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Some Further Notes on the Energy-Position/Momentum-Time Uncertainty Expressions for a Non-Relativistic Particle

Pavle I. Premović
Laboratory for Geochemistry, Cosmochemistry and Astrochemistry,
University of Niš, pavleipremovic@yahoo.com, Niš, Serbia

Abstract. A brief discussion related to the energy-position and the momentum-time uncertainty relations is presented.

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Derivation and Discussion. We recently introduced [1] the energy-position uncertainty and the momentum-time uncertainty expressions for a non-relativistic particle:

$\Delta E\Delta x < \hbar c$ (the energy-position uncertainty) … (1)
$\Delta p\Delta t > \hbar/c$ (the position-time uncertainty) … (2),

where $\hbar$ ($= h/2\pi = 1.05 \times 10^{-34}$ J sec)[1] is the reduced Planck's constant, h is the Planck's constant (6.63×10^{-34} J sec), and c ($\approx 3\times10^{8}$ m sec^{-1}) is the speed of light. These expressions are derived from two Heisenberg's uncertainty relations:

$\Delta p\Delta x \geq \hbar$ (the momentum-position uncertainty) ... (3)
$\Delta E\Delta t \geq \hbar$ (the energy-time uncertainty) ... (4).

Our derivation of the expression (1) is based on two assumptions: the speed of a non-relativistic particle $\upsilon < 1/2c$ and $\Delta p\Delta x$ ranges from $\hbar$ up to $2\hbar$.

If the energy uncertainty is defined as $\Delta E^2 = \upsilon\Delta p$, one can then easily derive the expression (1). According to Special relativity, $\upsilon < c$, so $\Delta x/v > \Delta x/c$. We can now write the following series of expressions: $\Delta E\Delta x/c < \Delta E\Delta x/\upsilon = \upsilon\Delta p\Delta x/\upsilon = \Delta p\Delta x$. So, $\Delta E\Delta x < c\Delta p\Delta x$. Since $\Delta p\Delta x \geq \hbar$, we get

Energy-position uncertainty relation: $\Delta E\Delta x < \hbar c$ … (5).

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[1] The values $\hbar$, $h/2$ and h are the most common values but the absolute minimum value is $\hbar/2$.

[2] The total energy of a non-relativistic particle $E = mc^2 + E_k$, where m and E_k ($= 1/2m\upsilon^2$) are, respectively, its rest mass and kinetic energy. Hence, $\Delta E = \Delta E_k < 1/2m\upsilon^2$. Thus, for a non-relativistic particle with a speed $\upsilon < 1/2c \rightarrow \Delta E < 1/8\, mc^2$.

The strong nuclear force is one of the four basic forces in nature (the others being gravity, the electromagnetic force, and the weak nuclear force) and it is the strongest one. However, this force also has the shortest range, holding together nucleons of deuteron: a proton and a neutron. The strong nuclear force is created between these nucleons by the exchange of particl*e* called the pion. The pion has a rest mass energy $\Delta E = mc^2 = 140$ MeV $= 2.24 \times 10^{-11}$ J. When the proton transfers its pion to the neutron, the neutron "borrows" the pion rest mass energy of 140 MeV. That energy produces a force between these two nucleons. According to the expression (5), the distance between them must then be $< 1.5 \times 10^{-15}$ m. For comparison, the charge radius of the deuteron is about 2.1×10^{-15} m.

Dividing the inequality (5) with Δx, we get $\Delta E < \hbar c/\Delta x$. Obviously, $\hbar c/\Delta x$ cannot be equal to zero because then ΔE would be less than zero or negative Thus, $0 < \Delta E < \hbar c/\Delta x$. Similar reasoning indicates that $0 < \Delta x < \hbar c/\Delta E$.

The inequalities (3) and (5) can be written as $\Delta p\Delta x = a\hbar$, where $a \geq 1$ and $\upsilon\Delta p\Delta x < \hbar c$. Substituting $\Delta p\Delta x$ with $a\hbar$ in $\upsilon\Delta p\Delta x < \hbar c$ we get $a\hbar v < \hbar c$ or $av < c$. For a non-relativistic particle with a speed $\upsilon < 1/2c$, $\upsilon = bc/2$, where $b < 1$. Now $a\upsilon = abc/2 < c$ or $ab < 2$. This last inequality is only possible if and only if $a \leq 2$. Thus, for a non-relativistic particle the following inequality is right:

$$2 \geq a \geq 1 \quad \dots (6).$$

In general, for a non-relativistic particle with a speed $\upsilon < c/q$, where $q < 2$, $\Delta E\Delta x < \hbar c$ is only possible for $q \geq a \geq 1$.

Depending on the values of various Planck's constants, now we can write for $\Delta p\Delta x$ the following limits ($\hbar/2 \leq \Delta p\Delta x \leq \hbar$; $\hbar \leq \Delta p\Delta x \leq 2\hbar$; $h/2 \leq \Delta p\Delta x \leq h$, and $h \leq \Delta p\Delta x \leq 2h$) and inequalities ($\Delta p\Delta x \geq h > h/2 > \hbar > \hbar/2$).

Dividing the expressions (3) and (4), we get $\Delta p\Delta x/\Delta E\Delta t \approx \upsilon/\upsilon^3 \approx 1$. So, for $\upsilon < 1/2c$, $\Delta E\Delta t$ also ranges $\hbar$ to $2\hbar$, as $\Delta p\Delta x$. In this case, we can write the following limits ($\hbar/2 \leq \Delta E\Delta t \leq \hbar$; $\hbar \leq \Delta E\Delta t \leq 2\hbar$; $h/2 \leq \Delta E\Delta t \leq h$, and $h \leq \Delta E\Delta t \leq 2h$) and inequalities ($\Delta E\Delta t \geq \hbar > \hbar/2$ and $\Delta E\Delta t \geq h > h/2$).

In our previous report [1], we derive the expression (2) using a simple mathematical procedure. For the sake of readers, we will repeat this derivation. The expression (4) can be written as $\Delta E\Delta t = (\upsilon\Delta p)\Delta t \geq \hbar$. Dividing $\upsilon\Delta p\Delta t \geq \hbar$ with υ we get $\Delta p\Delta t \geq \hbar/\upsilon$. Because $\upsilon < c$, $\hbar/\upsilon > \hbar/c$, we arrive at

$$\text{Momentum-time uncertainty relation: } \Delta p\Delta t > \hbar/c \quad \dots (7).$$

As we stated above, we can write the inequality (3) as $\Delta p\Delta x = a\hbar$, where $2 \geq a \geq 1$. The inequality (7) can be written as $\Delta p\Delta t = d\hbar/c$, where $d > 1$. Dividing these new equations, we get

[3] The uncertainty in momentum and position can be written as $\Delta p\Delta x \approx \hbar$. Nature prefers to keep the energy change to a minimum or $\Delta E\Delta t \approx \hbar$. Thus, $\Delta p\Delta x/\Delta E\Delta t \approx 1$. Since $\Delta p\Delta E = 1/\upsilon$ then $\Delta x/\Delta t \approx \upsilon$ {see also [2]}.

that $\upsilon = (a/d)c$. In the case of a non-relativistic particle with a speed $\upsilon < 1/2c$, after a little algebra, we get $d > 4$. So, for a non-relativistic particle:

$$\Delta p \Delta t > 4\hbar/c \quad \ldots (8).$$

Thus, depending of the values of various Planck's constants, now we can write for $\Delta p\Delta t$ the following limits ($\Delta p\Delta t > 2\hbar/c$; $\Delta p\Delta t > 4\hbar/c$; $\Delta p\Delta t > 2h$, and $\Delta p\Delta t > 4h$) and inequalities ($> 4\hbar/c > 2\hbar/c$ and $\Delta p\Delta t > 4h/c > 2h/c$).

In general,

$$\Delta p\Delta t > h/c \text{ or } \Delta p\Delta t > \hbar/c.$$

References

[1] P. I. Premović, *The Energy-position and the momentum-time uncertainty expressions.* The General Science Journal, December 2021.
[2] M. C. Jain, Quantum Mechanics: A Textbook for Undergraduates. (Second edition). PHI Learning, Delhi, 2017.