$ and lasts a time t_{va} to reach C after moving a distance d_{va} at a velocity c_{va}. According to the Lorentz Transformation we will have:

$$t_{va} = \gamma t_o + \frac{\gamma v (L_o/2)}{c^2} \tag{3}$$

$$d_{va} = \gamma^{-1}(L_o/2) + v t_{va} \tag{4}$$

$$c_{va} = \frac{v + c/n_o}{1 + \dfrac{c/n_o}{c}\dfrac{v}{c}} \tag{5}$$

$$= c\frac{(n_o v + c)/n_o}{1 + v/n_o c} \tag{6}$$

$$= \frac{(n_o v + c)/n_o}{(n_o c + v)/n_o c} \tag{7}$$

$$= c\,\frac{n_o v + c}{n_o c + v} \tag{8}$$

Thus, from the perspective of RF_v, t_{va} is the time it takes the photon a^* to travel the RF_v-distance d_{va} with a velocity c_{va}.

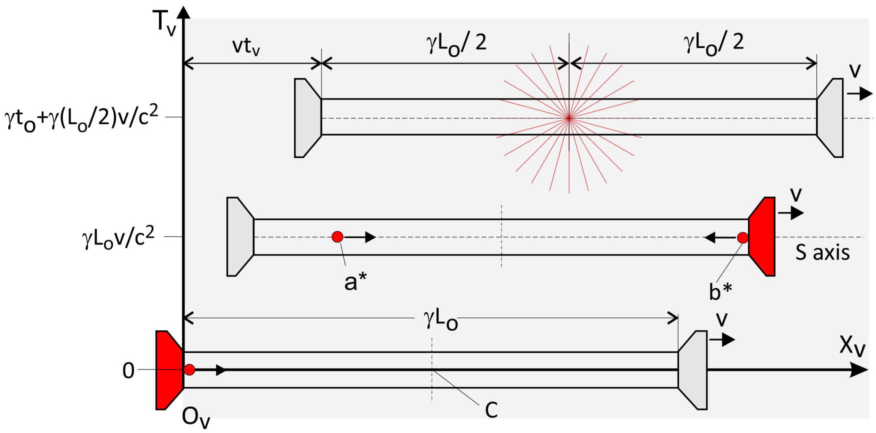

Figure 27.2 – The photons a^* and b^* moving through R from the perspective of RF_v.

In both frames RF_o and RF_v the collision of a^* and b^* takes place at C, which means that in both frames it is observed that a^* and b^* traverse the same distance through R. However, in RF_v, and due to the phase difference in synchronization, the photon b^* does not begin to move at $t = 0$, when a^* does, but at $t = \gamma v L_o/c^2$. Then, for this photon b^* we will have:

$$t_{vb} = \gamma t_o - \frac{\gamma v(L_o/2)}{c^2} \tag{9}$$

$$d_{vb} = \gamma^{-1}(L_o/2) - v t_{vb} \tag{10}$$

$$c_{vb} = \frac{c/n_o - v}{1 - \dfrac{c/n_o}{c}\dfrac{v}{c}} \tag{11}$$

$$= \frac{(c - n_o v)/n_o}{1 - v/n_o c} \tag{12}$$

$$= \frac{(c - n_o v)/n_o}{(n_o c - v)/n_o c} \tag{13}$$

$$= c\,\frac{c - n_o v}{n_o c - v} \tag{14}$$

Thus, t_{vb} is the time it takes the photon b^* to traverse the RF_v-distance d_{vb} at a velocity c_{vb}.

In RF_o each photon traverses the same distance $L_o/2$ in the same time t_o. Consequently, they move at the same speed. As expected, the refractive index of the isotropic material m is the same when light moves from left to right as when it moves from right to left.

P8 However, from the perspective of RF_v:

a) The photon a^* moves through R (inside R) from S_a to C, and the photon b^* from S_b to C.

b) In consequence, both photons traverse the same distance through R (once again, note we are not dealing here with distances with respect to RF_v, but with respect to R) just $\gamma^{-1}L_o/2$.

c) Both photons reach C simultaneously and collide.

d) The photon a^* is emitted before the photon b^*. Thus, a^* lasts a time greater than b^* in traversing the same distance inside R. Therefore, a^* moves through R slower than b^*.

e) In consequence the refractive index of the isotropic medium m is not the same when light propagates from left to right as when it does from right to left.

Before discussing its consequences, we will detail the relative variation of the refractive index referred to in P8, page 312. For the sake of simplicity we will make use of the following notation. Although in opposite senses, both photons a^* and b^* will always move along the same direction with respect to R. From now on, this direction will be referred to as S axis. We will also denote by S^+ the direction along S in the sense from S_a to S_b, i.e. the direction of the increasing x_v; and by S^- the direction along S in the sense from S_b to S_a, i.e. the direction of the decreasing x_v (Figure 27.3). From now on, the set formed by the solid R and the sources S_a and S_b will be referred to as R.

Since optical anisotropy will be crucial in the next discussion, we will also change the notations of the double relative velocities in order to express directions. So, to denote the double relative velocity of a^* and b^* with respect to R (which is initially in a horizontal position), we will now use c_{vh+} and c_{vh-} in the place of c_{vam} and c_{vbm} respectively, where the subindex h stands for 'horizontal orientation' and the subscripts $+$, $-$ for the S^+ and S^- directions respectively. If R is oriented vertically we will write c_{vv+} and c_{vv-}; and if it is oriented at any angle φ we will write $c_{v\varphi+}$ and $c_{v\varphi-}$.

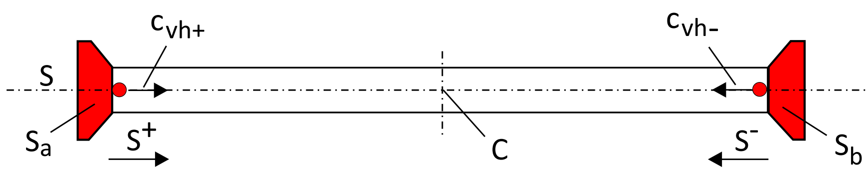

Figure 27.3 – The S axis along which photons a^* and b^* always move. The relative velocities of a^* and b^* with respect to R (in RF_v) are denoted by c_{vh+} and c_{vh-} respectively.

By way of example let us calculate, from the perspective of RF_v, the index of refraction n_{vh+} of m in the S^+ direction when, as above, R is placed horizontally, in the direction of the X_o axis, and assuming the refractive index is defined in the same way as in the proper case, i.e. as the ratio between the speed of light in a vacuum and the speed of light in m:

$$n_{vh+} = \frac{c}{c_{vh+}} \tag{15}$$

where c_{vh+} is the velocity of the photon a^* with respect to R as calculated from RF_v. For the observers in this frame, the photon a^* moves from S_a to C, which means a distance d_{vh+} given by:

$$d_{vh+} = \gamma^{-1} L_o/2 \tag{16}$$

To traverse d_{vh+} inside R takes the photon a^* the same time t_{va} (3) it takes it to traverse the distance d_{va} (4) with respect to RF_v. Or in other words, in the same time a^* moves a distance d_{va} with respect to RF_v, it traverses a distance d_{vh+} inside R. Thus we can write:

$$c_{vh+} = \frac{\gamma^{-1} L_o/2}{t_{va}} \tag{17}$$

$$= \frac{\gamma^{-1} L_o/2}{\gamma t_o + \dfrac{\gamma v L_o/2}{c^2}} \tag{18}$$

Now, taking into account that:

$$t_o = \frac{L_o/2}{c/n_o} \tag{19}$$

we can write:

$$c_{vh+} = \frac{\gamma^{-1} L_o/2}{\gamma t_o + \dfrac{\gamma v L_o/2}{c^2}} \tag{20}$$

$$= \frac{\gamma^{-1} L_o/2}{\gamma \dfrac{L_o/2}{c/n_o} + \dfrac{\gamma v L_o/2}{c^2}} \tag{21}$$

$$= \frac{\gamma^{-1}}{\gamma \left(\dfrac{n_o}{c} + \dfrac{v}{c^2} \right)} \tag{22}$$

$$= \frac{c^2}{\gamma^2 (n_o c + v)} \tag{23}$$

The refractive index n_{vh+} is therefore:

$$n_{vh+} = \frac{c}{c_{vh+}} \tag{24}$$

$$= \frac{c}{c^2/\gamma^2(n_o c + v)} \tag{25}$$

$$= \frac{\gamma^2 (n_o c + v)}{c} \tag{26}$$

$$= \gamma^2 \left(n_o + \frac{v}{c} \right) \tag{27}$$

The same argument applied to the photon b^* leads to the refractive index n_{vh-} of m in the S^- direction from the same perspective of RF_v, when R is placed horizontally in the direction of the X_o axis (Figure 27.4). Now we would obtain:

$$n_{vh-} = \gamma^2 \left(n_o - \frac{v}{c} \right) \tag{28}$$

Let us now examine the general case in which R is placed in any

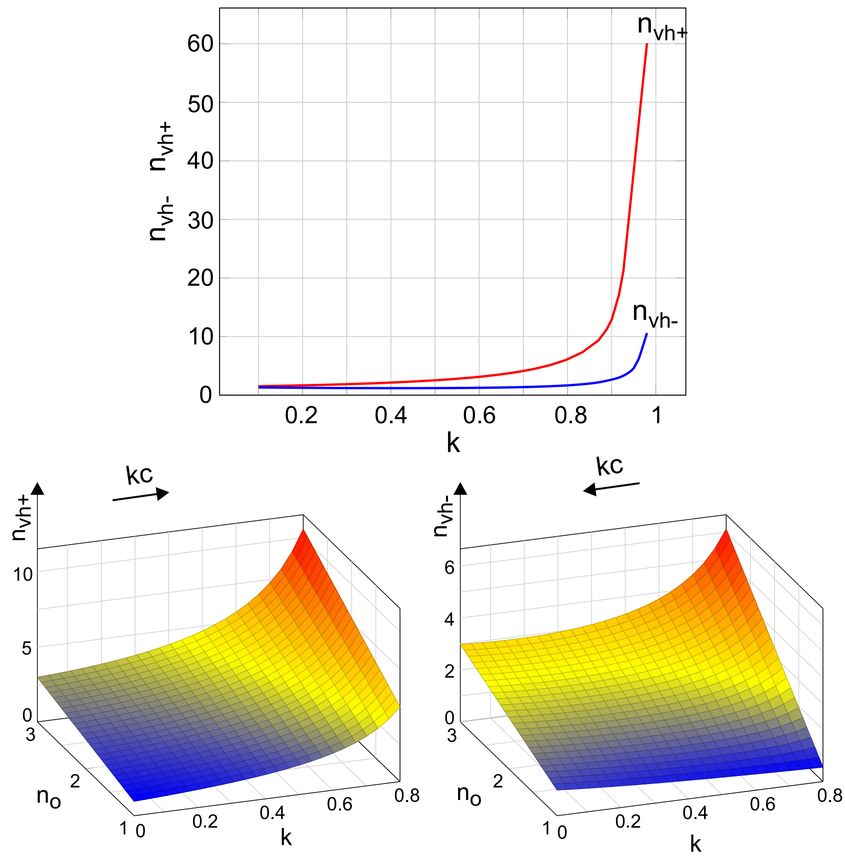

Figure 27.4 – Top: Relativity of the refractive indexes n_{vh+} and n_{vh-}. Bottom: Relativistic anisotropy of the refractive index of an optically isotropic medium when it is observed in relative motion from left to right (left) and from right to left (right). According to the laws of optical crystallography, in both cases the corresponding warped surfaces (whose points represent the values of the refractive index) should be the same horizontal plane.

orientation in the X_oY_o plane of its proper frame RF_o, for instance at an angle φ_o with respect to X_o. In RF_v, the relative index of refraction $n_{v\varphi_o+}$ of R when the photon a^* moves in the S^+ direction is now:

$$n_{v\varphi_o+} = \frac{c}{c_{v\varphi_o+}} \tag{29}$$

where $c_{v\varphi_o+}$ is the speed of a^* with respect to R as calculated from RF_v, i.e. the distance $d_{v\varphi_o+}$ it traverses inside R divided by the time $t_{v\varphi_o+}$ it lasts in the trip. Thus $d_{v\varphi_o+}$ is the distance from S_a to C, the geometrical center of R where the collision of both photons takes place. Taking into account FitzGerald-Lorentz contraction in the X_o

direction, we will have (Figure 27.5):

$$d_{v\varphi_o+} = (\gamma^{-2}x_o^2 + y_o^2)^{1/2} \tag{30}$$

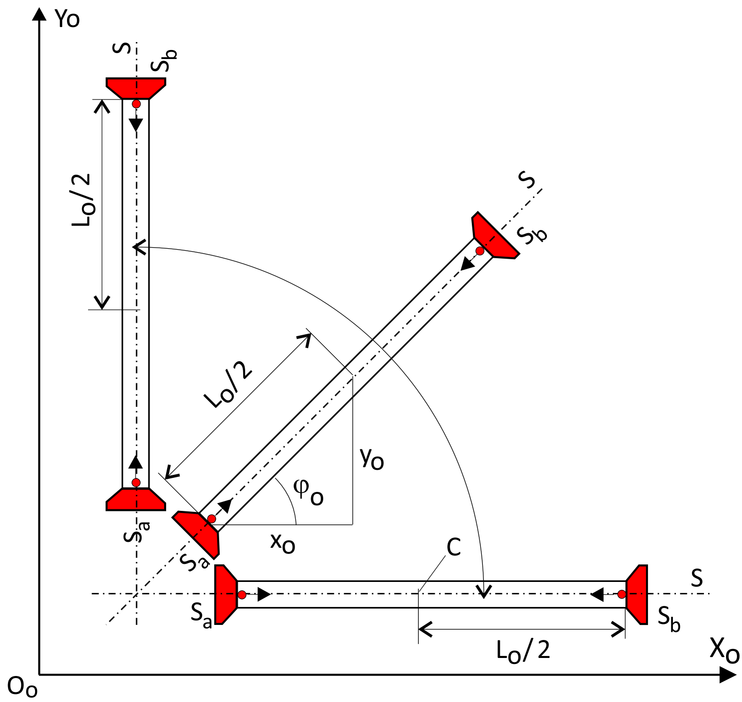

Figure 27.5 – The isotropic rod R and its emitting sources placed in different orientations in its proper frame RF_o.

Since the start point and the end point of of the trajectory followed by the photon a^* are now separated a proper distance x_o in the direction of the relative motion, the phase difference in synchronization will be $\gamma v x_o/c^2$ in terms of RF_v time. Therefore, the time $t_{v\varphi_o+}$ is:

$$t_{v\varphi_o+} = \gamma t_o + \frac{\gamma v x_o}{c^2} \tag{31}$$

Thus:

$$c_{v\varphi_o+} = \frac{(\gamma^{-2}x_o^2 + y_o^2)^{1/2}}{\gamma t_o + \dfrac{\gamma v x_o}{c^2}} \tag{32}$$

Now taking into account that:

$$t_0 = \frac{(x_o^2 + y_o^2)^{1/2}}{c/n_o} \tag{33}$$

we will have:

$$c_{v\varphi_o+} = \frac{(\gamma^{-2}x_o^2 + y_o^2)^{1/2}}{\gamma\dfrac{(x_o^2 + y_o^2)^{1/2}}{c/n_o} + \dfrac{\gamma v x_o}{c^2}} \tag{34}$$

This last equation leads immediate to the general case for any orientation φ_o as well as for the particular cases in which R is placed horizontally or vertically.

For the case in which R is placed horizontally and parallel to the X_o axis we will have: $y_o = 0$, $x_o = Lo/2$ and then equation (34) becomes:

$$c_{vh+} = \frac{(\gamma^{-2}x_o^2 + y_o^2)^{1/2}}{\gamma\dfrac{(x_o^2 + y_o^2)^{1/2}}{c/n_o} + \dfrac{\gamma v x_o}{c^2}} \tag{35}$$

$$= \frac{\gamma^{-1}x_o}{\gamma\dfrac{x_o}{c/n_o} + \dfrac{\gamma v x_o}{c^2}} \tag{36}$$

$$= \frac{\gamma^{-1}}{\dfrac{\gamma nc + \gamma v}{c^2}} \tag{37}$$

$$= \frac{c}{\gamma^2(n_o + v/c)} \tag{38}$$

So that:

$$n_{vh+} = \frac{c}{\dfrac{c}{\gamma^2(n_o + v/c)}} \tag{39}$$

$$= \gamma^2\left(n_o + \frac{v}{c}\right) \tag{40}$$

which is the same expression (27). The case of the photon b^* moving in the S^- direction differs only in the phase difference of synchronization, that is now $-\gamma x_o v/c^2$. The corresponding refractive index n_{vh-} will be:

$$n_{vh-} = \gamma^2\left(n_o - \frac{v}{c}\right) \tag{41}$$

If R is placed vertically, parallel to Y_o, and then orthogonal to the direction of the relative motion, we will have $x_o = 0$ and the phase difference in synchronization would also be null. Thus equation (34) becomes:

$$c_{vh-} = \frac{y_o}{\gamma \dfrac{y_o}{c/n_o}} \tag{42}$$

$$= \frac{c}{\gamma n_o} \tag{43}$$

And then:

$$n_{vv+} = \frac{c}{c_{vv+}} \tag{44}$$

$$= \frac{c}{\dfrac{c}{\gamma n_o}} \tag{45}$$

$$= \gamma n_o \tag{46}$$

Things are identical for the photon b^* moving in the direction S^-: there is neither length contraction in the direction of the relative motion nor phase difference in synchronization. So we will obtain:

$$n_{vv-} = \frac{c}{c_{vv+}} \tag{47}$$

$$= \frac{c}{\dfrac{c}{\gamma n_o}} \tag{48}$$

$$= \gamma n_o \tag{49}$$

This proves that in the case of the vertical orientation, light travels at the same speed in the S^+ direction as in the S^- direction.

Finally, let R be inclined at any angle φ_o with respect to the X_o axis. In this case $x_o > 0$ and $y_o > 0$, being not null the phase difference in synchronization. So, equation (34) can be written as:

$$c_{v\varphi_o+} = \frac{(\gamma^{-2}x_o^2 + y_o^2)^{1/2}}{\gamma \dfrac{(x_o^2 + y_o^2)^{1/2}}{c/n_o} + \dfrac{\gamma v x_o}{c^2}} \tag{50}$$

Dividing by x_o and writing α in the place of $\tan^2\varphi_o$:

$$c_{v\varphi_o+} = \frac{(\gamma^{-2}+\alpha)^{1/2}}{\dfrac{\gamma n_o(1+\alpha)^{1/2}}{c}+\dfrac{\gamma v}{c^2}} \tag{51}$$

$$= \frac{(\gamma^{-2}+\alpha)^{1/2}}{\dfrac{\gamma n_o c(1+\alpha)^{1/2}+\gamma v}{c^2}} \tag{52}$$

$$= \frac{c^2(\gamma^{-2}+\alpha)^{1/2}}{\gamma(cn_o(1+\alpha)^{1/2}+v)} \tag{53}$$

$$= \frac{c(\gamma^{-2}+\alpha)^{1/2}}{\gamma(n_o(1+\alpha)^{1/2}+v/c)} \tag{54}$$

Therefore:

$$n_{v\varphi_o+} = \frac{c}{c_{v_x\varphi_o+}} \tag{55}$$

$$= \frac{c}{\dfrac{c(\gamma^{-2}+\alpha)^{1/2}}{\gamma(n_o(1+\alpha)^{1/2}+v/c)}} \tag{56}$$

$$= \gamma\frac{n_o(1+\alpha)^{1/2}+v/c}{(\gamma^{-2}+\alpha)^{1/2}} \tag{57}$$

that, taking into account that $\alpha = \tan^2\varphi_o$, we will have:

$$1+\alpha = 1+\tan^2\varphi_o = \cos^{-2}\varphi_o \tag{58}$$

and we can write:

$$n_{v\varphi_o+} = \frac{n_o(1+\alpha)^{1/2}+k}{(1-k^2)^{1/2}(1-k^2+\alpha)^{1/2}} \tag{59}$$

$$= \frac{n_o\cos^{-1}\varphi_o+k}{\sqrt{(1-k^2)(\cos^{-2}\varphi_o-k^2)}} \tag{60}$$

where $k = v/c$. A similar reasoning on photon b^* leads to:

$$n_{v\varphi_o-} = \frac{c}{c_{v_x\varphi_o-}} \tag{61}$$

$$= \frac{c}{\dfrac{c(\gamma^{-2}+\alpha)^{1/2}}{\gamma(n_o(1+\alpha)^{1/2}-v/c)}} \quad (62)$$

$$= \gamma\frac{n_o(1+\alpha)^{1/2}-v/c}{(\gamma^{-2}+\alpha)^{1/2}} \quad (63)$$

$$= \frac{n_o(1+\alpha)^{1/2}-k}{(1-k^2)^{1/2}(1-k^2+\alpha)^{1/2}} \quad (64)$$

$$= \frac{n_o\cos^{-1}\varphi_o-k}{\sqrt{(1-k^2)(\cos^{-2}\varphi_o-k^2)}} \quad (65)$$

From the point of view of optical crystallography, it is worth noting the fact that whatever be the angle φ_o between R and the X_o axis, light always travels inside R in a direction parallel to the S axis (S^+ and S^- directions). In consequence, it should always move with the same speed through R, but this is not what is deduced from the Lorentz Transformation (Figure 27.6).

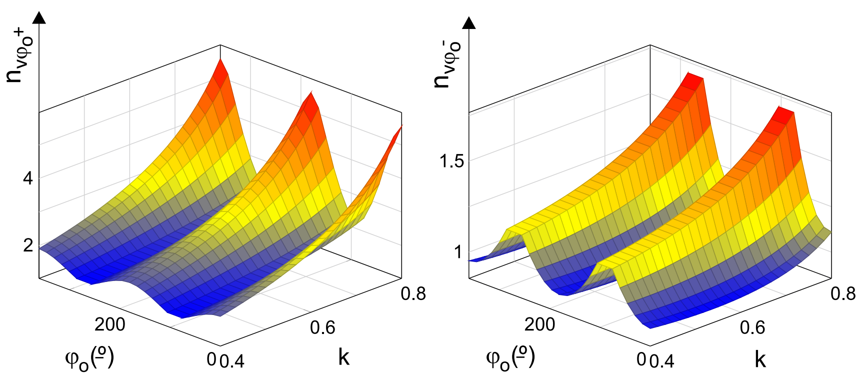

Figure 27.6 – Optical anisotropy of the isotropic medium m when observed in RF_v, assuming its refractive index is 1.2, and that the S axis (direction of the motion of light within R) forms an angle φ_o with the X_o axis of the proper reference frame RF_o of R, being RF_o moving relative to RF_v with a uniform velocity kc parallel to X_v from left to right. Left: light moves in the direction S^+. Right: light moves in the direction S^- (see text).

The above experience could be repeated for different orientations of the source of photons while R maintains its orientation, as Figure 27.7 shows. The conclusions would be the same, although in this case the observers in RF_v would have to conclude R is doubly anisotropic with respect to the refractive index. This index not only changes with the orientation of the trajectory AB that light follows inside R, but also with the direction along that trajectory: they will observe a different index of refraction for each opposite direction AB and BA. For the observers in RF_v all isotropic materials in RF_o are anisotropic and exhibit bipolar anisotropy.

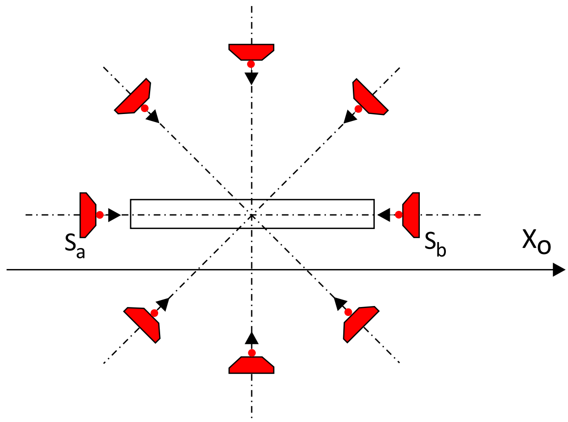

Figure 27.7 – All optically isotropic media in RF_v are seen as anisotropic in RF_v.

In RF_o things go as expected: light always travels at the same speed through the isotropic medium m; its velocity does not change when it moves in a direction or in the opposite one. This also applies to anisotropic media: if AB is any direction of the optical indicatrix (an ellipsoid whose points, together with its geometrical center, define the refractive index of a transparent medium for each possible spatial direction) of any anisotropic medium, then light moves at the same speed in the direction from A to B as in the direction from B to A.

In RF_v, on the contrary, the conclusions on the velocities of light through R disagree with the laws of optical crystallography:

a) Light travels at different velocities in opposite senses of the same direction: if AB is a straight line inside R (non-perpendicular to relative motion), then light moves with different speed when it travels from A to B than when it does from B to A. Optical crystallography states they have to move at the same velocity.

b) While maintaining the direction of propagation inside R, the speed of light is different in different orientations of R with respect to X_v. This is as if the determination of the refractive index would depend upon the orientation of the petrographic microscope with respect to the laboratory.

27.3 Consequences

The above discussion on the relativity of the refractive index leads to the following two alternatives (Lorentz symmetry breaks):

a) The Lorentz Transformation makes isotropic materials behave as if they were anisotropic, and in a sense that is not compatible with the laws of optical crystallography (bipolar anisotropy).

b) The Lorentz Transformation converts between reality and apparent realities. So, the observations and measurements on the refractive index of isotropic materials are only possible in their proper reference frames, which makes proper references frames special with respect to non-proper ones, and questions the universality of the Principle of Relativity.