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Review

Gravitational Forces in Vectorial Relativity

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ABSTRACT: It is known that Kepler's Laws can be derived from the Newton's Law of Universal Gravitation. For doing this, mass is considered as an invariable parameter. Although this consideration works wonderfully to solve most of problems in astronomy calculations, as well as in all physics, when astral body's speeds are so high and very precise measurements are required, the referred Kepler Laws do not cope enough what is expected. That's why the General Theory of Relativity materialized. As it was indirectly pointed out by Einstein in 1905, Newton and Kepler Laws do not consider the relativistic variation of mass with its velocity. The purpose of this review is to discuss the mathematical and theoretical physics underlying the development of the classical model as well as the presentation of an approach that naturally concludes with a modification of the Newton's Universal Force of Gravitation in order to adapt its conception to a mass-variation modern view (relativistic). This work considers a generic variable mass, without establishing any cause of its variation (Einstenian, or any other). It is worth mentioning that although this approach was not worked under the Einstein's General Theory of Relativity, but under a three-dimensional environment, it can be applied inside any relativistic gravitation theory, we discuss the conservation laws and the equations of motion in detail, and provide a number of (in our opinion) interesting and relevant applications.

KEYWORDS: Universal Gravitation Law, Kepler Laws, Angular Momentum, Vectorial Relativity.

INDEX

I INTRODUCTION

This review provides an introduction to the modeling of gravitation by taking mass varying with velocity, or in a more general way, varying with time. In this sense we have made an effort to keep the presentation pedagogical. The article will (hopefully) also be useful to researchers who work in areas discussing the exact validity of Einstein's General Relativity applied to gravitation.

Throughout the article we will assume that Relativity is the proper description of nature in the sense of mass variation, though we don't say anything about what kind of Relativity in order to

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allow the possibility of application of any theory. As we hope that the article will be used by students and researchers who are not necessarily experts in relativity, we have developed the discussion with the simplest mathematical tools required to build models of relativistic objects. Given the potential for future applications of this formalism, we have opted to base much of our description on Franco's work on the same topic [1].

II GRAVITATION FOR CONSTANT MASSES. KEPLER LAWS.

This topic can be easily found in the literature, but we are going to present a brief description of it in order to introduce the theme.

For this, we assume that gravitation is a central force, which means that the direction of forces between bodies attracting each other are co-linear with the line joining their centers of masses. For simplicity we consider that one of the bodies attracting the other body is fixed. In this way trajectory of moving body can be given in polar coordinates. We will introduce the variable unit vectors $\mathbf{U_r}$, in the direction of the attracting force, and $\mathbf{U_\theta}$, normal to $\mathbf{U_r}$, which can be defined as:

Fig. 1. Representation in polar coordinates of unit vectors taking aphelion or perihelion as initial point.

$$\mathbf{U_r} = \mathbf{i}.\cos\theta + \mathbf{j}.\sin\theta; \qquad \mathbf{U}_\theta = -\mathbf{i}.\sin\theta + \mathbf{j}.\cos\theta \qquad \Rightarrow \quad \begin{cases} \dfrac{d\mathbf{U_r}}{d\theta} = -\mathbf{i}.\sin\theta + \mathbf{j}.\cos n\theta = U_\theta \\ \dfrac{d\mathbf{U_\theta}}{d\theta} = -\mathbf{i}.\cos\theta - \mathbf{j}.\sin\theta = -U_r \end{cases}$$

Letting $\mathbf{r} = r.\mathbf{U_r}$, the following expressions can be obtained:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}.\mathbf{U_r} + r.\frac{d\theta}{dt}.\mathbf{U}_\theta \quad \text{and} \quad \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right].\mathbf{U_r} + \left[r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}\frac{d\theta}{dt}\right].\mathbf{U}_\theta \qquad (1)$$

By applying Newton's Universal Gravitation Law to two attracting bodies of constant masses M and m, we can write that:

$$\frac{d\mathbf{p}}{dt} = m.\mathbf{a} = -F.\mathbf{U_r} = -\frac{G.M.m}{r^2}.\mathbf{U_r} \qquad \Rightarrow \qquad \mathbf{a} = -\frac{G.M}{r^2}.\mathbf{U_r}$$

Substituting the expression of acceleration in (1) we arrive at:

$$\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right].\mathbf{U_r} + \left[r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}\frac{d\theta}{dt}\right].\mathbf{U}_\theta = -\frac{G.M}{r^2}\mathbf{U_r}$$

This vectorial equation originates the following two scalar equations:

1) $r.\dfrac{d^2\theta}{dt^2} + 2.\dfrac{dr}{dt}\dfrac{d\theta}{dt} = 0$

2) $\dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 = -\dfrac{G.M}{r^2}$

By remembering that angular velocity is defined as $\omega = \dfrac{d\theta}{dt}$, The first equation can be written as:

$$r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}\frac{d\theta}{dt} = 0 \Rightarrow \frac{\frac{d^2\theta}{dt^2}}{\frac{d\theta}{dt}} + 2.\frac{\frac{dr}{dt}}{r} = \frac{\frac{d\omega}{dt}}{\omega} + 2.\frac{\frac{dr}{dt}}{r} = 0 \Rightarrow \frac{d\omega}{\omega} + 2.\frac{dr}{r} = 0 \Rightarrow Log\,\frac{\omega}{\omega_0} = Log\,\frac{r_0^2}{r^2}$$

$$\omega.r^2 = \omega_0.r_0^2 = \text{Constant} \qquad \Rightarrow \qquad \frac{d\theta}{dt}.r^2 = \text{Constant} \qquad \text{for } \omega_0,\ r_0 \text{ measured at perihelion}$$

From this last result can be inferred the second Kepler's Law for planets motion, because the differential element of area of the trajectory described by the moving planet in its movement around the sun, in polar coordinates, is given by:

$$dA = \frac{1}{2}r^2.d\theta$$

And the referred result implies that ***"the line joining a planet and the Sun sweeps out equal areas in equal intervals of time"***, and the conservation of angular moment:

$$\frac{dA}{dt} = \frac{1}{2}\frac{d\theta}{dt}.r^2 = \frac{1}{2}\omega.r^2 = \text{Constant} \qquad \textbf{(Second Kepler's Law)}$$

For perihelion and Aphelion, this law can be put as:

$$\frac{dA}{dt} = \frac{1}{2}\omega_0.r_0^2 = \frac{1}{2}v_0.r_0 \qquad \frac{dA}{dt} = \frac{1}{2}\omega_a.r_a^2 = \frac{1}{2}v_a.r_a \tag{2}$$

The development of the second scalar equation: $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{G.M}{r^2}$, drives to the following relation:

$$\left(\frac{dr}{dt}\right)^2 = r_0^2.v_0^2.\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) - 2.G.M.\left(\frac{1}{r_0} - \frac{1}{r^{\cdot}}\right). \tag{3}$$

Which, evaluated at aphelion starting from perihelion, and remembering that $r_0.v_0 = r_a.v_a$, gives:

$$r_0^2.v_0^2 = \frac{2.G.M}{\left(\frac{1}{r_0} + \frac{1}{r_a}\right)} \Rightarrow r_0.v_0.r_a.v_a = \frac{2.G.M}{\left(\frac{1}{r_0} + \frac{1}{r_a}\right)} \Rightarrow v_0.v_a = \frac{2.G.M}{(r_a + r_0)} \Rightarrow r_a + r_0 = \frac{2.G.M}{v_0.v_a}$$

Evaluated at perihelion starting from aphelion, it gives the same constant result for the addition of radiuses at perihelion and aphelion:

$$r_a^2.v_a^2 = \frac{2.G.M}{\left(\frac{1}{r_a} + \frac{1}{r_0}\right)} \Rightarrow r_a.v_a.r_0.v_0 = \frac{2.G.M}{\left(\frac{1}{r_a} + \frac{1}{r_0}\right)} \Rightarrow v_0.v_a = \frac{2.G.M}{(r_a + r_0)} \Rightarrow r_a + r_0 = \frac{2.G.M}{v_0.v_a}$$

After this parenthesis, the equation in (2) has the following solution:

$$r = \frac{1/h}{1 + e.\cos\theta} \quad \text{for} \quad h = \frac{G.M}{\omega_0.r_0^2}, \quad e = \frac{1}{r_0.h} - 1 \quad \text{and} \quad h = \frac{1}{r_0.(1+e)} \tag{4}$$

The value of the radius given by the first expression in (4) provides the path of the planet and it represents a conic with eccentricity e, with the sun in one of the focus. This fact implies ***the first Kepler's law: "the orbit of a planet about the Sun is an ellipse with the Sun's center of mass at one focus"***.

As it can be observed in the first expression in (4) the only variables are the radius and the angle. The maximum radius r_a, which occurs at aphelion, for $\theta = \pi$, can be calculated as:

$$r_a = \frac{1/h}{1-e} = \frac{r_0.(1+e)}{1-e} \qquad \Rightarrow \qquad \frac{r_a}{r_0} = \frac{(1+e)}{(1-e)} \tag{5}$$

Thus, the length of the major axis will be: $2a = r_a + r_0 = \dfrac{2.r_0}{1-e} \quad \Rightarrow \quad r_0 = a.(1-e), \quad r_a = a.(1+e)$

The length of the minor axis will be, from a triangle rectangle whose sides are: a, the hypothenus e, b the opposite side and $c = e.a$, the base of triangle : $b = \sqrt{a^2 - (e.a)^2} = a.\sqrt{1-e^2}$.

Let T be the period corresponding to one complete orbit. So, the radio-vector will sweep out the area $\pi.a.b$ of the ellipse in T units of time. In this way, from second law,

$$\frac{dA}{dt} = \frac{\pi.a.b}{T} \quad \Rightarrow \quad T = \frac{2.\pi.a.b}{r_0.v_0} = \frac{2.\pi.a^2.\sqrt{1-e^2}}{r_0.v_0} \tag{6}$$

Now, in order to check the third law, let's obtain the expression for the speed of the planet at perihelion, v_0, in function of the eccentricity e. For this we use the correct definition of Energy K:

$$dK = \frac{d\mathbf{p}}{dt}.d\mathbf{r} = -F.\mathbf{U_r}.d\mathbf{r} = -\frac{G.M.m}{r^2}.\mathbf{U_r}.d\mathbf{r} \quad \Rightarrow \quad dK = mdv.v = -\frac{G.M.m}{r^2}.dr$$

Doing the integration between perihelion and aphelion, we obtain the Energy conservation law:

$$\int dK = \int mdv.v = -\int \frac{G.M.m}{r^2}.dr \;\Rightarrow\; \frac{1}{2}m(v_a^2 - v_0^2) = G.M.m.\left(\frac{1}{r_a} - \frac{1}{r_0}\right)$$

Simplifying, and substituting from (4),

$$\begin{aligned} &(v_0^2.\frac{r_0^2}{r_a^2} - v_0^2) = 2.G.M.\left(\frac{1}{r_a} - \frac{1}{r_0}\right) \quad \Rightarrow \quad v_0^2.\left(\frac{(1-e)^2}{(1+e)^2} - 1\right) = 2.G.M.\left(\frac{1}{a.(1+e)} - \frac{1}{a.(1-e)}\right) \\ &v_0^2.\left(\frac{-4.e}{(1+e)^2}\right) = \frac{-4.G.M.e}{a.(1-e^2)} \quad \Rightarrow \quad v_0 = \sqrt{\frac{G.M.(1+e)}{a.(1-e)}} \end{aligned} \tag{7}$$

Substituting (7) in (6), and remembering that $r_0 = a.(1-e)$, we finally obtain **Kepler's third law**:

$$T = \frac{2.\pi.a.b}{r_0.v_0} = \frac{2.\pi.a^2.\sqrt{1-e^2}}{a.(1-e).\sqrt{\dfrac{G.M.(1+e)}{a.(1-e)}}} = \frac{2.\pi.a.\sqrt{1-e^2}}{\sqrt{\dfrac{G.M.(1-e).(1+e)}{a}}} = \frac{2.\pi.\sqrt{a^3}}{\sqrt{G.M}} \quad \Rightarrow \quad \frac{T^2}{a^3} = \frac{4.\pi^2}{G.M} \tag{8}$$

The squares of the orbital periods of planets are directly proportional to the cubes of the semi-major axis of the orbits: $T^2 \propto a^3$**.**

As it has been previously demonstrated, Kepler's Laws can be obtained from the Newton's Universal Law of Gravitation. But as it has been accepted worldwide in modern physics, value of mass depends on its velocity and previous analysis does not take into account this fact, it is not correct, at least for high velocities of planets. Another important aspect not predicted by this analysis is the advance of perihelion observed in planets motion, or trajectories of planets are not perfect ellipses. Nevertheless, it has worked as a good approximation for practical uses.

Next section deals with the investigation of how a variable mass as a photon behaves under the influence of the gravitational field created by a massive body.

III EFFECT OF GRAVITATION FIELD ON LIGHT.

In this part it is shown that analysis of the mass of a photon, treated as a variable mass (or relativistic), attracted by a massive body with a gravitational force given by the Newton's Universal Law of Gravitation, leads to an erroneous and fallacious result. This could indicate that the treatment of gravitation when variable masses are taken into account either is not so simple as it was with constant masses, or indicates that something is lacking in Newton's Universal Law of Gravitation.

It is known that although photon does not have rest mass when it is traveling at speed c, it has a non-zero mass given by the correct relationship, $m = \frac{E}{c^2}$ [2]. Given this feature of photons, they can be attracted by the gravitational field of a massive body and take a curvilinear path as any other body. Let's recall that when a photon changes its mass only changes its frequency, so, its velocity magnitude, c, remains as it is: an universal constant.

Let's start setting up the problem in the following way: Let a photon be attracted by a massive body, approaching onto it at a minimum distance denoted by a radio-vector R_0, measured from center of mass of the massive body to a point P_0, located at this minimum distance, such that radio-vector R_0, forms an angle of 90 degrees with the photon's vector velocity $\mathbf{c}$, at P_0. Let's try to find the deflection of the photon, produced onto its variable mass by the gravitational field of the massive body.

By considering gravitational force as central, given by Newton's Universal Law and putting photon mass m as the quotient between the linear Momentum p, and the speed of light c, $m = \frac{p}{c}$, we can write that:

$$\frac{d\mathbf{p}}{dt} = -F.\mathbf{U}_r = -\frac{G.M.p}{c.r^2}.\mathbf{U}_r \tag{9}$$

where, G is the gravitational constant, M the massive body's mass, and $\mathbf{U}_r$ the unit vector on the direction pointing towards gravitational central force. Minus sign indicates the contrary sense of centrifugal force, $\frac{d\mathbf{p}}{dt}$. Thus, by making: $\mathbf{p} = \left(\frac{p}{c}\right).\mathbf{c}$:

$$\frac{d\mathbf{p}}{dt} = \frac{d\left(\frac{p}{c}.\mathbf{c}\right)}{dt} = \left(\frac{p}{c}.\frac{d(\mathbf{c})}{dt} + \frac{\mathbf{c}}{c}.\frac{dp}{dt}\right) = -\frac{G.M.p}{c.r^2}.\mathbf{U_r} \tag{10}$$

Expressing, as before, in polar form for plane curvilinear motion the vector velocity of light , $\mathbf{c}$, and its acceleration vector, $\frac{d\mathbf{c}}{dt}$, as function of the unit vectors $\mathbf{U}_r$ ($\mathbf{U}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$) and $\mathbf{U}_\theta$ ($\mathbf{U}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$). The angle θ, swept by radio-vector $\mathbf{r}$ of the photon's mass in its movement with origin in the center of mass of the massive body, beginning at P_0 for $\theta = t = 0$, until a generic point of the trajectory, P. Operating on $\mathbf{r} = r.\mathbf{U}_r$, we obtain:

$$\frac{d\mathbf{r}}{dt} = \mathbf{c} = \frac{dr}{dt}.\mathbf{U}_r + r.\frac{d\theta}{dt}.\mathbf{U}_\theta = \frac{dr}{dt}.\mathbf{U}_r + r.\omega.\mathbf{U}_\theta$$

$$\mathbf{c} = \frac{dr}{dt}.\mathbf{U}_r + r.\omega.\mathbf{U}_\theta \tag{11}$$

Also, we can get:

$$\frac{d\mathbf{c}}{dt} = \left[\frac{d^2r}{dt^2} - r.\left(\frac{d\theta}{dt}\right)^2\right].\mathbf{U}_r + \left[r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}.\frac{d\theta}{dt}\right].\mathbf{U}_\theta \tag{12}$$

Substituting these results in (2), dividing by $\frac{p}{c}$, and simplifying, it follows that:

$$\left[\frac{d^2r}{dt^2} - r.\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{p}.\frac{dp}{dt}.\frac{dr}{dt}\right].\mathbf{U}_r + \left[r.\frac{d^2\theta}{dt^2} + 2.\frac{dr}{dt}.\frac{d\theta}{dt} + \frac{r}{p}.\frac{dp}{dt}.\frac{d\theta}{dt}\right].\mathbf{U}_\theta = -\frac{G.M}{r^2}.\mathbf{U}_r \tag{13}$$

this vectorial equation originates the following two conditions:

a) What is multiplied by $\mathbf{U}_\theta$, must be zero because gravitational force only have component on $\mathbf{U}_r$, and

b) What is multiplied by $\mathbf{U}_r$ equals $-\frac{G.M}{r^2}$.

From the first condition, we obtain:

$$.\left(r.\frac{d^2\theta}{dt^2}+2.\frac{dr}{dt}.\frac{d\theta}{dt}+\frac{r}{p}.\frac{dp}{dt}.\frac{d\theta}{dt}\right)=0 \quad\Rightarrow\quad \frac{d^2\theta}{dt^2}=-\left(\frac{2}{r}.\frac{dr}{dt}.\frac{d\theta}{dt}+\frac{1}{p}.\frac{dp}{dt}.\frac{d\theta}{dt}\right)$$

$$\frac{\frac{d^2\theta}{dt^2}}{\frac{d\theta}{dt}}=-\left(\frac{2}{r}.\frac{dr}{dt}+\frac{1}{p}.\frac{dp}{dt}\right) \Rightarrow \text{for,}\,\omega=\frac{d\theta}{dt} \Rightarrow \frac{\frac{d\omega}{dt}}{\omega}=-\left(2.\frac{dr}{r}+\frac{dp}{p}\right).\frac{1}{dt} \Rightarrow \frac{d\omega}{\omega}=-\left(2.\frac{dr}{r}+\frac{dp}{p}\right)$$

By integrating this last expression from (ω_0, r_0, p_0) to (ω, r, p), between P_0 and another generic point P of the photon trajectory, it becomes:

$$\Rightarrow \ \mathbf{ln}\frac{\omega}{\omega_0}=-2.\mathbf{ln}\frac{r}{r_0}-\mathbf{ln}\frac{p}{p_0} \qquad \Rightarrow \qquad \omega.r.p=\omega_0.r_0^2.p_0=\text{Constant} \tag{14}$$

If we divide both members by c, the speed of light, we recognize in equation (14) another version of the angular momentum for light, constant, ratifying the conservation of Angular Momentum Law. Thus, we have obtained a consistent relationship (14) working on the first condition.

Applying the second condition and substituting $\frac{d\theta}{dt}$ by ω, we have:

$$\frac{d^2r}{dt^2}-r.\omega^2+\frac{1}{p}.\frac{dp}{dt}.\frac{dr}{dt}=-\frac{G.M}{r^2} \tag{15}$$

Before working on this expression, let's try to obtain an equivalent one to $\frac{1}{p}.\frac{dp}{dt}$, through the following general Energy definition and known relations (10) and (11):

$$K=\frac{d\mathbf{p}}{dt}.d\mathbf{r}=-G.\frac{M.p}{c.r^2}.\mathbf{U}_r.d\mathbf{r} \ \Rightarrow\ d\mathbf{p}.\frac{d\mathbf{r}}{dt}=d\mathbf{p}.\mathbf{c}=dp.c=-G.\frac{M.p}{c.r^2}.\mathbf{U}_r.(dr.\mathbf{U}_r+\omega.r.\mathbf{U}_\theta)$$

$$dp.c=-G.\frac{M.p}{c.r^2}.dr \qquad\Rightarrow\qquad \frac{dp}{p.dt}=-G.\frac{M}{c^2.r^2}.\frac{dr}{dt} \tag{16}$$

On the other hand, light speed can be put as: $c^2=\left(\frac{dr}{dt}\right)^2+(\omega.r)^2=q^2+(\omega.r)^2$, for $q=\frac{dr}{dt}$ (17)

By introducing equations (16) and (17) in (15), we obtain:

$$\frac{d^2r}{dt^2}-r.\omega^2-\frac{G.M}{c^2r^2}\left(c^2-r^2.\omega^2\right)=-\frac{G.M}{r^2}.\omega^2$$

Substituting by $\omega=\frac{d\theta}{dt}$: $\quad \frac{d^2r}{dt^2}-r.\omega^2=-\frac{G.M}{c^2}.\omega^2 \quad\Rightarrow\quad \frac{d^2r}{d\theta^2}-r=-\frac{G.M}{c^2}$ (18)

This second order differential equation, inhomogeneous, has as general solution:

$$r = A.e^{\theta} + B.e^{-\theta} - \frac{G.M}{2.c^2}\left[e^{\theta} + e^{-\theta} - 2\right]$$

Where initial conditions for obtaining the values of constants A and B, are: for angle $\theta = 0$, $r = R_0$, and $\frac{dr}{d\theta} = 0$, because radius is minimum at P_0. Thus, the explicit and general solution for any generic point P, on the trajectory, becomes:

$$r = \frac{R_0}{2}.e^{\theta} + \frac{R_0}{2}.e^{-\theta} - \frac{G.M}{2.c^2}\left[e^{\theta} + e^{-\theta} - 2\right] = R_0.\cosh\theta - \frac{G.M}{c^2}\left[\cosh\theta - 1\right] \qquad (19)$$

Let's analyze this result. If instead of a massive body a mathematical point existed single ($M = 0$) "attracting" the photon, instead of a curvilinear path the photon will describe a rectilinear movement,

Fig. 2. Differences between classical equation (12) and equation (11)

and its mass, energy and frequency will remain constants. In such situation the well-known first Kepler's law applies and also its general solution:

$$r = \frac{1}{h + \left(\frac{1}{R_0} - h\right).\cos\theta}; \quad \text{where} \quad h = \frac{G.M}{c^2.R_0^2}; \ \text{for } M = 0 \ \Rightarrow \ r = R_0.\sec\theta \qquad (20)$$

But, the solution given by equation (19) is $r = R_0.\cosh\theta$.

Then, equation (19), for $M = 0$, for any θ, gives as result for radius: $r = R_0.\cosh\theta$, which is inconsistent and not valid. In fact, its solution differs from the correct one given by equation (20).

Thus, we have to do an inventory of present assumptions and postulates in this development:

1) Speed of light is constant as in rectilinear as in curvilinear motion. It a proved postulate. It will be maintained.
2) Definition of photon mass, variable. It comes from the previous postulate.
3) Gravitational forces are central. It has wonderfully worked for constant moving masses. We will sustain it.
4) Definition of Newton's Gravitational Force has been accepted as valid for movable variable masses. In Author's opinion, as it has been shown in previous development, it can be presumed to be valid only between static attracting bodies, or for bodies that move at constant speed under the attraction of a gravitational field (circular motion).

Now, let's look for a gravitational law expression, so that logical and expected results are obtained.

IV DERIVATION OF THE GRAVITATIONAL LAW FOR PHOTONS

Let's try to find out where the weakness of Newton's Gravitational Force could be.

Thus, let's restart the analysis of a photon under the effect of a gravitational field produced by a massive body. Let's work with general equations trying to avoid as possible, assumptions. Let's start supposing that we don't know the expression of the Newton's universal gravitational force. The second condition in equation (15) keeping as central the unknown gravitational force, F :

$$\frac{d^2r}{dt^2} - r.\omega^2 + \frac{1}{p}.\frac{dp}{dt}.\frac{dr}{dt} = -\frac{F.c}{p} \qquad (21)$$

By preserving the definition of the gravitational field:

$$\frac{d\mathbf{p}}{dt} = -F.\mathbf{U_r} \qquad \Rightarrow \qquad \frac{d\mathbf{p}.d\mathbf{r}}{dt} = -F.d\mathbf{r}.\mathbf{U_r} \qquad \Rightarrow \qquad dp.c = -F.dr$$
$$\frac{dp}{p} = -\frac{F}{p.c}.dr = -\frac{\mathrm{G}}{c^2}.dr \quad \text{for, } F = \frac{\mathrm{G}.p}{c}; \quad \text{or, } \mathrm{G} = \frac{F.c}{p} \qquad (22)$$

Where G, denotes the gravitational field. From (17) we can put (21) as:

$$\frac{d^2r}{dt^2} - r.\omega^2 - \frac{F}{p.c}.\frac{dr}{dt}.\frac{dr}{dt} = \frac{d^2r}{dt^2} - r.\omega^2 - \frac{\mathrm{G}}{c^2}.\left(c^2 - \omega^2.r^2\right) = -\mathrm{G}$$

$$\frac{d^2r}{dt^2} - r.\omega^2 + \frac{\mathrm{G}}{c^2}\omega^2.r^2 = 0 \qquad (23)$$

Let's obtain other general relationships, in order to compare them with the original Newton' expressions. We know, by the law of conservation of the angular momentum, that:

$$\frac{dp}{p} = -\frac{d(\omega.r^2)}{\omega.r^2}$$

So, from (22) we can establish the following relation:

$$\frac{d(\omega.r^2)}{\omega.r^2} = \frac{\mathrm{G}}{c^2}.dr \qquad \Rightarrow \qquad \omega.r^2.d(\omega.r^2) = \left(\omega.r^2\right)^2 \frac{\mathrm{G}}{c^2}.dr$$

Making the integration from the minimum radius (perihelion) to a generic radius:

$$\omega^2.r^4 - \omega_0^2.r_0^4 = 2.\int_{r_0}^{r} \left(\omega.r^2\right)^2 \frac{\mathcal{G}}{c^2}.dr$$

What is just like to write:

$$\frac{1}{p^2}-\frac{1}{p_0^2}=\frac{2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2\frac{\mathrm{G}}{c^2}.dr \qquad \Rightarrow \qquad p^2=\frac{p_0^2}{1+\frac{2.p_0^2}{c^2.K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr}$$

From here we can obtain:

$$p^2=\frac{p_0^2}{1+\frac{2.p_0^2}{c^2.K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathcal{G}.dr}=\frac{c^2.p_0^2}{c^2+\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathcal{G}.dr}=\frac{\omega_0^2.r_0^2.p_0^2}{c^2+\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathcal{G}.dr}$$

$$p^2=\frac{\omega_0^2.r_0^2.p_0^2}{c^2+\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathcal{G}.dr}=\frac{\omega_0^2.r_0^4.p_0^2.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)+\frac{\omega_0^2.r_0^4.p_0^2}{r^2}}{c^2+\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathcal{G}.dr}$$

$$p^2.\left[c^2-\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr\right]=\omega_0^2.r_0^4.p_0^2.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)+\frac{\omega_0^2.r_0^4.p_0^2}{r^2}$$

$$p^2.\left[c^2-\frac{\omega_0^2.r_0^4.p_0^2}{p.^2r^2}+\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr\right]=\omega_0^2.r_0^4.p_0^2.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)$$

$$p^2.\left[c^2-\omega^2.r^2+\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathcal{G}.dr\right]=\omega_0^2.r_0^4.p_0^2.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)$$

$$p^2.\left[q^2+\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathcal{G}.dr\right]=\omega_0^2.r_0^4.p_0^2.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)$$

By remembering that $c^2=q^2+\omega^2.r^2$, this takes us to a relation similar to that of the solution for the classical case of constant mass:

$$\Rightarrow \qquad q^2=\frac{\omega_0^2.r_0^4.p_0^2}{p^2}.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)-\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathrm{G}.dr$$

On the other hand,

$$\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\mathcal{G}.dr=\frac{2.p_0^2}{K^2}\int_{r_0}^{r}\frac{K^2}{p^2}.(-c^2\frac{dp}{p})=-p_0^2.c^2.\left(\frac{1}{p_0^2}-\frac{1}{p^2}\right)$$

In this way, we can write down the following **exact** expression for photon's motion:

$$q^2 = \omega^2 .r^4 .\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) + p_0^2 .c^2 .\left(\frac{1}{p_0^2} - \frac{1}{p^2}\right) \tag{24}$$

A similar expression is found in classical analysis for constant mass (except for the second term at right):

$$q^2 = v_0^2 .r_0^2 .\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) - 2.G.M.\left(\frac{1}{r_0} - \frac{1}{r}\right) \tag{25}$$

Let's work on this classical expression to establish some criteria to orientate our research. When in elliptic motion a constant mass, coming from perihelion, r_0, pass through aphelion, q becomes null. That is what the classical equation (25) means. Evaluating (25) at aphelion we obtain:

$$v_0^2 .r_0^2 = \frac{2.G.M}{\dfrac{1}{r_0} + \dfrac{1}{r_A}}$$

Changing the roles of parameters or which is just like to say that we are evaluating (25) at perihelion but coming from aphelion. Thus, we obtain:

$$v_A^2 .r_A^2 = \frac{2.G.M}{\dfrac{1}{r_A} + \dfrac{1}{r_0}}$$

From these two results, it follows that for any constant mass that moves elliptically around a mass M :

$$v_0^2 .r_0^2 = v_A^2 .r_A^2 \;\Rightarrow\; \omega_0^2 .r_0^4 = \omega_A^2 .r_A^4$$

Observe that this last expression is the angular momentum for constant mass, evaluated at aphelion and perihelion: $m^2 .\omega_0^2 .r_0^4 = m^2 \omega_A^2 .r_A^4$. By taking this classical solution as a clue or as a starting point for developing our study in a similar manner for photon's motion, and remembering from the result obtained by applying the first condition for photon, $\omega_0 .r_0^2 .p_0 = \omega .r^2 .p$, allow the second term at the right in equation (16) being modified in a similar way as:

$$q^2 = \omega^2 .r^4 .\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) - 2.G.M.\left(\frac{1}{r_0} - \frac{1}{r}\right) \tag{26}$$

In this way, we would be in the same situation as before, when we were in the classical case. For an elliptic motion of the variable mass of **photon**, coming from perihelion, r_0, and just passing through aphelion, the radial velocity q becomes null at this moment. That is what is deduced from equation (26). Evaluating (26) at aphelion and at perihelion, but now for photon, we obtain:

$$\omega_A^2.r_A^4 = c^2.r_A^2 = \frac{2.G.M}{\frac{1}{r_0}+\frac{1}{r_A}} \qquad \omega_0^2.r_0^4 = c^2.r_0^2 = \frac{2.G.M}{\frac{1}{r_A}+\frac{1}{r_0}} \quad \Rightarrow \quad c^2.r_A^2 = c^2.r_0^2 \quad \Rightarrow \quad r_A = r_0 \qquad \textbf{(?)}$$

As we see, these two equations lead to the contradiction that radiuses at perihelion and aphelion are forced to be equal given the constant speed of photon, namely, only circular motion is allowed by equation (26), and worst of all, conservation of angular moment is not preserved, because if $p_A.c^2.r_A^2 = p_0.c^2.r_0^2 \Leftrightarrow p_A = p_0$. How to break this illogical limitation? From analyzing what happened with classical case, we conclude that our goal should be to obtain a factor inside equation (26) that solves the inconsistency of radiuses, preserving the constancy of angular momentum for photon's motion. Then, we have to look for an expression slightly different to that of (26) such that after evaluating, radiuses contradiction becomes broken and we can arrive at the equality of angular momentum, at perihelion and aphelion, i.e.:

$$c^2.r_0^2.p_0^2 = c^2.r_A^2.p_A^2$$

We have found that the following general expression for q^2, with a factor of the linear momentums quotient multiplying the second term at right, fulfills the requirement of the angular momentum's conservation and radiuses' limitation is broken:

$$q^2 = \frac{\omega_0^2.r_0^4.p_0^2}{p^2}.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right) - 2.G.M.\frac{p_0}{p}.\left(\frac{1}{r_0}-\frac{1}{r}\right) \tag{27}$$

Previous one is the general expression starting from perihelion. By considering as starting from aphelion, the following general expression shows up:

$$q^2 = \frac{\omega_A^2.r_A^4.p_A^2}{p^2}.\left(\frac{1}{r_A^2}-\frac{1}{r^2}\right) - 2.G.M.\frac{p_A}{p}.\left(\frac{1}{r_A}-\frac{1}{r}\right) \tag{28}$$

Evaluating equations (27) at aphelion and (28) at perihelion, we obtain the expected expression of constant angular momentum, which was our corrected and expected goal:

$$c^2.r_0^2.\frac{p_0}{p_A} = c^2.r_A^2.\frac{p_A}{p_0} = \frac{2.G.M}{\frac{1}{r_0}+\frac{1}{r_A}} \quad \Rightarrow \quad K^2 = c^2.r_0^2.p_0^2 = p_0.p_A.\frac{2.G.M}{\frac{1}{r_0}+\frac{1}{r_A}} = c^2.r_A^2.p_A^2$$

$$c.r_0.p_0.c.r_A.p_A = p_0.p_A.\frac{2.G.M}{\frac{1}{r_0}+\frac{1}{r_A}} = c.r_A.p_A.c.r_0.p_0 \quad \Rightarrow \quad c^2 = \frac{2.G.M}{r_0 + r_A} = c^2$$

Which implies that addition of radiuses, starting from perihelion or from aphelion, has always the same value: $r_0 + r_A = 2.G.M/c^2$. A similar situation is observed for constant masses in equation (3). For circular motion of photon around M, where, $r_A = r_0$; $p_A = p_0$, the following result arises:

$$c^2 = \frac{G.M}{r_0} \qquad \Rightarrow \qquad r_0 = \frac{G.M}{c^2}$$

Observe carefully this probable exact result, where, circular motion of a photon of light depends on the mass M *of the massive body, and has a radius that has a unique value:* $r_0 = G.M / c^2$*. For instance, if the boundary of the body, at* $r = r_B$*, is inside the radius of the circular motion of the photon, i.e.,* $r_B < r_0 = G.M / c^2$*, then this body is known as black hole, because photon of light whose perihelion,* r_P*, is between these two values* $r_B < r_P < r_0$ *only can have an elliptical motion around the black hole. If its perihelion will be into the black hole border then it will strike the body surface and it will be absorbed. Also, it is worth noting that obtained result,* $r_0 = G.M / c^2$*, is half of the radius known in the relativistic jerk as Schwarzschild radius [3]. This result could be also obtained from the very first differential and exact equation (23), though. For instance, for constant radius (circular motion) Newton's Gravitation holds and it readily arises:*

$$\frac{d^2 r_0}{dt^2} - r_0.\omega_0^2 + \frac{G.M}{c^2}\omega_0^2 = 0 \qquad \Rightarrow \quad \text{For} \quad \frac{d^2 r_0}{dt^2} = 0 \;\Rightarrow\; r_0.\omega_0^2 = \frac{G.M}{c^2}\omega_0^2 \Rightarrow \quad r_0 = \frac{G.M}{c^2}$$

After this necessary comment, let's continue. Observe that the expression (27) takes implicit assuming the equality given next, which will be continuously checked throughout all this work.

$$p_0^2.c^2.\left(\frac{1}{p_0^2} - \frac{1}{p^2}\right) = -2.G.M.\frac{p_0}{p}.\left(\frac{1}{r_0} - \frac{1}{r}\right) \tag{29}$$

Starting from the expression (27), let's try now to obtain the expression for the gravitational field. For that, our strategy will be to take derivatives relative to time, in order to construct an equation similar to the equation (21), where the force and therefore the gravitational field are unknowns. Operating on equation (27):

$$q^2 = \frac{\omega_0^2.r_0^4.p_0^2}{p^2}.\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) - 2.G.M.\frac{p_0}{p}.\left(\frac{1}{r_0} - \frac{1}{r}\right) = \frac{K^2}{p^2}.\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) - 2.G.M.\frac{p_0}{p}.\left(\frac{1}{r_0} - \frac{1}{r}\right)$$

$$2.q.dq = \frac{2.K^2}{p^2}.\frac{dr}{r^3} - 2.\left(\frac{1}{r_0^2} - \frac{1}{r^{2.}}\right)\frac{K^2}{p^3}.dp + 2.G.M.\left(\frac{1}{r_0} - \frac{1}{r^{\cdot}}\right).\frac{p_0}{p^2}.dp - 2.G.M.\frac{p_0}{p}.\frac{dr}{r^2}$$

$$q.dq = \frac{K^2}{p^2}.\frac{dr}{r^3} - \left(\frac{1}{r_0^2} - \frac{1}{r^{2.}}\right)\frac{K^2}{p^3}.dp + G.M.\left(\frac{1}{r_0} - \frac{1}{r^{\cdot}}\right).\frac{p_0}{p^2}.dp - G.M.\frac{p_0}{p}.\frac{dr}{r^2}$$

$$q.\frac{dq}{dr} = \omega^2.r + \frac{1}{r_0^2}\frac{K^2}{p^2}.\frac{\mathcal{G}}{c^2} - \frac{1}{r^2}\frac{K^2}{p^2}.\frac{\mathcal{G}}{c^2} - G.M.\frac{p_0}{p}.\frac{\mathcal{G}}{c^2}\left(\frac{1}{r_0} - \frac{1}{r}\right) - G.M.\frac{p_0}{p}.\frac{1}{r^2}$$

Thus, replacing and regrouping, by observing that: $q.\frac{dq}{dr} = \frac{dr}{dt}.\frac{d}{dr}\left(\frac{dr}{dt}\right) = \frac{d^2 r}{dt^2}$

$$\frac{d^2r}{dt^2}=\omega^2.r-\frac{\omega^2.r^2}{c^2}.\mathcal{G}+\left[\mathcal{G}.\frac{p_0^2}{p^2}-G.M.\frac{p_0}{p}.\frac{\mathrm{G}}{c^2}\left(\frac{1}{r_0}-\frac{1}{r}\right)-\frac{G.M}{r^2}.\frac{p_0}{p}\right]$$

Here we must make the term between parenthesis null, because equation (23), for the case of the photon, does not contain this term. Since within the parenthesis it appears the Gravitation Field like variable, this condition takes us to obtain it. In fact, from:

$$\mathcal{G}.\frac{p_0^2}{p^2}-G.M.\frac{p_0}{p}.\frac{\mathcal{G}}{c^2}\left(\frac{1}{r_0}-\frac{1}{r}\right)-\frac{G.M}{r^2}.\frac{p_0}{p}=0$$

We finally obtain the expression for the gravitation Field:

$$\mathcal{G}=\frac{\frac{G.M}{r^2}.\frac{p_0}{p}}{\frac{p_0^2}{p^2}-\frac{G.M}{c^2}.\frac{p_0}{p}.\left(\frac{1}{r_0}-\frac{1}{r}\right)}$$

If we substitute in this obtained expression, the expression previously assumed by that exact one in equation (29), we obtain a more simplified expression for Gravitational Field:

$$\mathcal{G}=\frac{\frac{G.M}{r^2}.\frac{p_0}{p}}{\frac{p_0^2}{p^2}-\frac{G.M}{c^2}.\frac{p_0}{p}.\left(\frac{1}{r_0}-\frac{1}{r}\right)}=\frac{\frac{G.M}{r^2}.\frac{p_0}{p}}{\frac{p_0^2}{p^2}+\frac{p_0^2}{2}.\left(\frac{1}{p_0^2}-\frac{1}{p^2}\right)}\quad\Rightarrow\quad\mathcal{G}=\frac{\frac{2.G.M}{r^2}}{\frac{p}{p_0}+\frac{p_0}{p}}\tag{30}$$

Observe that, according to this work, we have arrived at an exact expression of the Gravitational field for photons. Also, it can be observed that gravitational field G depends, not only on the radius but on the linear momentum of photon. This means also that Newton's Gravitational Force for photons will have the following new presentation:

$$F=\frac{\frac{2.G.M.m}{r^2}}{\frac{p}{p_0}+\frac{p_0}{p}}=\frac{\frac{2.G.M.p}{r^2.c}}{\frac{p}{p_0}+\frac{p_0}{p}}\tag{31}$$

A first conclusion of this study is that the original Newton's Law of Universal Gravitation, as it is known and applied for the case of photon, from expression (31), according to this development is valid only for circular motion of photon, where, $r=r_0;\ \ p=p_0$. By using another process of independent derivation, in order to check consistency of the assumption in (29), we should obtain another way to get to the expression of the Field. Repeating equation (29):

$$-2.G.M.\left(\frac{1}{r_0}-\frac{1}{r}\right).\frac{p_0}{p}=p_0^2.c^2.\left(\frac{1}{p_0^2}-\frac{1}{p^2}\right)$$

Simplifying and separating variables, with the advisable multiplications, we obtain:

$$\Rightarrow\quad \frac{-2.G.M}{c^2.r_0}+\frac{2.G.M}{c^2.r}=\left(\frac{p}{p_0}-\frac{p_0}{p}\right)$$

Differentiating both members, multiplying by -1 and replacing:

$$\frac{2.G.M}{c^2.r^2}.dr=-\left(\frac{dp}{p_0}+\frac{p_0.dp}{p^2}\right)=-\frac{dp}{p}.\left(\frac{p}{p_0}+\frac{p_0}{p}\right)=\frac{\mathcal{G}}{c^2}.dr.\left(\frac{p}{p_0}+\frac{p_0}{p}\right)$$

$$\frac{2.G.M}{r^2}=\mathcal{G}.\left(\frac{p}{p_0}+\frac{p_0}{p}\right)\quad\Rightarrow\quad \mathcal{G}=\frac{\dfrac{2.G.M}{r^2}}{\left(\dfrac{p}{p_0}+\dfrac{p_0}{p}\right)}$$

We obtain that both expressions obtained for the Gravitation Field are equal, and therefore the expected consistency in the assumed relation is obtained. This is a valuable result in this search for the expression of the Gravitational Field, because different processes of differentiation originate linearly independent equations. Yet we will continue checking the expression assumed through other routes. We will try now to obtain the relationship between radius r and linear momentum p. From the assumption (29), we can easily obtain the expression for p, and also for p_0:

$$p=p_0.\left[\sqrt{\left(\frac{G.M}{c^2}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right)^2+1}-\frac{G.M}{c^2}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right]$$

$$p_0=p.\left[\sqrt{\left(\frac{G.M}{c^2}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right)^2+1}+\frac{G.M}{c^2}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right]\qquad(32)$$

$$r=r_0\Big/1-\frac{r_0.c^2}{2.G.M}\left(\frac{p_0}{p}-\frac{p}{p_0}\right)$$

Let's make some conceptual checks on these results. We should expect that for $r=\infty$, the linear momentum be a positive value, finite and less than the momentum p_0 at the closest point at r_0, where photon has its maximum energy:

$$p=p_0.\left[\sqrt{\left(\frac{G.M}{c^2}.\left(\frac{1}{r_0}\right)\right)^2+1}-\frac{G.M}{c^2}.\left(\frac{1}{r_0}\right)\right]>0\qquad\Rightarrow\qquad p<p_0$$

In fact, thus it is, which indicates a coherent result.
Additionally, it is also expected that for an infinite radius $r=\infty$, because angular velocity is null, $\omega=0$, and according to the general equation for velocities (17), the value of q should be equal to the velocity of light c. From the exact expression in (24), obviously, we should have this result. Thus for $r=\infty$, we have:

$$q^2=\frac{\omega_0^2.r_0^4.p_0^2}{p^2}.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)+p_0^2.c^2.\left(\frac{1}{p_0^2}-\frac{1}{p^2}\right)=\frac{c^2.p_0^2}{p^2}+c^2-\frac{c^2.p_0^2}{p^2}=c^2$$

Thus, as it was expected, we have:

$$q^2=c^2$$

On the other hand, operating with the assumed expression in (29), by evoking the conservation of the angular momentum and equation of velocities (17) we have:

$$q^2=\frac{\omega_0^2.r_0^4.p_0^2}{p^2}.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)-2.G.M.\frac{p_0}{p}.\left(\frac{1}{r_0}-\frac{1}{r}\right)=\left(\omega^2.r^2\right).r^2.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)-2.G.M.\frac{p_0}{p}.\left(\frac{1}{r_0}-\frac{1}{r}\right)$$

$$q^2=\left(c^2-q^2\right).\left(\frac{r^2}{r_0^2}-1\right)-2.G.M.\frac{p_0}{p}.\left(\frac{1}{r_0}-\frac{1}{r}\right)$$

$$q^2.\left(1+\frac{r^2}{r_0^2}-1\right)=\frac{r^2}{r_0^2}.q^2=c^2.r^2.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)-2.G.M.\frac{p_0}{p}.\left(\frac{1}{r_0}-\frac{1}{r}\right)$$

$$q^2=c^2.r_0^2.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)-2.G.M.\frac{p_0}{p}.\frac{r_0^2}{r^2}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\xrightarrow{r\to\infty}\quad q^2=\frac{c^2.r_0^2}{r_0^2}=c^2$$

Namely, the assumed expression is consistent as in circular motion as in elliptic, or parabolic and respects the conservation of angular momentum. In sum, until now in the checking process the assumption has successfully passed.

IV GRAVITATIONAL LAW FOR ANY PAIR OF MASSES.

Let's try to follow a similar procedure for developing gravitational attraction between two generic variable masses for obtaining a general expression similar to that of equation (27). Repeating derivation process:

$$\frac{d\mathbf{p}}{dt}=-F.\mathbf{U}_r\quad\Rightarrow\quad\frac{dm.\mathbf{v}}{dt}=m.\frac{d\mathbf{v}}{dt}+\mathbf{v}.\frac{dm}{dt}=-F.\mathbf{U}_r$$

$$\mathbf{r}=r.\mathbf{U}_r\quad\Rightarrow\quad\mathbf{v}=\frac{dr}{dt}.\mathbf{U}_r+r.\omega.\mathbf{U}_\theta\quad\Rightarrow\quad\frac{d\mathbf{v}}{dt}=\left[\frac{d^2r}{dt^2}-r.\left(\frac{d\theta}{dt}\right)^2\right].\mathbf{U}_r+\left[r.\frac{d^2\theta}{dt^2}+2.\frac{dr}{dt}.\frac{d\theta}{dt}\right].\mathbf{U}_\theta$$

Substituting properly,

$$\left[m.\frac{d^2r}{dt^2}-r.m.\left(\frac{d\theta}{dt}\right)^2+\frac{dm}{dt}.\frac{dr}{dt}\right].\mathbf{U}_r+\left[r.m.\frac{d^2\theta}{dt^2}+2.m.\frac{dr}{dt}.\frac{d\theta}{dt}+r.\frac{dm}{dt}.\frac{d\theta}{dt}\right].\mathbf{U}_\theta=-F.\mathbf{U}_r$$

By preserving the definition of the gravitational field as quotient between Force and mass:

$$\frac{d\mathbf{p}}{dt}=-F.\mathbf{U_r}\ \Rightarrow\ \frac{d\mathbf{p}.d\mathbf{r}}{dt}=-F.d\mathbf{r}.\mathbf{U_r}\ \Rightarrow\ d\mathbf{p}.\mathbf{v}=d(m.\mathbf{v}).\mathbf{v}=m.\mathbf{v}.d\mathbf{v}+\mathbf{v}.\mathbf{v}.dm$$

$$\frac{dp}{p}=-\frac{F}{p.v}.dr=-\frac{\mathcal{G}}{v^2}.dr,\ \text{for, } F=\mathcal{G}.m;\ \text{or, } \mathcal{G}=\frac{F}{m};\ m.v.dv+v^2.dm=dp.v=-F.dr \qquad (33)$$

$$\frac{dm}{m.dt}=\frac{dp}{p.dt}-\frac{dv}{v.dt}$$

Where $\mathcal{G}$, denotes the gravitational field. The second condition applied to a mass moving around another one originates an equation similar to that of following equation (21).

$$\frac{d^2r}{dt^2}-r.\left(\frac{d\theta}{dt}\right)^2+\frac{dm}{m.dt}.\frac{dr}{dt}=\frac{d^2r}{dt^2}-\omega^2.r+\left(\frac{1}{p}\frac{dp}{dt}-\frac{1}{v}\frac{dv}{dt}\right)\frac{dr}{dt}=-\frac{F}{m}=-\mathcal{G}$$

$$\frac{d^2r}{dt^2}-\omega^2.r+\left(-\frac{\mathrm{G}}{v^2}.\frac{dr}{dt}-\frac{1}{v}\frac{dv}{dt}\right)\frac{dr}{dt}=\frac{d^2r}{dt^2}-\omega^2.r-\frac{\mathrm{G}}{v^2}.\left(\frac{dr}{dt}\right)^2-\frac{1}{v}\frac{dv}{dt}\frac{dr}{dt}=-\mathrm{G}$$

$$\frac{d^2r}{dt^2}-\omega^2.r-\frac{\mathrm{G}}{v^2}.\left(v^2-\omega^2.r^2\right)-\frac{1}{v}\frac{dv}{dt}\frac{dr}{dt}=-\mathrm{G}\quad\Rightarrow\quad\frac{d^2r}{dt^2}-\omega^2.r+\frac{\mathrm{G}}{v^2}\omega^2.r^2-\frac{1}{v}\frac{dv}{dt}\frac{dr}{dt}=0 \qquad (34)$$

By following the same procedure done for photon, starting from the conservation of angular momentum law, we have:

$$\frac{d(\omega.r^2)}{\omega.r^2}=-\frac{dm}{m}=-\left(\frac{dp}{p}-\frac{dv}{v}\right)=-\left(-\frac{\mathcal{G}}{v^2}.dr-\frac{dv}{v}\right)=\frac{\mathcal{G}}{v^2}.dr+\frac{dv}{v}$$

$$\omega.r^2.d(\omega.r^2)=\left(\omega.r^2\right)^2\frac{\mathcal{G}}{v^2}.dr+\left(\omega.r^2\right)^2.\frac{dv}{v}$$

$$\omega^2.r^4-\omega_0^2.r_0^4=2.\int_{r_0}^{r}\left(\omega.r^2\right)^2\frac{\mathcal{G}}{v^2}+2.\int_{r_0}^{r}\left(\omega.r^2\right)^2\frac{dv}{v}$$

$$m^2=\frac{m_0^2}{1+\frac{2.m_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\frac{\mathcal{G}}{v^2}.dr+\frac{2.m_0^2}{K^2}.\int_{r_0}^{r}\left(\omega.r^2\right)^2\frac{dv}{v}}=\frac{m_0^2}{1-\frac{2.m_0^2}{K^2}\int_{r_0}^{r}\left(\omega.r^2\right)^2.\frac{dm}{m}}$$

$$m^2 = \frac{m_0^2}{1 - \frac{2}{V_0^2 . r_0^2} \int_{r_0}^{r} \left(\omega . r^2\right)^2 . \frac{dm}{m}} = \frac{V_0^2 . m_0^2}{V_0^2 + \frac{2}{r_0^2} \int_{r_0}^{r} \left(\omega . r^2\right)^2 . \frac{dm}{m}} = \frac{\omega_0^2 . r_0^2 . m_0^2}{V_0^2 - \frac{2}{r_0^2} \int_{r_0}^{r} \left(\omega . r^2\right)^2 . \frac{dm}{m}}$$

$$m^2 = \frac{m_0^2}{1 - \frac{2}{V_0^2 . r_0^2} \int_{r_0}^{r} \left(\omega . r^2\right)^2 . \frac{dm}{m}} = \frac{V_0^2 . m_0^2}{V_0^2 + \frac{2}{r_0^2} \int_{r_0}^{r} \left(\omega . r^2\right)^2 . \frac{dm}{m}} = \frac{\omega_0^2 . r_0^2 . m_0^2}{V_0^2 - \frac{2}{r_0^2} \int_{r_0}^{r} \left(\omega . r^2\right)^2 . \frac{dm}{m}}$$

$$m^2 = \frac{\omega_0^2 . r_0^2 . m_0^2}{V_0^2 - \frac{2}{r_0^2} \int_{r_0}^{r} \left(\omega . r^2\right)^2 . \frac{dm}{m}} = \frac{\omega_0^2 . r_0^4 . m_0^2 . \left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) + \frac{\omega_0^2 . r_0^4 . m_0^2}{r^2}}{V_0^2 - \frac{2}{r_0^2} \int_{r_0}^{r} \left(\omega . r^2\right)^2 . \frac{dm}{m}}$$

$$m^2 . \left[V_0^2 - \frac{2}{r_0^2} \int_{r_0}^{r} \left(\omega . r^2\right)^2 . \frac{dm}{m} \right] = \omega_0^2 . r_0^4 . m_0^2 . \left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) + \frac{\omega_0^2 . r_0^4 . m_0^2}{r^2}$$

$$m^2 . \left[-\frac{\omega_0^2 . r_0^4 . m_0^2}{m.^2 r^2} + V_0^2 - \frac{2}{r_0^2} \int_{r_0}^{r} \left(\omega . r^2\right)^2 . \frac{dm}{m} \right] = \omega_0^2 . r_0^4 . m_0^2 . \left(\frac{1}{r_0^2} - \frac{1}{r^2}\right)$$

$$m^2 . \left[-\frac{\omega_0^2 . r_0^4 . m_0^2}{m.^2 r^2} + V_0^2 - \frac{2}{r_0^2} \int_{r_0}^{r} \left(\omega . r^2\right)^2 . \frac{dm}{m} \right] = \omega_0^2 . r_0^4 . m_0^2 . \left(\frac{1}{r_0^2} - \frac{1}{r^2}\right)$$

$$m^2 . \left[q^2 + V_0^2 - v^2 - \frac{2}{r_0^2} \int_{r_0}^{r} \left(\omega . r^2\right)^2 . \mathcal{G} . dr \right] = \omega_0^2 . r_0^4 . m_0^2 . \left(\frac{1}{r_0^2} - \frac{1}{r^2}\right)$$

$$q^2 = \frac{K^2}{m^2} . \left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) + \frac{2}{r_0^2} \int_{r_0}^{r} \left(\omega . r^2\right)^2 . \frac{dm}{m} - \left(V_0^2 - v^2\right)$$

$$q^2 = \frac{K^2}{m^2} . \left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) + \frac{2}{r_0^2} . \int_{r_0}^{r} \frac{K^2}{m^2} \frac{dm}{m} - \left(V_0^2 - v^2\right)$$

The **exact** expression obtained for masses, analogous to that of photon's equation (24), by following a similar procedure to that of photon's, is given by:

$$q^2 = \frac{K^2}{m^2} . \left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) + \frac{K^2}{r_0^2} . \left(\frac{1}{m_0^2} - \frac{1}{m^2}\right) - \left(V_0^2 - v^2\right) \qquad (35)$$

And the expression to assume, analogous to that of equation (27) that ensures conservation of angular momentum becomes:

$$q^2 = \frac{K^2}{m^2}.\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) - 2.G.M.\left(\frac{1}{r_0} - \frac{1}{r^{\cdot}}\right).\frac{m_0}{m} \tag{36}$$

In fact, previous one is the general expression starting from perihelion. By considering as starting from aphelion, the following general expression shows up:

$$q^2 = \frac{K^2}{m^2}.\left(\frac{1}{r_A^2} - \frac{1}{r^2}\right) - 2.G.M.\frac{m_A}{m}.\left(\frac{1}{r_A} - \frac{1}{r}\right) \tag{36 bis}$$

Evaluating equations (36) at aphelion and (36 bis) at perihelion, we obtain the expected expression of constant angular momentum:

$$v_0^2.r_0^2.\frac{m_0}{m_A} = \frac{2.G.M}{\frac{1}{r_0} + \frac{1}{r_A}} = v_A^2.r_A^2.\frac{m_A}{m_0} \quad \Rightarrow \quad v_0^2.r_0^2.m_0^2 = m_0.m_A.\frac{2.G.M}{\frac{1}{r_0} + \frac{1}{r_A}} = v_A^2.r_A^2.m_A^2$$

$$\Rightarrow \quad v_0.r_0.m_0.v_A.r_A.m_A = m_0.m_A.\frac{2.G.M}{\frac{1}{r_0} + \frac{1}{r_A}} = v_A.r_A.m_A.v_0.r_0.m_0 \Rightarrow \quad v_0.v_A = \frac{2.G.M}{r_A + r_0} = v_A.v_0$$

Which implies that the addition of radiuses, starting from perihelion or from aphelion, has always the same value: $r_0 + r_A = \frac{2.G.M}{v_0.v_A}$. This expression is the exactly the same observed for constant masses and similar to that for photons, as a consequence of equation (3).

We see that the assumption in this case involves the following terms:

$$-2.G.M.\left(\frac{1}{r_0} - \frac{1}{r^{\cdot}}\right).\frac{m_0}{m} = \frac{K^2}{r_0^2}.\left(\frac{1}{m_0^2} - \frac{1}{m^2}\right) - \left(V_0^2 - v^2\right)$$

Simplifying this assumption:

$$-2.G.M.\frac{m_0}{m}.\frac{1}{r_0} + 2.G.M.\frac{m_0}{m}.\frac{1}{r^{\cdot}} = \frac{K^2}{r_0^2}.\frac{1}{m_0^2} - \frac{K^2}{r_0^2}.\frac{1}{m^2} - V_0^2 + v^2 = -\frac{K^2}{r_0^2}.\frac{1}{m^2} + v^2$$

$$-2.G.M.\frac{1}{r_0} + 2.G.M.\frac{1}{r^{\cdot}} = -\frac{K^2}{r_0^2}.\frac{1}{m^2}.\frac{m}{m_0} + \frac{m}{m_0}.v^2 = -\frac{V_0^2.m_0^2}{r_0^2}.\frac{r_0^2}{m^2}.\frac{m}{m_0} + \frac{m}{m_0}.v^2$$

$$-2.G.M.\frac{1}{r_0} + 2.G.M.\frac{1}{r^{\cdot}} = -\frac{m_0}{m}.V_0^2 + \frac{m}{m_0}.v^2$$

By taking derivatives for obtaining the corresponding equation for later equaling to that in (34), it will be obtained the general expression of the gravitational field for any pair of masses:

$$-2.G.M.\frac{dr}{r^{2\cdot}}=\frac{m_0}{m}.V_0^2.\frac{dm}{m}+\frac{dm}{m_0}.v^2+\frac{m_0}{m}.2.v.dv$$

$$-2.G.M.\frac{dr}{r^{2\cdot}}=\frac{dm}{m}.\left(\frac{m_0}{m}.V_0^2+\frac{m}{m_0}.v^2\right)+2.\frac{m_0}{m}.v.dv$$

$$-2.G.M.\frac{dr}{r^{2\cdot}}=\frac{dp}{p}.\left(\frac{m_0}{m}.V_0^2+\frac{m}{m_0}.v^2\right)-\frac{dv}{v}.\left(\frac{m_0}{m}.V_0^2+\frac{m}{m_0}.v^2\right)+2.\frac{m_0}{m}.v.dv$$

$$-2.G.M.\frac{dr}{r^{2\cdot}}=-\frac{\mathcal{G}.dr}{v^2}.\left(\frac{m_0}{m}.V_0^2+\frac{m}{m_0}.v^2\right)-\frac{dV}{v}.\frac{m_0}{m}.V_0^2-\frac{dV}{v}.\frac{m}{m_0}.v^2+2.\frac{m_0}{m}.v.dv$$

$$-2.G.M.\frac{dr}{r^{2\cdot}}=-\frac{\mathcal{G}.dr}{v^2}.\left(\frac{m_0}{m}.V_0^2+\frac{m}{m_0}.v^2\right)-\frac{dV}{v}.\frac{m_0}{m}.V_0^2+\frac{dV}{v}.\frac{m}{m_0}.v^2$$

$$\frac{2.G.M}{r^{2\cdot}}=\frac{\mathcal{G}}{v}.\left(\frac{m_0}{m}.V_0^2+\frac{m}{m_0}.v^2\right)+\frac{dV}{V.dr}.\left(\frac{m_0}{m}.V_0^2-\frac{m}{m_0}.v^2\right)$$

$$\frac{2.G.M}{r^{2\cdot}}=\frac{\mathcal{G}}{v^2}.v^2.\frac{V_0}{v}\left(\frac{m_0}{m}.\frac{V_0}{v}+\frac{m}{m_0}.\frac{v}{V_0}\right)+\frac{dv}{v.dr}.v^2.\frac{V_0}{v}.\left(\frac{m_0}{m}.\frac{V_0}{v}-\frac{m}{m_0}.\frac{v}{V_0}\right)$$

$$\frac{2.G.M}{r^{2\cdot}}=\mathcal{G}.\frac{V_0}{v}\left(\frac{m_0}{m}.\frac{V_0}{v}+\frac{m}{m_0}.\frac{v}{V_0}\right)+\frac{dv.V_0}{dr}.\left(\frac{m_0}{m}.\frac{V_0}{v}-\frac{m}{m_0}.\frac{v}{V_0}\right)$$

$$\frac{2.G.M}{r^{2\cdot}}.\frac{v}{V_0}=\mathcal{G}.\left(\frac{m_0}{m}.\frac{V_0}{v}+\frac{m}{m_0}.\frac{v}{V_0}\right)+\frac{v.dv}{dr}.\left(\frac{m_0}{m}.\frac{V_0}{v}-\frac{m}{m_0}.\frac{v}{V_0}\right)$$

$$\mathcal{G}=\frac{\frac{2.G.M}{r^2}.\frac{v}{V_0}-v.\frac{dv}{dr}.\left(\frac{m_0}{m}.\frac{V_0}{v}-\frac{m}{m_0}.\frac{v}{V_0}\right)}{\left(\frac{m_0}{m}.\frac{V_0}{v}+\frac{m}{m_0}.\frac{v}{V_0}\right)}$$

Simplifying again, the searched for expression arises:

$$\mathrm{G}=\frac{\frac{2.G.M}{r^2}.\frac{v}{V_0}-v.\frac{dv}{dr}.\left(\frac{p_0}{p}-\frac{p}{p_0}\right)}{\left(\frac{p}{p_0}+\frac{p_0}{p}\right)} \qquad (37)$$

Observe that expression (37) for the gravitational field reduces to that of photon, for $v=c$, given the constancy of the speed of light. Expression (37) for circular movement of a mass m around another mass M, where for $r=r_0$, $p=p_0$, $m=m_0$ and $v=V_0$, namely for a constant mass, also reduces consistently to the known expression for the Newtonian gravitational field: $\mathrm{G}=\frac{G.M}{r^2}$.

At this moment we can reach the conclusion that the original Newton Universal Law of Gravitational is valid only for constant masses which is the same as saying that it is valid only for fixed masses or at constant speed in circular motion or in a free rectilinear motion. In this way, the general expression of the gravitational force F between two masses: m and M, that corrects that of Newton, is finally obtained as:

$$F=\frac{\frac{2.G.M}{r^2}.\frac{m.v}{V_0}-m.v.\frac{dv}{dr}.\left(\frac{p_0}{p}-\frac{p}{p_0}\right)}{\left(\frac{p}{p_0}+\frac{p_0}{p}\right)}=\frac{\frac{2.G.M}{r^2}.\frac{p}{V_0}-p.\frac{dv}{dr}.\left(\frac{p_0}{p}-\frac{p}{p_0}\right)}{\left(\frac{p}{p_0}+\frac{p_0}{p}\right)} \qquad (38)$$

Other relationships that can be obtained are:

$$m=m_0.\left[\sqrt{\left(\frac{G.M}{v^2}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right)^2+\frac{V_0^2}{v^2}}-\frac{G.M}{v^2}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right]$$

$$m_0=m.\left[\sqrt{\left(\frac{G.M}{V_0^2}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right)^2+\frac{v^2}{V_0^2}}+\frac{G.M}{V_0^2}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right] \qquad \text{(Mass)} \qquad (39)$$

$$p=p_0.\left[\sqrt{\left(\frac{G.M}{v.V_0}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right)^2+1}-\frac{G.M}{v.V_0}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right]$$

$$p_0=p.\left[\sqrt{\left(\frac{G.M}{v.V_0}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right)^2+1}+\frac{G.M}{v.V_0}.\left(\frac{1}{r_0}-\frac{1}{r}\right)\right] \qquad \text{(Momentum)} \qquad (40)$$

$$r=\frac{r_0}{1-\frac{r_0.v.V_0}{2.G.M}\left(\frac{p_0}{p}-\frac{p}{p_0}\right)} \qquad \text{(Radius)} \qquad (41)$$

$$q=\sqrt{\frac{K^2}{m^2}.\left(\frac{1}{r_0^2}-\frac{1}{r^2}\right)-2.G.M.\left(\frac{1}{r_0}-\frac{1}{r^{\cdot}}\right).\frac{m_0}{m}} \qquad \text{(Radial velocity)} \qquad (42)$$

$$v^2=V_0^2.\frac{m_0^2}{m^2}-2.G.M.\left(\frac{1}{r_0}-\frac{1}{r^{\cdot}}\right).\frac{m_0}{m} \qquad \text{(Generic velocity)} \qquad (43)$$

We should say that the only aspect that we have not considered yet here and that completes this study in the sense of defining relativistic gravitation, is the different presentations of mass appearing in this development, which uses the following mass definitions, measured by an observer at the fixed point P_0: the rest mass $\overset{0}{m}$ (change of notation is to distinguish it from the massive mass M), the initial mass m_0, when it is moving at velocity V_0 and is passing by P_0 at the minimum distance r_0 from M; and the generic value of mass m, measured when it has another generic velocity v and another generic radius r, as so it was obtained in equation (4) in the Review presented in past issue [3]: "Mass in Vectorial Relativity", based on Franco's work [2], where, recalling (with new notation), they were related among them as:

$$m = \frac{\overset{0}{m}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \qquad m_0 = \frac{\overset{0}{m}}{\left(1 - \frac{V_0^2}{c^2}\right)^{\frac{3}{2}}} \tag{44}$$

V CONCLUSIONS

As in previous definitions of mass or energy, again we are in the same situation: It is necessary to check the experimental validation of this approach. For example, until now we have not found any relation between gravitation and time as it is in Einstein's General Theory of Relativity or between the speed of light and gravitation as it is found in the speed of a radar signal when it goes opposite to or parallel to the gravitational force. The accuracy of new definitions of mass, energy or Gravitational Force rigorously obtained, according to us, in this work and in previous ones, referred in the Reviews published in this and previous issues, will probably require further research and complex experiments with known rest masses accelerated at speeds close to that of light in order to establish the correctness of our work. These tasks probably could be possible to achieve by next years. See in the "News", in the first part of previous Journal, the article **"European new particle accelerator: The Large Hadron Collider (LHC)".**

It is also relevant to recall the checks done by professor Pritchard and Professor Thompson at MIT of the measurements to masses distinct of photons in where it proves that mass varies with velocity as it was one of the major outcomes of the Special Theory of Relativity. The consequences of these controls ratify that relativity is a correct vision of nature. The complete article "**E=mc² passes tough MIT test"** can be read in this issue in the section "**NEWS**".

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