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On Bell's Theorem

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Abstract

Bell's Theorem implies that quantum correlation of entangled particles as calculated in quantum mechanics violates elementary probabilistic inequalities. It is shown that the reason is a problem in scaling of detector directions.

Keywords: Bell's Theorem, CHSH inequality, Born rule

Bell's Theorem [1] considers measurements of spin for entangled particles. The entangled wave function studied in this teorem is

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|\psi_{z+}\rangle \otimes |\psi_{z-}\rangle - |\psi_{z-}\rangle \otimes |\psi_{z+}\rangle)$$

where $|\psi_{j+}\rangle$ and $|\psi_{j-}\rangle$, $j \in \{x, y, z\}$, are the eigenvectors of the Pauli matrices σ_j corresponding to the eigenvalues 1 and -1 respectively. The spin of the first particle is detected with two detector directions a and a' and the spin of the second particle is detected in the second measurement with two detector parameters b and b'. Measurement of the spin means applying operators

$$(\sigma \cdot a) \otimes Id \quad , \qquad Id \otimes (\sigma \cdot b).$$

In both cases the spin measurement can only give the results spin up or spin down and collapses the wave function to direct products of eigenvectors of Pauli matrices. The quantum correlation

$$C(a,b)_q = \langle\phi| Id \otimes (\sigma \cdot b)(\sigma \cdot a) \otimes Id |\phi\rangle$$

gives the expected value for the empirical correlation

$$C(a,b)_e = \frac{N_{++} + N_{--} - N_{+-} - N_{-+}}{N_{++} + N_{--} + N_{+-} + N_{-+}}$$

where $N_{\alpha,\beta}$, $\alpha, \beta \in \{+,-\}$, is the number of cases when the first particle is measured spin α and the second spin β. A direct calculation shows that

$$C(a,b)_q = -b \cdot a.$$

The measurements of entangled pairs in the directions a, b, a, b', a', b and a', b' form four time series that have only values ± 1. We can define binary probability variables A, B, A' and B' and assign to them probabilities of being 1 or -1 from these time series. The corretion between these variables is then the expectation value of the product of the variables, thus

$$C(a,b)_p = E(AB).$$

Binary variables satisfy certain inequalities, called Bell's inequalities. The CHSH inequality is a convenient Bell's inequality for proving Bell's theorem:

$$C(a,b)_p + C(a,b')_p + C(a',b)_p - C(a',b')_p \leq 2. \tag{1}$$

Assiging detector directions as

$$a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad a' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad b = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \quad b' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{2}$$

gives

$$C(a,b)_q + C(a,b')_q + C(a',b)_q - C(a',b')_q = 2\sqrt{2} > 2 \tag{3}$$

showing that $C(a,b)_q \neq C(a,b)_p$. Bell's inequality violations have been observed in experiments. However, the reason is quite simple. The detector values are normalized to give

$$a \cdot b = a' \cdot a' = b \cdot b = b' \cdot b' = 1 \tag{4}$$

which may initially seem correct, but it is not. The correct scaling is

$$\sum_i |a_i| = \sum_i |a'_i| = \sum_i |b_i| = \sum_i |b'_i| = 1. \tag{5}$$

Scaling the directions according to (5) removes the Bell's inequality violation in (2).

In order to show that the scaling (4) is incorrect, let the directions be scaled as in (4) and let us consider the first particle

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_{z+}\rangle - |\psi_{z-})$$

without focusing on the entanglement. It is measured by applying the operator $\sigma \cdot a$. The wave function $(\sigma \cdot a)|\psi\rangle$ collapses to one of the eigenvectors $|\psi_{m\alpha}\rangle$, $m \in \{x, y, z\}$, $\alpha \in \{+, -\}$, of Pauli matrices and the corresponding eigenvalue is α. If the first particle collapses to $|\psi_{m\alpha}\rangle$ the second particle collapses to $|\psi_{m\beta}\rangle$, $\beta \neq \alpha$. In the second measurement this collapsed second particle collapses to one of the eigenvectors $|\psi_{n\gamma}\rangle$, $n \in \{x, y, z\}$, $\gamma \in \{+, -\}$. Because $|\langle\psi_{n\gamma}|\psi_{m\beta}\rangle|^2 = \frac{1}{2}$ if $n \neq m$, the probability of $|\psi_{m\alpha}\rangle$ collapsing to $|\psi_{n+}\rangle$ is the same as the probability that it collapses to $|\psi_{n-}\rangle$. As the eigenvalues for $|\psi_{n+}\rangle$ and $|\psi_{n-}\rangle$ are opposites, these contributions to the correlation of the first and the second particle cancel. There remains the collapse of $|\psi_{m\beta}\rangle$ to $|\psi_{m\gamma}\rangle$, $\gamma \in \{+, -\}$. As $|\langle\psi_{m\gamma}|\psi_{m\beta}\rangle|^2 = \delta_{\beta=\gamma}$ if can only collapse to $|\psi_{m\beta}\rangle$.

In the scaling (4) the wave function $|\psi\rangle$ of the first particle collapses to $|\psi_{m\alpha}\rangle$ with the probability a_m. Because of entanglement, the second particle collapses after the first measurement to $|\psi_{m\beta}\rangle$, $\beta \neq \alpha$, with the same probability a_m. In the second measurement this collapsed second particle collapses to either $|\psi_{n+}\rangle$ or $|\psi_{n-}\rangle$ with the weight $|b_n|$. The sum of these weights must be one. Thus, the probability of the collapse is

$$\frac{b_n}{\sum_i |b_i|.}$$

Thus, the quantum correlation is not $-b \cdot a$. It is

$$C(a, b)_q = \frac{-b \cdot a}{\sum_i |b_i|}. \tag{8}$$

Correcting the quantum correlation removes Bell's inequality violation in (2).

However, this is not the logical way to correct it. We certainly want that

$$\langle\psi|(\sigma \cdot a)(\sigma \cdot a)|\psi\rangle = a \cdot a.$$

This holds if we scale as in (5):

$$\sum_i |a_i| = 1.$$

Adopting this scaling we see that

$$\sum_i a_i^2 < 1 \tag{9}$$

and may wonder where the missing probability is. It is in the contributions that cancelled in the calculation of quantum correlation and which also cancel in the calculation of autocorrelation (8). Indeed, the probability of $|\psi\rangle$ collapsing to either $|\psi_{m+}\rangle$ or $|\psi_{m-}\rangle$ is not a_m^2. That is only the part that is seen. Calculating autocorrelation (8) can be understood as two measurements, as with the correlation of two particles. The first measurement collapses $|\psi\rangle$ to eigenvectors $|\psi_{m+}\rangle$ or $|\psi_{m-}\rangle$ with the probability $|a_m|$ in the scaling (5). In the second measurement these eigenvectors are further collapsed to $|\psi_{n\gamma}\rangle$ and the probability of collapses to either $|\psi_{n+}\rangle$ or $|\psi_{n-}\rangle$ is $|a_n|$. Thus, the sum of the probabilities of eigenvectors to which $|\psi_{m+}\rangle$ or $|\psi_{m-}\rangle$ collapse is

$$|a_m|(|a_x| + |a_y| + |a_z|) = |a_m|$$

of which we see only the part $|a_m|^2$ as the other parts cancel in the measurement. The total probability of scaling (5) is one, though it appears, as in (9), that probabilities do not sum to one. This phenomenon explains why experiments have verified violations of Bell's inequality. In these experiments the probabilities have been derived from the numbers of detected particles and their sum has been scaled to 1 as in (9). This ignores the probability of contributions that cancel.

Bell's Theorem is sometimes explained by the following example. Assume that in a test of entangled particles the detectors in both sides are perfectly aligned, $a_z = b_z = 1$. We see perfect anticorrelation. Then move the detector of the first particle in the (x,z)-plane to a small angle $\alpha = \gamma/2$ so that there are 1% errors in detection. Moving the detector of the second particle in the (x,y)-plane to the angle $\beta = -\gamma/2$ must also introduce 1% errors. Thus, there should be 2% errors, but according to quantum mechanics, and experiments, there will be 4% errors. The number of errors was $\sin^2(\gamma/2)$ when the (small) angle between the detectors was $\gamma/2$ and it grows to

$$\sin^2(\gamma) = (2\sin(\gamma/2)\cos(\gamma/2)^2 \approx 4\sin^2(\gamma/2)),$$

to four times as large when the angle between the detectors is γ.

In reality, there is no mystery. The number of errors is related to the x-coordinate of the detector as $\sin^2(\gamma/2) = a_x^2$ thus $a_x = \sin(\gamma/2)$. In a communication system analogy we can think that the fraction a_x of the bits have errors. When β is set to $\gamma/2$ the errors in the communication analogy grow to about $2a_x$. The number of noticed errors is thus about $(2a_x)^2$, that is, four times as many errors as when the angle was only in one side.

The mathematical form of the quantum correlation $-b \cdot a$ can be expressed with the angle θ between the detectors as

$$-b \cdot a = -b_z a_z - b_x a_x = -|b||a| \cos(\alpha) \cos(\beta) - |b||a| \sin(\alpha) \sin(\beta)$$

$$= -\cos(\alpha - \beta) = -\cos(\theta)$$

as $|a| = |b| = 1$ by the norming (4), which is used when deriving this mathematical form of the correlation and also in experiments that have confirmed the form $-\cos(\theta)$.

The function $-\cos(\theta)$ is -1 if $\theta = 0$, zero if $\theta = \pi/2$ and 1 if $\theta = \pi$. Sometimes it is argued that as a classical correlation should be a linear function and the linear function fitting to these three points is $\frac{2}{\pi}\theta$, but as experiments confirm that $-\cos(\theta)$ is the correct form, this is a demonstration that quantum mechanics differs from classical physics.

Quantum mechanics certainly differs from classical physics, there is e.g. the collapse of wave functions, but this particular mathematical form of quantum correlation does not touch those issues. The correlation should indeed classically be a linear function, but a linear function of $|a_x|$. As $|a_x| = \sin(\theta)$ we have to look for a linear function agreeing on those three points. Two free parameters is not enough to fit the three points. We have to still take a linear shift of θ in the inside function $\sin(\theta)$. Thus, we look for a solution of the type

$$C(a,b)_q = k \sin(\theta + \gamma) + r$$

where k, r and γ are to be determined. From the three points we get three equations and the solution is $k = -1$, $r = 0$ and $\gamma = \frac{\pi}{2}$ giving the correct quantum correlation $C(a,b)_q = -\cos(\theta)$.

For clarity, I will go through the problem in Bell's theorem again. We have expressed a vector in a base e_i as $a = \sum_i a_i e_i$. The square a_i^2 is interpreted as a probability. Then this probability is divided into parts by weights $p_{ij} \geq 0$:

$$a_i^2 = \sum_j p_{ij} a_i^2$$

These weights are real and nonnegative numbers and their sum must be one. The square root of the probability is also divided into parts by these weights

$$a_i = \sum_j p_{ij} a_i$$

and again the weights must sum to one or the probability does not stay constant. In Bell's theorem the weights p_{ij} are the settings b_j of the other detector. These b_j are real and nonnegative numbers. They divide the probability a_i^2 and they must sum to one in order to the probability to stay constant. In Bell's theorem the numbers a_i and b_j are scaled as wave functions by the Born rule and division of probability is made by a projection. The projection is done correctly, the way projections are usually done, but a projection in a complex space is not a valid way of dividing probability and here the question is division of probability: a violation of Bell's inequality is caused by the loss of probability in this division of probability by taking a projection.

It may seem natural that taking a projection should be the correct way as it is similar to the Born rule, but doing so leads to a contradiction: an elementary theorem from the probability theory is violated. Elementary mathematics cannot be violated in quantum mechanics just like they cannot be violated in any other field. Indeed, if basic mathematics would turn out false, we could just as well through away all science as it depends on basic mathematics. The usual explanation is that the elementary probability theorem cannot be applied because the particles coming to the detectors do not have probabilities. This explanation is false, as I show in [2]. There are two base vectors (one was forgotten by Bell) and the particles coming to the detectors have probabilities in the normal way. The reason for the failure of the Bell's inequality can only be the scaling of the detector values and we can exactly see that the problem comes when dividing a_i by the weights b_j. The problem appear if the particles have probabilities when coming to the detectors and it appears in the same way if the particles do not have probabilities, so the usual explanation does not solve the contradiction.

In Born's rule a wave function is expressed in the base e_i and the squares of the norms of the coefficients sum to one. No contradicion has arisen from this rule and it is logical: the wave function is expressed in the basis of wave functions. Consequently a wave function is zero if all coefficients in this basis are zero. That is, the probability of a wave function giving the zero vector in this basis is zero. The detector values are not expressed in the basis of detector values. They are expressed in the basis of wave functions. The sum of components of the detector value in the basis of wave functions is a projection of the detector value to the basis of wave functions. The probability of the zero vector need not be zero and therefore the sum of the squares of the norms in this basis does not need to be one. If in doubt that some rule in quantum mechanics states that detector values must be scaled as wave functions try this: turn the detector head away from the beam of particles. Then the projection of the detector value in the basis of wave functions is zero, but the detector value has some value and is not zero. Keep the head in this position for half of the experient and ask the audience what might be the probability of a zero vector for a detector value in this basis.

There is no valid reason to claim that detector values must be scaled as wave functions. The scaling of detector values must be selected and in Bell's experiment it must be selected in such a way that probability is not lost and an elementary probability theorem holds.

Hopefully this short note has helped to demystify Bell's Theorem. The reason for the failure of Bell's inequalities seems to be incorrect scaling of detector directions and it has nothing to do with hidden parameters in the EPR paradox. For a discussion of the EPR paradox, see [2].

References:

[1] John Bell,*On the Einstein Podolsky Rosen Paradox. Physics.* 1 (3):195-200, 1964. [2] Jorma Jormakka, *A Hidden Variable Solution to the EPR Problem* vixra:1810.0503, 2018.