P4. *Proper Velocity and Relativistic 4-Velocity, Mechanics in term of Proper velocities*

P. K. Vizminiska and C. P. Viazminsky

Abstract
The transformation relations and the basic mechanical results obtained in earlier works [1-3] are expressed in terms of proper velocities. The relation between proper and the relativistic 4-velocity is explored. Contrary to what was said in previous articles, the proper velocity is not directly measured but rather calculated from the measured familiar velocity or inertial velocity.

1.*Introduction*

In the first part P1 [3] of this work the mass-energy equivalence relation emerged as a natural consequence of the concept of intrinsic units. In P2 [3] we introduced the universal frame as a stationary synchronous inertial frame and derived the scaling transformation (ST) which relates the geometric lengths of a light trip in the universal frame and a moving one. The ST relates also the geometric time of a trip with its true or optical time. The Euclidean form of the ST, which is best illustrated by the observer-body triangle, reveals that the proper (or universal) velocity that must be attached to a moving object is distinct from the velocity of the inertial frame which accommodates the object (or inertial velocity). The proper (or universal) velocity of an object is unbound and can exceed the velocity of light c whereas the inertial velocity is bound by c. In P3 the basic mechanical relations concerning the momentum and energy of a body were derived; they coincides with the relativistic results.

In this part we express the scaling transformations and the basic mechanical equations in terms of the proper velocity. We also correct what we stated in earlier works regarding velocity measurements and conclude that proper velocity is calculated from the measured inertial velocity. We also show that employing proper velocities for light and the body emitting it yields the familiar Galilean forms.

2. The Body-Observer Triangle

Let b be an object moving in S at velocity $\boldsymbol{v}$ and assume that b passes by the point $B \in S$ at an instant $t = 0$. It was shown [1,2] that if two light's waves 1 and 2 emanate simultaneously (figure 1) from the moving source of light b when at $B \in S$ and from $O \in S$ respectively, then when the former arrives at O the latter hits the moving body b (at some point $b' \in S$). In other words the events:

(i) Wave 1 arrives at O and,

(ii) Wave 2 hits the body b,

are simultaneous in S. Equivalently, the light trips $(b\ at\ B \rightarrow O)$ or $(O \rightarrow b\ (at\ b'))$ have the same time length. The duration t of the light trip $(b\ at\ B \rightarrow O)$ or $(O \rightarrow b\ (at\ b'))$ is given by

$$t = \Gamma(\theta, \beta) T \tag{2.1}$$

where

$$\Gamma(\theta, \beta) = \frac{\beta cos\theta + \sqrt{1 - \beta^2 sin^2\theta}}{\sqrt{1 - \beta^2}} \tag{2.2}$$

Note that wave 1 which appears to follow the path $\boldsymbol{BO}$ in S follows the path $\boldsymbol{bo}$ in s which coincides when light arrives at O with $\boldsymbol{b'O}$ if s is considered moving relative to S. The relation (2.1), which is called the scaling transformation (*ST*), can be understood in either of the following ways:

(i). It determines *the light trip's duration t in terms of its geometric length T*. In this concern the light trip meant is $(b\ at\ B \rightarrow O)\ or\ (O \rightarrow b)$. The geometric length refers to the distance between the beginning and end points in S at the initial instant of time, say $t = 0$. The geometric length for the trip 1 is the distance between b and O at $t = 0$, which is $|\boldsymbol{BO}| = R = cT$. For the trip 2 it is the distance between O and b at $t = 0$, which is $|\boldsymbol{OB}| = R = cT$.

(ii) The relation between the geometric lengths of the light trip in S and s when either frame is considered stationary [1,2].

The ST may be written in the Euclidean form

$$t = \sqrt{1-\beta^2}E(\pi-\theta,\beta)T, \tag{2.3a}$$

or

$$t' \equiv \frac{t}{\sqrt{1-\beta^2}} = E(\pi-\theta,\beta)T. \tag{2.3b}$$

where

$$E(\pi-\theta,\beta) = \frac{\Gamma(\theta,\beta)}{\sqrt{1-\beta^2}} \tag{2.4}$$

is named the Euclidean factor.

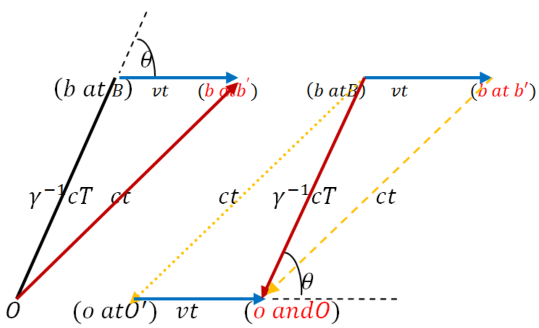

Figure 1. In both figures the frame S is stationary while s is moving to the right at velocity $\boldsymbol{v}$. **On left**, the stationary observer O emanates, when the moving object $b \in s$ is at $B \in S$, a pulse of light that intercept the object b at the point $b' \in S$. **On right**, the object b emits when at $B \in S$ a pulse of light which arrives at $(O \in S\ and\ o \in s)$ (which also means $(o \in s\ at\ O{\in}S$. By the fact mentioned above the events $o{\in}s\ and\ O{\in}S$ and $(b{\in}s\ at\ b'{\in}S)$ are simultaneous. **On right**, at the initial instant of time which corresponds to $(b\ at\ B)$ we have $(o \in s\ at\ O' \in S)$. In the frame s the pulse emanating from $b \in s$ heads towards $o \in s$. The path of the pulse is $(B \rightarrow O)$ in the stationary frame S and $(b \rightarrow o)$ in the moving frame s. The latter appears in S to coincide with $\boldsymbol{BO'}$ and $\boldsymbol{b'O}$ at the initial and final times respectively.

The initial and final positions of the moving body in the universal frame S together with the observer's position O form a triangle OBb' which we call the *body-observer* *triangle* (**figures 1 and 2).** The different values assigned in the two figures to the sides' lengths will be explained in section 3. The Euclidean form (2.3a) or (2.3b) enable one to calculate the time duration t of the light trip $(b\ at\ B \to O)$ or $(O \to b\ (at\ b'))$ and hence the distance $|\boldsymbol{Bb'}|$ traveled by the body and the distance $|\boldsymbol{Ob'}|$ traveled by the light signal till meeting each other, using the Euclidean trigonometry.

3. Time Duration and Geometric Time Distance, Proper Velocity Measurement.

The sides, lengths of the body-observer triangle in figure 2 are

$$T,\ T_{body} = \frac{\beta t}{\sqrt{1-\beta^2}},\quad t' = \frac{t}{\sqrt{1-\beta^2}} \equiv T_{wave}. \tag{3.1}$$

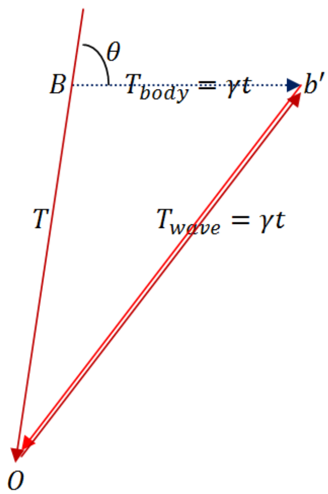

Figure 2. the body-observer triangle

The unit used for these lengths is second, and these values should be multiplied by c to get the lengths in meter. The given lengths yield a value t for the time duration as it is prescribed by the ST .

The quantity $t' = E(\beta, \pi - \theta)T$ appearing in (2.3b) is the geometric time distance between $O \in S$ and $b' \in S$. From (2.3b) we obtain the relation

$$t = t'\sqrt{1-\beta^2}, \tag{3.2}$$

which has the same form as the time dilation formula in special relativity, but bears an absolutely different meaning. Indeed, equation (3.1) states that the time period t associated with the journey $\boldsymbol{Bb'}$ of the moving body b is a fraction $\sqrt{1-\beta^2}$ of the corresponding geometric time distance $t' \equiv td(b', O)$. Equivalently, the geometric time is γ times greater than the duration of the corresponding light trip to the moving body: $O \to b(\ at\ b')$. During the period t of its journey, $\boldsymbol{Bb'}$, the moving body b covers a distance

$$(\gamma v)t = v(\gamma t) = vt'. \tag{3.3}$$

The latter relation, implies that the proper (or universal) velocity of the body is γv. The description "proper" here has nothing to do with the concept of "proper …" in special relativity. The latter relation also suggests to name the velocity v which multiplies geometric time to get the true distance, geometric or (inertial) velocity.

Contrary to what I have said in previous articles about its measurement, the proper velocity is not directly measured, but rather calculated from the measured familiar velocity or inertial velocity in classical mechanics.

To determine the speed v of an inertial frame s relative to an inertial frame S it is sufficient to measure the velocity of one point $b \in s$ in S. Indeed, by the basic concept of an inertial frame s, the distances between all its points remain fixed and s translates at a uniform velocity v relative to S. Suppose that the inertial frame s is moving in the direction of the X-axis of S and that the point $o \in s$ passes by $O \in S$ at $t = 0$. If the point $o \in s$ passes by a point $O_1 \in S$ at $t = 1$ then the speed of s is $\boldsymbol{v} = |OO_1|\hat{x}$. The timer at $O_1 \in S$ is synchronous with the timer at $O \in S$. An object b, whatever far from $o \in s$ belong to s if it is at rest in s, or equivalently if its velocity in S is $\boldsymbol{v}$. The velocity of b is measured locally as that of $o \in s$.

4. Undoing Space Contraction

The transformations (2.3a) or (2.3b) assert that the pulse of light and the body meet after t seconds, but they differ in the values assigned to the corresponding sides' lengths of the body-observer triangle OBb' (figures 1 and 2).

The body-observer triangle $(c\sqrt{1-\beta^2}T, \beta ct, ct)$ (figure 1) representing (2.3a) shows that: assigning the value v to the body's velocity and the constant value c to light's pulse velocity requires contracting the initial geometric length of $\boldsymbol{OB}$ by $\gamma^{-1} = \sqrt{1-\beta^2}$. This means that *in conjunction with the latter values for light's and body's velocities, the geometric distance in the space is contracted by* γ^{-1}. Therefore: *If the duration t of a light pulse from an observer $O \in S$ to a moving body b is adopted to measure the distance to the moving body, simply as ct, then the result will be a fraction* $\sqrt{1-\beta^2}$ *of that obtained through Euclidean trigonometry. The distances*

$$dlight = ct \quad and \quad dgeometry = E(\pi - \theta, \beta)T = \gamma ct,$$

are identical only for $\beta = 0$, *i.e. when the body b is at rest in S.*

The aforementioned space contraction may formally be undone by replacing in the concerned values, $[meter]$ by $[\gamma\ meter]$, which amounts to replacing ct by γct and vt by γvt. For instance, in deriving one type of Doppler effect, we follow a classical treatment and use v and c for the body's and light's velocities respectively, but eventually we expand the value calculated for the wavelength by γ.

It should be kept in mind that the encountered space contraction is due to using the erroneous speeds. The proper speeds to be used for the body and light in order to preserve the metric of the space are $V \equiv \gamma v$ and $C \equiv \gamma c$ respectively.

5. The Scaling Transformation in Terms of Proper Velocity

From the expression of proper velocity

$$V = \frac{v}{\sqrt{1-(v/c)^2}} \quad or\ \beta_V = \frac{\beta}{\sqrt{1-\beta^2}}, \tag{5.1}$$

where

$$\beta = v/c, \quad \beta_V = V/c, \tag{5.2}$$

v and V are the inertial and proper velocities respectively, we obtain following useful relations

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \sqrt{1+\beta_V^2},\ \ \beta = \frac{\beta_V}{\sqrt{1+\beta_V^2}}, \tag{5.3}$$

$$\beta_V^2 \equiv \gamma^2\beta^2 = \gamma^2 - 1. \tag{5.4}$$

The scaling transformation

$$t = \Gamma(\theta,\beta)T, \tag{5.5}$$

in which the scaling factor is

$$\Gamma(\theta,\beta) = \frac{\beta cos\theta + \sqrt{1-\beta^2 sin^2\theta}}{\sqrt{1-\beta^2}}, \tag{5.6}$$

determines the light trip's duration in terms of its geometric length.The ST may be written in the Euclidean form

$$t \equiv \sqrt{1-\beta^2}E(\pi-\theta,\beta)T, \tag{5.7a}$$

or

$$\frac{t}{\sqrt{1-\beta^2}} = E(\pi-\theta,\beta)T. \tag{5.7b}$$

where

$$E(\pi-\theta,\beta) = \frac{\Gamma(\theta,\beta)}{\sqrt{1-\beta^2}} \tag{5.8}$$

is named the Euclidean factor. In terms of the proper velocity β_V the ST is written thus

$$t = \Gamma(\theta,\beta_V)T \equiv E(\pi-\theta,\beta_V)\frac{T}{\sqrt{1+\beta_V^2}}, \tag{5.9a}$$

or

$$\sqrt{1+\beta_V^2}\,t = E(\pi-\theta,\beta_V)T, \tag{5.9b}$$

where

$$E(\pi-\theta,\beta_V) = \sqrt{1+\beta_V^2}\left(\beta_V cos\theta + \sqrt{1+\beta_V^2\,cos^2\theta}\right), \tag{5.10}$$

$$\Gamma(\theta,\beta_V) = \beta_V cos\theta + \sqrt{1+\beta_V^2\,cos^2\theta}. \tag{5.11}$$

The following figure depicts the same triangle but in terms of the universal velocity.

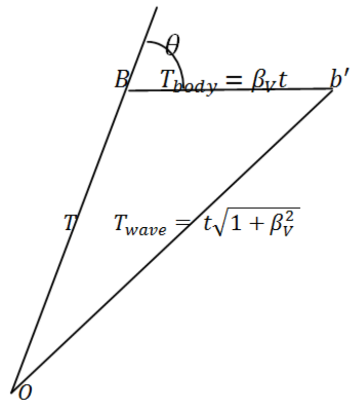

Figure 3. The body-observer triangle,
Sides' lengths n terms of the β_V

The inertial velocity is

$$\beta = \frac{T_{body}}{T_{wave}} = \frac{\beta_V}{\sqrt{1+\beta_V^2}} = \frac{V}{\sqrt{c^2+V^2}} \tag{5.12}$$

Also, from figure 1 the two sides T_{body} and T_{wave} satisfy the equation,

$$\frac{T_{body}}{\beta} = \frac{T_{wave}}{1} = \frac{t}{\sqrt{1-\beta^2}}, \tag{5.13}$$

In terms of β_V

$$\frac{T_{body}}{\beta_V}\sqrt{1+\beta_V^2} = T_{wave} = t\sqrt{1+\beta_V^2} \tag{5.14}$$

The following equality

$$(T_{wave})^2 - \left(T_{body}\right)^2 = \gamma^2(1-\beta^2)t^2 = t^2, \tag{5.15}$$

holds on the account of the identity $\gamma^2(1-\beta^2) = 1$

6. Momentum and Total Energy

The *momentum* of the particle b is defined by the product of its mass m and proper velocity:

$$\boldsymbol{p} = m\boldsymbol{V} = \frac{m\boldsymbol{v}}{\sqrt{1-(\boldsymbol{v}/c)^2}}. \tag{6.1}$$

Multiplying both sides of the identity $\gamma^2(\beta^2-1) = -1$ by m^2 we obtain $c^{-2}p^2 - \gamma^2 m^2 = -m^2$, or

$$\frac{p^2}{c^2} = \gamma^2 m^2 - m^2. \tag{6.2}$$

In the reduced system of units (RSUI$\equiv MKM$) mass, energy and momentum are all measured by the same unit, kilogram. The *right hand-side of (6.2) can be envisaged as a difference between the squares of two values of the mass or energy of the moving body corresponding to the states of motion and rest respectively*. Denoting these values by E ($\equiv M$) and E_0($\equiv m$) respectively, i.e,

$$E(kg) \equiv M(kg) = \frac{m}{\sqrt{1-\beta^2}}\,(kg), \quad E_0(kg) \equiv m\,(kg), \tag{6.3}$$

we write (6.2) in MKM in the form

$$p^2 = E^2 - E_0^2 = M^2 - m^2. \tag{6.4}$$

The latter relation reads: the state of motion of a body with rest mass m that is characterized by a momentum of magnitude p is accompanied by a *total energy, or kinetic mass*,

$$E = \frac{m}{\sqrt{1-\beta^2}} = \sqrt{m^2 + p^2}. \tag{6.5}$$

When p goes to zero, the total energy (or kinetic mass) tends to the rest energy (or rest mass) $E_0 = m$.

The force acting on a particle is defined as in Newtonian mechanics by the time rate of the change in its momentum :

$$\boldsymbol{f} = \frac{d\boldsymbol{p}}{dt} = \frac{d}{dt}(mV). \tag{6.6}$$

Note that $V = \beta_V$ in *MKM*. Differentiating both sides of the equation $M^2 = m^2 + p^2$ with respect to time, we obtain

$$\frac{dM}{dt} = \frac{1}{M}\boldsymbol{p}.\boldsymbol{f} = \boldsymbol{v}.\boldsymbol{f} = \frac{\boldsymbol{V}.\boldsymbol{f}}{\sqrt{1+\beta_V^2}}. \tag{6.7}$$

The relation (6.7) determines the instantaneous rate at which the mass changes under the action of a force when moving at velocity $\boldsymbol{V}$. Looking on M as the total energy of the particle, the equation (6.7) also determines the power of the force, i.e. the rate at which it does work. The work done by the force during a displacement $d\boldsymbol{r} = \boldsymbol{V}dt$ is given by

$$dW = \frac{\boldsymbol{f}.d\boldsymbol{r}}{\sqrt{1+V^2/c^2}}. \tag{6.8}$$

7. The Equation of Motion

We found in P2 that the motion of the free body b in S is governed by the equation

$$\boldsymbol{r}(t) - \boldsymbol{r}(o) = \frac{\boldsymbol{v}t}{\sqrt{1-\beta^2}},$$

where,
$\boldsymbol{r}(o)$ is the initial position of b
$\boldsymbol{v}$ is the geometric velocity of b
$\boldsymbol{r}(t)$ is the position

In terms of the universal velocity the equation of motion takes the Galilean form

$$\boldsymbol{r}(t) = \boldsymbol{r}(o) + \boldsymbol{V}t.$$

8. Proper Velocity and 4-Velocity

We show here the formal connection between proper velocity $\boldsymbol{V} = \gamma(v)\boldsymbol{v}$ and the 4-velocity:

$$U \equiv (U_0, U_1, U_2, U_3) = \gamma(v)(c, \boldsymbol{v}) = \gamma(v)(c, v_1, v_2, v_3) \tag{8.1}$$

where $\gamma(v) = 1/\sqrt{1 - v^2/c^2}$. The norm of the 4-velocity $\boldsymbol{U}$ is

$$\|\boldsymbol{U}\|^2 \equiv U_0^2 - U_1^2 - U_2^2 - U_3^2 = \gamma^2(c^2 - v^2) = c^2 \tag{8.2}$$

Now, combining the proper velocities of the light pulse $C = \gamma c$ and the body $\boldsymbol{V} = \gamma\boldsymbol{v}$ in one entity

$$(C, \boldsymbol{V}) \equiv \gamma(c, \boldsymbol{v}) \tag{8.3}$$

yields what is formally identical to the 4-vector velocity (8.1). Moreover the norm of the 4-vector $(C, \boldsymbol{V})$ satisfies (8.2). Indeed

$$C^2 - V^2 = \gamma^2(c^2 - v^2) = c^2 \tag{8.4}$$

The proper velocity which we introduced endows $(C, \boldsymbol{V})$ with a meaning that circumvents and transcends its meaning as a 4-vector in Minkowski space. To see this we make use of figure 2 to write

$$t(V\boldsymbol{i} + C\boldsymbol{e_L}) = T\boldsymbol{e},$$

which reads: the combined displacements of the body and the pulse it emits is the negative of the initial position vector. The proper velocities $V\boldsymbol{i}$ and $C\boldsymbol{e_L}$ results in the spatial displacement $tV\boldsymbol{i}$ of the body and the pulse $tC\boldsymbol{e_L}$.

Substituting in the identity

$$\boldsymbol{Ob'^2} - \boldsymbol{Bb'^2} = (c\gamma t)^2 - (v\gamma t)^2 = c^2t^2 \quad (8.5)$$

for t from the ST we get an important formula

$$\boldsymbol{Ob'^2} - \boldsymbol{Bb'^2} = \Gamma^2(\theta, \beta)(cT)^2 = \Gamma^2(\theta, \beta)\boldsymbol{OB}^2 \quad (8.6)$$

that relates the distances travelled by body, the emitted pulse, and the geometric distance between the initial position and the observer, involving only the squares of the body-observer triangle sides and one of its angles. For $\theta = \frac{1}{2}\pi$, the relation (8.6) becomes

$$\boldsymbol{Ob'^2} - \boldsymbol{Bb'^2} = \boldsymbol{OB}^2 \quad (8.9)$$

We may also construct an analogue to the 4-momentum by multiplying the 4-velocity by the rest mass m of the moving body;

$$\boldsymbol{P} = m(C, \boldsymbol{V}) = m\gamma(c, \boldsymbol{v}) = (\frac{E}{c}, \boldsymbol{p}) \quad (8.10)$$

The norm of $\boldsymbol{P}$ is

$$\|\boldsymbol{P}\|^2 = \frac{E^2}{c^2} - p^2 = m^2c^2$$

which can also be gotten from (6.2) or (8.3) and (8.4).

References

[1] Viazminsky, C. and Vizminiska, P. (2014) On Universal Space and Time. *Applied Mathematics*, **5**, 2530-2546. doi: 10.4236/am.2014.516243.

[2] Viazminsky, C. and Vizminiska, P. (2014) On Universal Mechanics and Superluminal Velocities. *Applied Mathematics*, **5**, 2728-2738. doi: 10.4236/am.2014.517260.

[3] C. P. Viazminsky and P. K. Vizminiska **September 7, 2019:**

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