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The Energy-Position and the Momentum-Time Uncertainty Expressions

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Abstract.

The energy-position uncertainty and the momentum-time uncertainty expressions for a non-relativistic particle are derived from the two mathematical expressions of the Heisenberg uncertainty principle. These additional expressions are:

$$\Delta E \Delta x < \hbar c \text{ (the energy-position uncertainty)}$$

$$\Delta p \Delta t > \hbar / c \text{ (the position-time uncertainty)}$$

where $\hbar$ is the reduced Planck constant and c is the speed of light.

Keywords: Quantum mechanics, uncertainty principle, particle, energy-position, momentum-time.

Introduction

The Heisenberg uncertainty principle is one of the fundamental postulates of quantum mechanics[a]. This principle is typically expressed in either of two mathematical forms

$$\Delta p \Delta x \geq \hbar \text{ (the momentum-position uncertainty)} \quad \dots (1)$$

$$\Delta E \Delta t \geq \hbar \text{ (the energy-time uncertainty)} \quad \dots (2).$$

In essence, the formal uncertainty principle says that the momentum (Δp) times the uncertainty in the position (Δx) or the uncertainty in the energy (ΔE) times the uncertainty in the time (Δt) is greater or equal to $\hbar$[b]. The h is the Planck constant (6.63 x 10^{-34} J sec) or $\hbar = h/2\pi = 1.05 \times 10^{-34}$ J s. The uncertainty principle states that some pairs of physical observables (these are the quantities of a state that we can determine in the lab) cannot be precisely measured simultaneously to arbitrary accuracy [1].

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[a] Except of the Klein–Gordon and Dirac versions, quantum mechanics is non-relativistic [2-6].

[b] In practice, the absolute minimum uncertainty of $\hbar$ or $h/2$ *(or even h)* is far more common than the value $\hbar/2$ [7]. It is worth noting here that Budzik and Kizowski [8] reported that the single slit diffraction pattern of electrons is in accord with the following momentum-position uncertainty expression: $\Delta x \Delta p = \hbar$.

The purpose of this work is to present a very simple derivation of the energy-position uncertainty and the momentum-time uncertainty expressions for non-relativistic particles *via* the uncertainty principle inequalities (1) and (2). We assume that its content can be understood by a wide audience of readers acquainted with a calculus-based introduction to quantum mechanics.

Derivations and Discussion

Let us consider a non-relativistic particle of speed υ (say, approximately, $< 0.5c$), with the rest mass m_0, a momentum p (= $m_0\upsilon$) which moves along the x-axis. The total energy E of this particle is the following sum

$$E = m_0c^2 \text{ (the rest energy)} + p^2/2m_0 \text{ (the kinetic energy).}$$

Taking the first derivative of the total E (i. e. kinetic energy) with respect p, one get

$$\Delta E = (p/m_0)\Delta p$$

or

$$\Delta E = \upsilon\Delta p.$$

If we substitute Δp with $\Delta E/\upsilon$ in the expression (1), $\Delta p\Delta x$, and afterward reshuffle υ to the right side, we obtain

$$\Delta E\Delta x \geq \hbar\upsilon.$$

Suppose that $\Delta p\Delta x = a\hbar$[c] where $2 > a \geq 1$ then

$$\Delta E\Delta x = a\hbar\upsilon \quad ... (3).$$

Since Special relativity limits all particles to the speed of light c, we arrive at

$$\Delta E\Delta x < \hbar c \quad ... (4)$$

for $\upsilon < 0.5c$[d] or, in general, for $a(\upsilon/c) < 1$. This represents the energy-position uncertainty expression for a non-relativistic particle. Of course, if the speed of a non-relativistic particle $\upsilon << c$ then $\Delta E\Delta x \sim 0$ since the rest energy of this particle m_0c^2 is much greater than its kinetic energy $p^2/2m_0$.

The expression (4) informs us that the maximum of a non-relativistic particle is on the order of about 10^{-26} J m or that $\Delta E\Delta x$ of a particle ranges from ~ 0 J m - about 3.15×10^{-26} J m. It is worthy of note that the Planck length l_P is roughly 1.6×10^{-35} m so the maximum energy uncertainty ΔE is about 2×10^9 J or about the Planck energy E_P.

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[c] It appears that the formal momentum-position uncertainty equation (1), $\Delta p\Delta x \geq \hbar$, allows a very large change in Δp. In nature, any particle system tends to maintain a minimum of $\Delta p\Delta x \approx \hbar$ (or $\Delta E\Delta t \approx \hbar$) [see ref. 9].

[d] Of note, at $\upsilon/c \leq 0.5$ non-relativistic kinetic and relativistic kinetic energies are practically equal.

Multiplying both sides of the above expression $\Delta E = \upsilon \Delta p$ with Δt and rearranging terms, we obtain

$$\Delta E \Delta t = \upsilon \Delta p \Delta t \geq \hbar$$

or

$$\Delta p \Delta t \geq \hbar/\upsilon.$$

As $\hbar/\upsilon > \hbar/c$ then

$$\Delta p \Delta t > \hbar/c \quad \text{... (5).}$$

This represents the momentum-time uncertainty expression for a non-relativistic particle. The minimum of $\Delta p \Delta t$ is about 3×10^{-43} kg m. As in the case of the energy-position uncertainty expression (4), the expression (5) is only correct for non-relativistic limit $\upsilon < 0.5c$. It is worth of noting that the Planck time $t_P = 5.39 \times 10^{-44}$ sec so the minimum momentum uncertainty is roughly about 6 kg m sec^{-1}.

Thus, there are four expressions of the uncertainty relation for a single, non-relativistic particle which can be arranged to those related to energy

$\Delta E \Delta x < \hbar c$ (the energy-position uncertainty)
$\Delta E \Delta t \geq \hbar$ (the energy-time uncertainty)

and momentum

$\Delta p \Delta x \geq \hbar$ (the momentum-position uncertainty)
$\Delta p \Delta t > \hbar/c$ (the momentum-time uncertainty).

It is interesting to note that the two "new" expressions (4) and (5) link together three universal constants: the speed of light c, the Planck constant h and Ludolph's number π.

A derivation of the energy-position and momentum-time uncertainty expressions for a relativistic particle is a bit "fuzzy" (like space and time in quantum mechanics). The total energy E of the relativistic particle is given by the following equation

$$E = mc^2 = (p^2c^2 + m_0^2c^4)^{1/2}.$$

In this case for $c > \upsilon \geq 0.5c$

$$\Delta E \Delta x < 2\hbar c$$
$$\Delta p \Delta t > \hbar/c$$

only if the Heisenberg inequality (1) is non-relativistic (see, also, refs. 5 and 6].

In the case of a photon

$$E = pc.$$

After the first derivation of E with respect for p

$$\Delta E = c\Delta p.$$

Multiplying both sides with Δx

$$\Delta E\Delta x = c\Delta p\Delta x$$

The position Δx of a photon is its wavelength λ, i.e. $\Delta x = \lambda$. The momentum of a photon p is known to be the ratio h/λ. For a finite change of p, we can write $\Delta p = h/\lambda$. By substituting h/λ for Δp and λ for Δx in the last equation and after some simple algebra, we arrive at

$$\Delta E\Delta x = hc.$$

Moreover, since Δt of a photon is λ/c then

$$\Delta p\Delta t = \Delta p(\lambda/c) = h/c.$$

Applications. There are possibly many applications of the above expressions for the energy-position uncertainty (4) and the momentum-time uncertainty (5). It appears that in many applications the energy-position uncertainty expression (3) for a non-relativistic particle: $\Delta E\Delta x < \hbar c$ is more convenient and straightforward than the expression (1) of the momentum-position uncertainty: $\Delta p\Delta x \geq \hbar$. One of the illustrative examples is the case of the non-existence of the electron in the hydrogen (atom) proton.

The radius of the hydrogen proton is approximately 10^{-15} m. If an electron exists inside this proton, then the uncertainty in the position of the electron is given by $\Delta x \approx 10^{-15}$ m. According to the expression (3) $\Delta E\Delta x < \hbar c$. The uncertainty in energy is $\Delta E < \hbar c/\Delta x$ where $\Delta x \sim 10^{-15}$ m $\Delta E < 1.05\times10^{-34}\times3\times10^{8}/10^{-15} \sim 3\times10^{-11}$ J ~ 200 MeV. If this rough estimation is correct, then the energy (E) of the electron in the hydrogen proton should be in the same order of magnitude, 10^{-11} J or 100 MeV. The experimental results however indicate that the electron in the atom has $E < 4$ MeV. Therefore, an electron cannot exist in the hydrogen proton.

The second example is the case of the "zero-point" energy of liquid He (helium) at temperature $T = 0$ K. In this case, all other energy is removed from a system except the "zero-point" one. Indeed, according to the expression (4) if a He atom is confined within the smallest possible distance Δx at $T = 0$ K then it still has $\Delta E < \hbar c/\Delta x > 0$ at this temperature. In other words, since the location of this atom is not completely indefinite then its energy E cannot be zero. Hence the He atom must possess finite kinetic energy even at 0 K, so-called "zero-point energy". This is the kinetic energy of the He atoms at 0 K, coming from their vibrational motion.

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