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Compton scattering with particles in motion

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Abstract: In this article I will show a modified version of the classical Compton scattering law which contain a particle in constant motion and not only at rest. The modified law will have the correct limit if the particle is at rest in a system of reference. But the law also implies that the scattering effect is not only limited to particles with mass but also massless particle like photons.

1.1 Introduction

Compton scattering is one of the profound evidence of particle-wave duality. When a photon with wavelength λ_1 and momentum $\mathbf{p}_1$ hits an electron (at rest) the resulting photon will have a longer wavelength λ_2 than the incident photon because it has transferred some of its momentum to the electron. This classic scenario involves an electron (or another particle with mass for that matter) at rest in the moment of impact. By applying the conservation of energy and momentum and using the special theory of relativity one can derive the Compton scattering formula

$$\lambda_2 - \lambda_1 = \frac{h}{m_e c}\left(1 - \cos\theta\right) \tag{1}$$

Where h is Planck's constant. But what always has been "puzzling" is the assumption that the electron is always at rest. Could it not be in motion instead and still being impacted? Because after all not many things in nature can be considered being at "rest". I must point out that the validity of equation (1) is undisputed because of the countless experiments that confirm the equation.

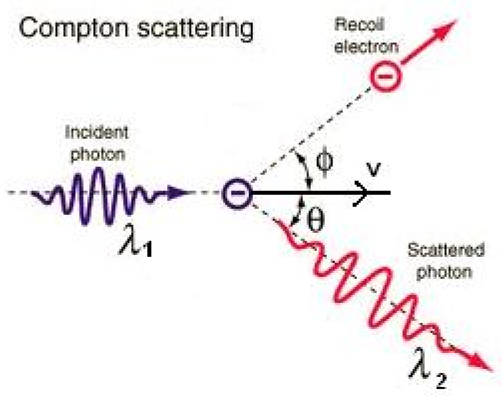

Figure 1

The figure illustrates a situation where electron is not at rest [1].

Before mentioned assumption is more correctly based on the fact that impact between the photon and the electron happens in a very short time interval and therefore can be considered too be at "rest". In my opinion this scenario is just one limit from a more general equation.

But how will the equation look like if we include a particle moving at a *constant* velocity? We could guess that the Compton wavelength will contain a term involving the momentum of the particle which is fair assumption because this will reduce to equation (1) when the particle is not moving. But let's see what the laws of physics tell us.

1.2 Conservation of energy and momentum

To make things familiar we will use a moving electron as our particle. Also we will denote bold letters as vectors and unbolt letters as scalars until other is mentioned.

With this little introduction to our mathematical notations, we will venture back to figure 1. The conservation of momentum tells us that

$$\mathbf{p}_1 + \mathbf{p}_{e1} = \mathbf{p}_2 + \mathbf{p}_{e2} \tag{2}$$

Where $\mathbf{p}_1$ is the momentum of the incident photon, $\mathbf{p}_{e1}$ is the momentum of the electron, $\mathbf{p}_2$ is the momentum of the scattered photon and last but not least $\mathbf{p}_{e2}$ is the momentum of the scattered electron. The conservation of energy tells us that

$$E_1 + E_{e1} = E_2 + E_{e2} \tag{3}$$

In terms of relativistic relations between momentum and energy we have

$$\sqrt{\left(m_e c^2\right)^2 + \left(p_{e1} c\right)^2} + \frac{hc}{\lambda_1} = \sqrt{\left(m_e c^2\right)^2 + \left(p_{e2} c\right)^2} + \frac{hc}{\lambda_2} \tag{4}$$

I will prefer to rewrite equation (4) as followed

$$\sqrt{\left(m_e c^2\right)^2 + \left(p_{e1} c\right)^2} + \left(\frac{hc}{\lambda_1} - \frac{hc}{\lambda_2}\right) = \sqrt{\left(m_e c^2\right)^2 + \left(p_{e2} c\right)^2} \tag{5}$$

Equation (2) can be rewritten as followed

$$\mathbf{p}_{e2} = \left(\mathbf{p}_1 - \mathbf{p}_2\right) + \mathbf{p}_{e1} \tag{6}$$

The square length of this vector is given by

$$p_{e2}^2 = \left(\mathbf{p}_1 - \mathbf{p}_2\right)\cdot\left(\mathbf{p}_1 - \mathbf{p}_2\right) + \mathbf{p}_{e1}\cdot\mathbf{p}_{e1} + 2\left(\mathbf{p}_1 - \mathbf{p}_2\right)\cdot\mathbf{p}_{e1} \tag{7}$$

The first term of the equation is given by

$$\begin{aligned} \left(\mathbf{p}_1 - \mathbf{p}_2\right)\cdot\left(\mathbf{p}_1 - \mathbf{p}_2\right) &= p_1^2 + p_2^2 - 2\left(\mathbf{p}_1\cdot\mathbf{p}_2\right) \Leftrightarrow \\ \left(\mathbf{p}_1 - \mathbf{p}_2\right)\cdot\left(\mathbf{p}_1 - \mathbf{p}_2\right) &= \left(\tfrac{h}{\lambda_1}\right)^2 + \left(\tfrac{h}{\lambda_2}\right)^2 - \tfrac{2h^2}{\lambda_1\lambda_2}\cos\theta \end{aligned} \tag{8}$$

Were θ is the scattering angle of the scattered photon. The second term is trivial so I won't comment on that. The third term is more interesting. By using the distributive properties of the dot product one can write

$$\begin{aligned} \left(\mathbf{p}_1 - \mathbf{p}_2\right)\cdot\mathbf{p}_{e1} &= \mathbf{p}_1\cdot\mathbf{p}_{e1} - \mathbf{p}_2\cdot\mathbf{p}_{e1} \Leftrightarrow \\ \left(\mathbf{p}_1 - \mathbf{p}_2\right)\cdot\mathbf{p}_{e1} &= p_{e1}\left(p_1\cos\delta - p_2\cos\left(\theta+\phi\right)\right) \end{aligned} \tag{9}$$

But since the angle between the incident photon and electron is zero ($\delta = 0^\circ$) then the equation will just reduce to

$$\left(\mathbf{p}_1 - \mathbf{p}_2\right)\cdot\mathbf{p}_{e1} = p_{e1}\left(p_1 - p_2\cos\left(\theta+\phi\right)\right) \tag{10}$$

But this may not always be the case even though figure 1 illustrates it. That's why we will leave it for now. By inserting equations (8) and (9) into equation (7) we have

$$p_{e2}^2 = \left(\tfrac{h}{\lambda_1}\right)^2 + \left(\tfrac{h}{\lambda_2}\right)^2 - \tfrac{2h^2}{\lambda_1\lambda_2}\cos\theta + p_{e1}^2 + 2p_{e1}\left(p_1\cos\delta - p_2\cos(\theta+\phi)\right) \tag{11}$$

By multiplying the equation by c^2 and adding the resting energy of the electron $\left(m_e c^2\right)^2$ equation (11) will become

$$\begin{aligned} E_{e2}^2 &= \left(\tfrac{hc}{\lambda_1}\right)^2 + \left(\tfrac{hc}{\lambda_2}\right)^2 - \tfrac{2(hc)^2}{\lambda_1\lambda_2}\cos\theta + E_{e1}^2 + 2c^2 p_{e1}\left(p_1\cos\delta - p_2\cos(\theta+\phi)\right) \Leftrightarrow \\ E_{e2}^2 - E_{e1}^2 &= \left(\tfrac{hc}{\lambda_1}\right)^2 + \left(\tfrac{hc}{\lambda_2}\right)^2 - \tfrac{2(hc)^2}{\lambda_1\lambda_2}\cos\theta + 2c^2 p_{e1}\left(p_1\cos\delta - p_2\cos(\theta+\phi)\right) \end{aligned} \tag{12}$$

We will now return to equation(5). By squaring this equation and performing the algebraic operations on the left side of the equation we will get

$$\begin{gathered} \sqrt{\left(m_e c^2\right)^2 + \left(p_{e1}c\right)^2} + \left(\tfrac{hc}{\lambda_1} - \tfrac{hc}{\lambda_2}\right) = E_{e2} \Leftrightarrow \\ \left(\tfrac{hc}{\lambda_1} - \tfrac{hc}{\lambda_2}\right)^2 + 2\left(\tfrac{hc}{\lambda_1} - \tfrac{hc}{\lambda_2}\right)E_{e1} = E_{e2}^2 - E_{e1}^2 \end{gathered} \tag{13}$$

The first term of the last equation in equation (13) can be rewritten as

$$\left(\tfrac{hc}{\lambda_1} - \tfrac{hc}{\lambda_2}\right)^2 = \left(\tfrac{hc}{\lambda_1}\right)^2 + \left(\tfrac{hc}{\lambda_2}\right)^2 - \tfrac{2(hc)^2}{\lambda_1\lambda_2} \tag{14}$$

Equation (13) may therefore be written as

$$E_{e2}^2 - E_{e1}^2 = \left(\tfrac{hc}{\lambda_1}\right)^2 + \left(\tfrac{hc}{\lambda_2}\right)^2 - \tfrac{2(hc)^2}{\lambda_1\lambda_2} + 2\left(\tfrac{hc}{\lambda_1} - \tfrac{hc}{\lambda_2}\right)E_{e1} \tag{15}$$

By comparing equation (12) with equation (15) we will see that there is a lot terms that will cancel each other out

$$-\tfrac{hc}{\lambda_1\lambda_2} + \left(\tfrac{1}{\lambda_1} - \tfrac{1}{\lambda_2}\right)E_{e1} = -\tfrac{hc}{\lambda_1\lambda_2}\cos\theta + \tfrac{p_{e1}c}{h}\left(p_1\cos\delta - p_2\cos(\theta+\phi)\right) \tag{16}$$

Note that the "kinetical part" of the electrons total energy is given by $K_{e1} = p_{ec}c$. By using this definition and rearranging terms even further we see that things get more and more familiar

$$-\tfrac{hc}{\lambda_1\lambda_2}+\left(\tfrac{1}{\lambda_1}-\tfrac{1}{\lambda_2}\right)E_{e1}=-\tfrac{hc}{\lambda_1\lambda_2}\cos\theta+\tfrac{K_{e1}}{h}\left(p_1\cos\delta-p_2\cos(\theta+\phi)\right)\Leftrightarrow$$
$$\lambda_2-\lambda_1=\tfrac{hc}{E_{e1}}(1-\cos\theta)+\tfrac{K_{e1}}{E_{e1}h}\lambda_1\lambda_2\left(p_1\cos\delta-p_2\cos(\theta+\phi)\right) \tag{17}$$

For the sake of simplicity we will define the ratio between the electrons kinetical energy and its total energy by

$$\kappa_{e1}:=\frac{K_{e1}}{E_{e1}} \tag{18}$$

The properties of this ratio are that it can be zero if the kinetical part is zero. That is $K_{e1}=0$ and therefore a particle with mass is at “rest” in some frame of reference. But it is also valid to assume that the kinetical part is sufficiently small in comparison to $m_e c^2 \gg p_{e1}c$ (its resting energy) and therefore $\kappa_{e1}\approx 0$. If the particle don’t have any rest mass (or $m_e c^2 \ll p_{e1}c$) then $\kappa_{e1}=1$ because the particle will have a velocity equal to the speed of light.

By using the relations between momentum and wavelength for the incident and scattered photon on the last term, we will get the equation

$$\lambda_2-\lambda_1=\tfrac{hc}{E_{e1}}(1-\cos\theta)+\tfrac{\kappa_{e1}}{h}\lambda_1\lambda_2\left(\tfrac{h}{\lambda_1}\cos\delta-\tfrac{h}{\lambda_2}\cos(\theta+\phi)\right)\Leftrightarrow$$
$$\lambda_2-\lambda_1=\tfrac{hc}{E_{e1}}(1-\cos\theta)+\kappa_{e1}\left(\lambda_2\cos\delta-\lambda_1\cos(\theta+\phi)\right) \tag{19}$$

Even though equation (19) reminds us of equation (1) we notice that the second term also involves the wavelength of the incident and scattered photon. This means we have to make further rearrangements of terms to complete the equation

$$\tfrac{hc}{E_{e1}}(1-\cos\theta)+\kappa_{e1}\left(\lambda_2\cos\delta-\lambda_1\cos(\theta+\phi)\right)=\lambda_2-\lambda_1\Leftrightarrow$$
$$(1-\kappa_{e1}\cos\delta)\lambda_2-\left(1-\kappa_{e1}\cos(\theta+\phi)\right)\lambda_1=\tfrac{hc}{E_{e1}}(1-\cos\theta) \tag{20}$$

The right side of equation (20) will have some coefficients that will *affect the difference* in wavelength between the incident and scattered photon and not the physical nature of the particles. I will define these numbers as *the wave coefficients*

$$\begin{aligned} A(\kappa_{e1},\delta) &:= 1-\kappa_{e1}\cos\delta \\ B(\kappa_{e1},\theta,\phi) &:= 1-\kappa_{e1}\cos(\theta+\phi) \end{aligned} \tag{21}$$

Notice the A and B depends mathematically speaking on the mentioned variables in the equation. Equation (20) can therefore be written as

$$A(\kappa_{e1},\delta)\lambda_2 - B(\kappa_{e1},\theta,\phi)\lambda_1 = \tfrac{hc}{E_{e1}}(1-\cos\theta) \tag{22}$$

Or just more simply

$$\Lambda = \frac{hc}{E_{e1}}(1-\cos\theta) \tag{23}$$

Where

$$\Lambda := A(\kappa_{e1},\delta)\lambda_2 - B(\kappa_{e1},\theta,\phi)\lambda_1 \tag{24}$$

If we compare equation (23) to equation (1) it is evidently clear that it will reduce to the classical Compton scattering equation, by considering that the electron is essentially at "rest" at the moment of impact by the incident photon. That is to say $\kappa_{e1} \approx 0$ which implies $A \approx 1$ and $B \approx 1$. Notice this implies that the coefficients are independent of the angles in the equations but not the scattered angle on the right side of (23). On the right side of equation (23) the Compton wavelength of electron will appear because $E_{e1} \approx m_e c^2$ and therefore it has the right limit.

Equation (23) also implies that particle that is being impacted doesn't necessarily need a rest mass (that is $m_e = 0$ an $\kappa_{e1} = 1$). For instance; if we have two photons with different wavelength and momentum, the equation tells us it must be possible to observe a scattering effect.

One crucial observation must be made if this scattering effect is possible. The angle present in coefficient A cannot be zero ($\delta \neq 0^{\circ}$) for massless particle moving with the speed of light. The reason for this is simple. The incident photon can never catch up to the photon where the scattering will occur because it also moves with the speed of light. For this to be possible, the incident photon will have to move with a velocity greater than the speed of light. And that is of course not possible.

1.3 An example and a useful relation

Equation (24) has a connection to matrices

$$\Lambda = \det W \Leftrightarrow \det W = \begin{vmatrix} A(\kappa_{e1},\delta) & B(\kappa_{e1},\theta,\phi) \\ \lambda_1 & \lambda_2 \end{vmatrix} \tag{25}$$

Where W is the *wave-coefficient matrix*

$$W := \begin{pmatrix} A(\kappa_{e1},\delta) & B(\kappa_{e1},\theta,\phi) \\ \lambda_1 & \lambda_2 \end{pmatrix} \tag{26}$$

Let's take an example. Imagine an electron moving at 80% of the speed of light. Then its total energy will be $E_{e1} \approx 0.851\,\text{MeV}$. Its kinetic energy will then be equal to $K_{e1} = 0.341\,\text{MeV}$. By equation (18) you will get a ratio of $\kappa_{e1} \approx 0.400$ By assuming that $\delta = 0^{\circ}$ the first wave-coefficient will have a value of $A \approx 0.598$. If we want a recoil effect of the scattered photon the angles will be $\theta = \pi$ and $\phi = 0^{\circ}$. This will give a wave coefficient of $B \approx 1.400$. The difference in wavelength is given by the right side of equation (23) and the value will be $\Lambda \approx 0.0029\,\text{nm}$. Let's "slam" a photon with a wavelength of 6 pm into the electron. What is the energy of scattered photon? Well, the answer is

$$E_2 = \frac{hcA}{\Lambda + B\lambda_1} \tag{27}$$

This will give a wavelength of $\lambda_2 \approx 0.01885\,\text{nm}$ and energy of $E_2 \approx 0.075\,\text{MeV}$ which is well within the limitations of measurements. The wave coefficient matrix will be

$$W = \begin{pmatrix} 0.6 & 1.4 \\ 6\cdot 10^{-3}nm & 0.01885nm \end{pmatrix} \tag{28}$$

By calculating the determinant we can compare the right and left side of equation (23) to see if the calculations are correct. We see that it has the value $\det W \approx 0.0029\,\text{nm}$. The energy of the scattered electron is the energy before the collision plus the amount added by the incident photon

$$E_{e2} = E_{e1} + \left(E_1 - E_2\right) \tag{29}$$

And this will give an energy of $E_{e2} \approx 0.9925\,\text{MeV}$. The consequence is that the electrons velocity will be greater than before the collision by an amount

$$\frac{v_2 - v_1}{c} = \sqrt{1 - \left(\frac{m_e c^2}{E_{e2}}\right)^2} - \sqrt{1 - \left(\frac{m_e c^2}{E_{e1}}\right)^2} \tag{30}$$

But its velocity will never exceed the speed of light. By inserting the numbers we can calculate an increase in velocity of around 5.7% or 85.7% of the speed of light.

The wave coefficients are very easy to compute if you know the energies. But mathematically, the coefficients are actually related because of equation (18).This means we can write one of the coefficients in term of the other. If chose coefficient A we see

$$A = 1 - \kappa_{e1}\cos\delta \Leftrightarrow \kappa_{e1} = \tfrac{1-A}{\cos\delta} \tag{31}$$

For $\delta \neq \left\{\tfrac{1}{2}\pi, \tfrac{3}{4}\pi\right\}$. By inserting equation (31) into wave coefficient B we get

$$B = 1 - \frac{1-A}{\cos\delta}\cos\left(\theta + \phi\right) \Leftrightarrow B = 1 - \frac{\cos(\theta+\phi)}{\cos\delta} + A\frac{\cos(\theta+\phi)}{\cos\delta} \qquad (32)$$

If we define

$$C := \frac{\cos(\theta+\phi)}{\cos\delta} \qquad (33)$$

We will get the following equation

$$B = 1 - C\left(1 - A\right) \qquad (34)$$

So in our example we know that $C = -1$ which means

$$B = 2 - A \qquad (35)$$

By inserting $A \approx 0.6$ we will get $B \approx 1.4$ which is the same value as in the example.

1.4 Conclusion

The purpose with this article was to derive the Compton scattering formula, which includes particles in motion. Though we could have "guessed" how the right side of equation (23) would look like the left side was only due to the physical assumptions.

The new formula adds extra factors (the wave coefficients) on the left side of the equation and affects only the difference in wavelength between the incident and scattered photon. So overall we can conclude that the formula looks almost identical to the classical scattering formula but has the right limit if the impacted particle is at rest.

1.5 References

[1] http://hyperphysics.phy-astr.gsu.edu/hbase/quantum/compton.html.
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