The Scaling Theory VIII: The Sagnac's Interference and Michelson and Gale Experiment.

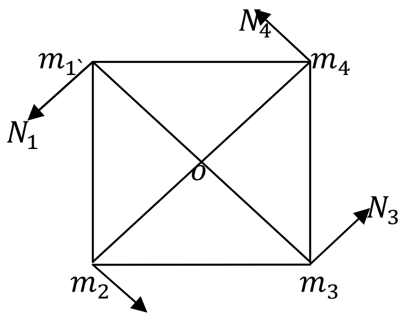
14.1. Introduction

Since its discovery in 1913 by Georges Sagnac many attempts to explain the Sagnac effect had been forwarded. Some attempts followed a purely classical approach based on the relative velocity addition[26,33], and some resorted to the ether model, or emission theory[32,34-36]. Relativists claim that this effect is explicable within the theory of relativity, but this claim is much disputed[26,32]. In an earlier work by the author[6], the Sagnac's effect corresponding to two pulses, each making a complete equatorial round in opposite directions, was explained on the basis of the scaling theory. The similarity in nature between this particular type of Sagnac's effect and that of two pulses covering equal distances in opposite direction on a line was also revealed[10]. In this work we employ the bound scaling transformations, to present a neat and elaborate explanation of the general type of this effect, as well as, of Michelson and Gale experiment. In a subsequent article[37], the translational type of Sagnac effect will be discussed in detail.

14.2. The Sagnac Interferometer

The Sagnac's interferometer consists of n plane mirrors $m_1, m_2, \dots, m_n$ occupying the vertices of a regular polygon $m_1 m_2 \dots m_n m_1$ with a center o, radius a, side L, and a source of light capable to send simultaneously two beams in opposite directions along the polygonal loop. The whole arrangement is set on a turntable. For a more expound account of Sagnac interferometer we refer to Faraj[32].

We assume that the enumeration of the mirrors increases counter clock-wise, which is also the positive sense of rotation, and that the normals to the diagonals point counter clock-wise. Some of the following easily obtained geometrical data are useful in subsequent discussions:



For a regular polygon

- The angle of a regular polygon with n sides is
$$\angle(\overrightarrow{m_{\iota-1}m_\iota}, \overrightarrow{m_\iota m_{\iota+1}}) = \pi(1 - 2/n)$$
- The angle between the diagonal and the corresponding side is
$$\emptyset = \angle(\overrightarrow{m_\iota o}, \overrightarrow{m_\iota m_{\iota+1}}) = \angle(\overrightarrow{m_\iota o}, \overrightarrow{m_\iota m_{\iota-1}}) = \frac{\pi}{2} - \frac{\pi}{n}$$
- The angles between the normal $\overrightarrow{m_\iota N_\iota}$ to the diagonal $\overrightarrow{om_\iota}$ and the corresponding sides are
$$\theta = \angle(\overrightarrow{m_\iota N_\iota}, \overrightarrow{m_\iota m_{\iota+1}}) = \frac{\pi}{n}, \quad \pi - \theta = \angle(\overrightarrow{m_\iota N_\iota}, \overrightarrow{m_\iota m_{\iota-1}}) = \pi - \frac{\pi}{n}$$
- The area of the polygon is
$$A = n.\frac{1}{2} L a sin\emptyset = \frac{1}{2} n\, a^2 sin2\emptyset = -\frac{1}{2} n\, a^2 sin2\theta,$$
where $L = 2acos\emptyset = 2asin\theta$ is the length of the polygon's side.

- As n tends to infinity the area A tends to πa^2.
- The area of a polygon (not necessarily regular) can be written in the form

$$A = \frac{1}{2}\left|\sum_{i=1}^{i=n} \overrightarrow{pm_i} \times \overrightarrow{pm_{i+1}}\right|, \tag{14.1}$$

where the vector $\overrightarrow{pm_{n+1}}$ is identified by $\overrightarrow{pm_1}$, and p is any point in the plane of the polygon.

Now, two pulses of light that set out at the same time from the same point m_1, to make closed trips in opposite directions along the perimeter of the polygon, return to $m_1$at the same time if the interferometer is at rest in the timed inertial frame *S*. If the interferometer is rotating, the two pulses do not arrive back at m_1 simultaneously. In fact the pulse moving against the rotation arrives earlier than the pulse moving in its direction. Here we shall employ the bound scaling transformations (BST) to calculate the time difference between the arrival of the two pulses back at m_1 when the interferometer rotates in the positive sense at an angular velocity ω about an axis through its center o and perpendicular to its plane.

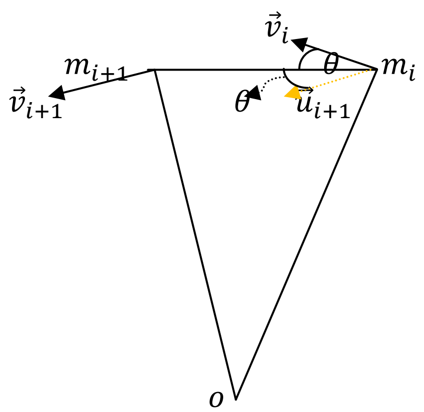
14.3. The Use of Bound Scaling Transformations

A frame attached to the terrestrial laboratory with respect to which the interferometer is rotating can be considered during the short period taken by the experiment a timed inertial frame *S* of fixed stars[38] . We assume that $\omega \gg 2\pi\, radian/day$, i.e. the angular velocity of the interferometer is much greater than that of the earth about its axis. In the frame *S* the velocity $\overrightarrow{u_i}$ of the mirror m_i when at $M_i \in S$ is perpendicular to the vector $\overrightarrow{oM_i}$, and hence

$$\angle\left(\overrightarrow{M_iM_{i+1}}, \overrightarrow{u_i}\right) = \theta = \frac{\pi}{n}, \quad \angle\left(\overrightarrow{M_{i+1}M_i}, \overrightarrow{u_{i+1}}\right) = \pi - \theta,$$

where M_i and M_{i+1} are the positions of the mirrors m_i and m_{i+1} in *S* at the instant of light emission.

If the system was stationary, light which has been reflected at a mirror would take a duration $T = L/c$ to reach either adjacent mirror. Now, the characters of the trip ($m_{i+1} at\ M_{i+1} \rightarrow m_i$) are the same whether the observer m_i was moving at his actual velocity $\overrightarrow{u_i}$ or at a velocity $\overrightarrow{u_i}'$ which is the mirror image of $\overrightarrow{u_i}$ with respect to the line m_im_{i+1}. Since $\overrightarrow{u'_i} \parallel \overrightarrow{u_{i+1}}$, the conditions of the applicability of the extended bound transformations are met thoroughly. Or we may instead use directly the BST in the rigid frame attached to the interferometer (see section (13.4)).



Next, we rewrite the bound scaling transformations in a most suitable form to use. If M_i and M_{i+1} are the laboratory, or *S*-observers, conjugate to m_i and m_{i+1} at the instant light is emitted from m_{i+1}, then

$$(14.2) \qquad r_{i,i+1} = \frac{-\vec{\beta}_i.\overrightarrow{R_{i,i+1}} + \sqrt{R_{i,i+1}^2 - \left|\vec{\beta}_i \times \overrightarrow{R_{i,i+1}}\right|^2}}{\sqrt{1-\beta_i^2}},$$

where

$$r_{i,i+1} = |\overrightarrow{m_i m_{i+1}}|, \qquad R_{i,i+1} = \left|\overrightarrow{M_i M_{i+1}}\right| = L, \ \ \vec{\beta}_i = \frac{\vec{u}_i}{c}.$$

For $max\beta_i \ll 1$, which is the case in Sagnac's effect, the latter transformation is approximated by

$$(14.3) \qquad r_{i,i+1} \approx R_{i,i+1} - \vec{\beta}_i.\overrightarrow{R_{i,i+1}} = L - \vec{\beta}_i.\overrightarrow{M_i M_{i+1}},$$

which is the length of the trip ($m_{i+1}\ at\ M_{i+1} \to m_i$). Similarly, the optical length of the opposite trip ($m_i\ at\ M_i \to m_{i+1}$) is approximated by

$$(14.4) \qquad r_{i+1,i} \approx L - \vec{\beta}_{i+1}.\overrightarrow{M_{i+1} M_i}.$$

14.4. Rotation About the Polygon's Center

When the polygon rotates about its center *o*, the velocity of the mirror m_i will be

$$(14.5) \qquad \vec{\beta}_i = \frac{1}{c}\vec{\omega} \times \overrightarrow{oM_i},$$

and the length of the counter rotation-wise trip ($m_{i+1}\ at\ M_{i+1} \to m_i$) is given then by

$$r_{i,i+1} = L - \vec{\omega} \times \overrightarrow{oM_i}.\overrightarrow{M_i M_{i+1}} = L - \vec{\omega}.\overrightarrow{oM_i} \times \left(\overrightarrow{oM}_{i+1} - \overrightarrow{oM_i}\right)$$

$$(14.6) \qquad L - \frac{\vec{\omega}}{c}.\overrightarrow{oM_i} \times \overrightarrow{oM}_{i+1}.$$

Similarly, the optical length of the opposite trip ($m_i\ at\ M_i \to m_{i+1}$) is

$$(14.7) \qquad r_{i+1,i} = L - \vec{\beta}_{i+1}.\overrightarrow{M_{i+1} M_i} = L - \frac{\vec{\omega}}{c}.\overrightarrow{oM}_{i+1} \times \overrightarrow{oM_i}.$$

The difference in paths lengths of the two trips for one side of the polygon is

$$(14.8) \quad \Delta r_i = r_{i+1,i} - r_{i,i+1} = \frac{\vec{\omega}}{c}.\left(\overrightarrow{oM_i} \times \overrightarrow{oM}_{i+1} - \overrightarrow{oM}_{i+1} \times \overrightarrow{oM_i}\right)$$

$$= \frac{2\vec{\omega}}{c}.\overrightarrow{oM_i} \times \overrightarrow{oM}_{i+1}.$$

The total difference in the durations of the two trips is therefore

$$(14.9) \qquad \Delta t = \sum_{i=1}^{i=n+1} \frac{\Delta r_i}{c} = \frac{2}{c^2}\vec{\omega}.\sum_{i=1}^{i=n+1} \overrightarrow{oM_i} \times \overrightarrow{oM}_{i+1} = \frac{4\omega}{c^2}A,$$

where we identify M_{n+1} by M_1, and *A* is the area of the polygon.

14.5. Rotation About an Arbitrary Point

We consider now the case in which the turntable rotates about its center *O* which is distinct from the center *o* of the polygon. During the short period taken by light to travel one side of the polygon the motion of the receiver does not deviate appreciably from uniformity. Also, since the velocities of all mirrors (i.e. receivers) are all much less than *c*, the conditions necessary for the applicability of the BST are met (see section (13.4)). [Note that these conditions are also fulfilled for rotation about the center].

Now, the velocity of the mirror m_i is

$$\vec{u}_i = \vec{\omega} \times \overrightarrow{OM_i}\,. \tag{14.10}$$

By (14.3) and (14.4), the difference in path length corresponding to one side is

$$\begin{aligned}\Delta r_i = r_{i+1,i} - r_{i,i+1} &= \vec{\beta}_i.\overrightarrow{M_iM_{i+1}} - \vec{\beta}_{i+1}.\overrightarrow{M_{i+1}M_i}\\ &= \frac{\vec{\omega}}{c}.\left[\overrightarrow{OM_i} \times \overrightarrow{M_iM_{i+1}} - \overrightarrow{OM_{i+1}} \times \overrightarrow{M_{i+1}M_i}\right]\\ &= \frac{\vec{\omega}}{c}.\left[\overrightarrow{OM_i} \times \overrightarrow{OM_{i+1}} - \overrightarrow{OM_{i+1}} \times \overrightarrow{OM_i}\right]\\ &= \frac{2\vec{\omega}}{c}.\overrightarrow{OM_i} \times \overrightarrow{OM_{i+1}}.\end{aligned} \tag{14.11}$$

The total difference between the durations of the two trips is therefore

$$\Delta t = \sum_{i=1}^{i=n+1} \frac{\Delta r_i}{c} = \frac{2}{c^2}\vec{\omega}.\sum_{i=1}^{i=n+1} \overrightarrow{OM_i} \times \overrightarrow{OM_{i+1}} = \frac{4\omega}{c^2}A. \tag{14.12}$$

It is obvious, on geometrical bases, that the magnitude of the sum in the last equation is twice the area enclosed by the polygon, and it is easy also to verify this result. Indeed

$$\begin{aligned}\overrightarrow{OM_i} \times \overrightarrow{OM_{i+1}} &= \left(\overrightarrow{Oo} + \overrightarrow{oM_i}\right) \times \left(\overrightarrow{Oo} + \overrightarrow{oM_{i+1}}\right)\\ &= \overrightarrow{Oo} \times \left(\overrightarrow{oM_{i+1}} - \overrightarrow{oM_i}\right) + \overrightarrow{oM_i} \times \overrightarrow{oM_{i+1}}\\ &= \overrightarrow{Oo} \times \overrightarrow{M_iM_{i+1}} + \overrightarrow{oM_i} \times \overrightarrow{oM_{i+1}},\end{aligned}$$

and hence

$$\sum_{i=1}^{i=n+1} \overrightarrow{OM_i} \times \overrightarrow{OM_{i+1}} = \sum_{i=1}^{i=n+1} \overrightarrow{Oo} \times \overrightarrow{M_iM_{i+1}} + \sum_{i=1}^{i=n+1} \overrightarrow{oM_i} \times \overrightarrow{oM_{i+1}} =$$

$$\sum_{i=1}^{i=n+1} \overrightarrow{oM_i} \times \overrightarrow{oM_{i+1}} = 2\vec{A}. \tag{14.13}$$

14.6. General Remarks.

(i)-It may be useful to mention that when light traces a loop in opposite directions, it is the ***area enclosed by this loop, but not its length***, that determines the magnitude of Sagnac effect. Although it is true that a regular polygon's area is expressible as a function of its circumference, but this function is not 1-1 correspondence. Indeed

$$A = \frac{cot\frac{\pi}{n}}{4n} \times (circumference)^2,$$

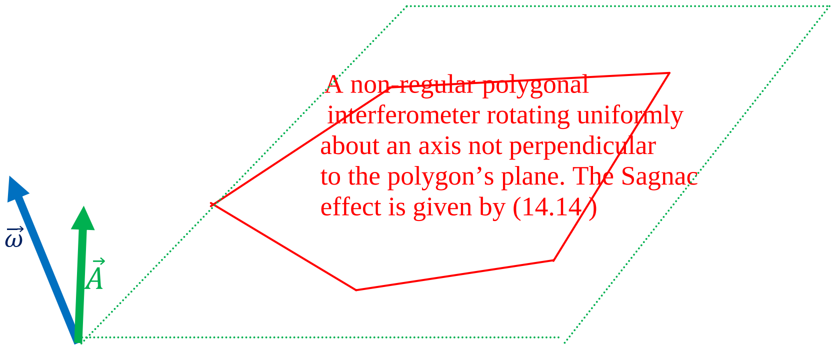
and one can shape a given length (say a fiber optical loop) in a variety of polygons that have the same circumference but differ in their areas. However, if *n* is fixed, which means that the type of regular polygon is fixed, the latter formula determines a 1-1 map between the area and the perimeter. The latter fact will be illuminated through a broader treatment when discussing translational Sagnac effect.

(ii) A fiber optics loop can assume a non-regular polygon, and the Sagnac effect, given by (14.12) remains *valid for any shape of polygon.* Indeed, the derivation of (14.9) or (14.12) do not make use at all of the regularity of the polygon, and the area of a polygon is given by (14.1), whether it was regular or non-regular.

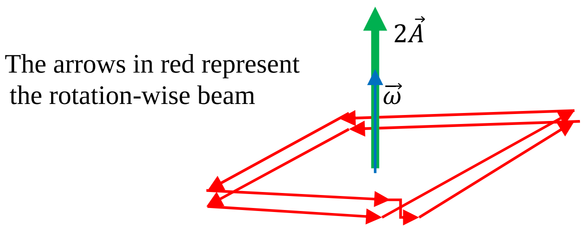
(iii) The magnitude of Sagnac effect can be expressed in the form

$$\Delta t = \frac{4}{c^2}\vec{\omega}.\vec{A}, \tag{14.14}$$

where $\vec{A}$ is given by (14.13). We may bypass equation (14.13) which determines correctly the direction of the area vector, and consistently impose on $\vec{A}$ to make an acute angle with $\vec{\omega}$. Under this convention the beam propagating rotation-wise rotates about $\vec{A}$ in a positive sense, and Δt will be positive as given by (14.14).

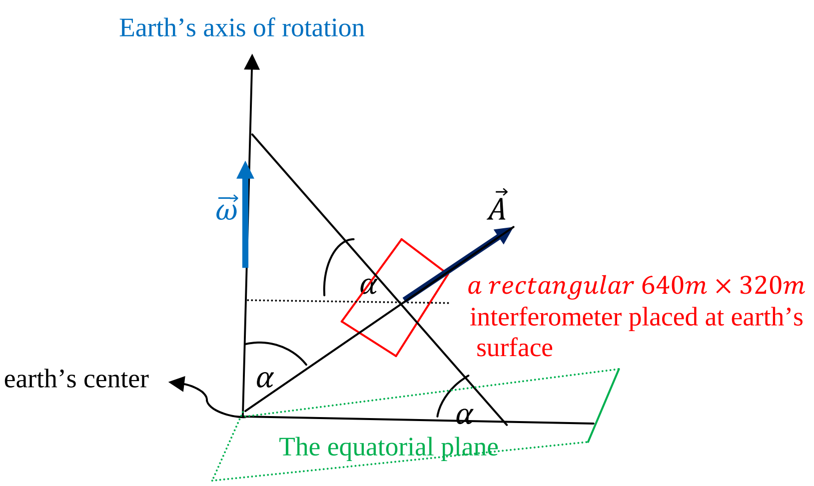


(iv) The optical circuit drawn below illustrates the idea of an optical coil, by which Sagnac effects add up.



14.6. The Michelson and Gale Experiment

This experiment is a version of an interferometer placed on a turntable, with the earth is the turntable. Here S is the frame of fixed stars and s is a frame attached to earth. The $640m \times 320m$ rectangular interferometer is set up horizontally on earth's surface. The rotation vector $\vec{\omega}$ of the earth is in the direction of the (south pole→north pole) axis, and its magnitude is 2π radian per sidereal day. The area vector $\vec{A}$ points upwards (downwards) in the northern (southern) hemisphere, and makes with the rotation vector $\vec{\omega}$ an angle α, which is the angle between the equatorial plane and the interferometer plane.



The Sagnac effect is given by

$$\Delta t = \frac{4}{c^2}\vec{\omega}.\vec{A}, \quad (14.14)$$

where:

$$\vec{\omega}.\vec{A} = \omega\, proj_{\vec{\omega}}\vec{A} = \omega \times \text{ projection of the area } A \text{ on the equatorial plane}$$
$$= \omega A\, cos\alpha.$$

The latter result constitutes an explanation of the Michelson and Gale experiment.

The relation (14.14) shows that Sagnac effect

(i)Assumes maximal values at the north and south poles, where $\alpha = 0$.
(ii) Decreases towards the equator till vanishing at it.
(iii) Not influenced by the orientation of the interferometer in its horizontal site, as well as, by the longitude of the later.
(iv) Is nil wherever the interferometer is placed parallel to the earth's axis of rotation.
(v) Has the same maximum magnitude measured at a pole if the interferometer is placed vertically at the equatorial plane.