the general science journal

Solution of the Relativistic Quantum Hydrodynamic Fundamental Equations for Free and Bound Photons

Halil Güveniş, Istanbul
E-mail: guevenis@rocketmail.com

Abstract

In the present paper we solve the relativistic quantum hydrodynamic fundamental equations in the simplest case for free and bound photons. We obtain as solution spherical harmonics and spherical Bessel functions. We put the density distribution of free photons into the density distribution of bound photons and derive thereby the total energy equation for free and bound photons. For a single bound photon we calculate the Casimir energy using the Euler-MacLaurin sum formula. The resulting total energy of a single bound photon has two different forces, centrifugal and Casimir forces keeping each other in balance. The fact that the total energy equation can be split into two independent energy equations is interpreted as the effect that the single bound photon is a binding state of a particle and its antiparticle.

Key words: Fundamental equations of relativistic quantum hydrodynamics, free and bound photons, Casimir energy, binding state of a particle and its antiparticle.

1. Introduction

Although nearly a century has passed since Madelung [1] and de Broglie [2] developed the hydrodynamic formulation of quantum mechanics, no attempt has been made in the meantime to systematically solve the quantum hydrodynamic fundamental equations for each known quantum physical problem. – An important exception in this context is the pioneering work of H. E. Wilhelm [3] who solved the nonrelativistic, quantum hydrodynamic fundamental equations 1. for a particle in a box, 2. for the harmonic oscillator, 3. for the hydrogen atom and 4. for the free motion of a particle. But he did not succeed to clearly identify the quantum potential occurring in quantum hydrodynamic picture [4] with the effect of any physical object – with "hidden variables" –, because, in our opinion, he started from a nonrelativistic, hydrodynamic, fundamental equation equivalent to the Schrödinger equation and he did not use the Klein-Gordon or the Dirac equation for the relativistic hydrodynamic formulation.

In the present paper, we want to start with the systematic solution of the relativistic quantum hydrodynamic fundamental equations and solve them in the simplest case for free and bound photons. For this purpose, we postulate in Section 2 the relativistic quantum hydrodynamic fundamental equations equivalent to the Dirac equation and solve them in Section 3 for bound and in Section 4 for free photons. In Section 5, the total energy equation for free and bound photons is derived and then in Section 6, the Casimir energy of a single bound photon is calculated. In Section 7, we calculate the total energy of a single bound photon. Section 8 is provided for summary and conclusions.

2. The relativistic quantum hydrodynamics

As fundamental equations of relativistic quantum hydrodynamics (RQH) we postulate the hydrodynamic formulation of the Dirac equation [5]:

$$\partial(\rho(E - e\Phi)/c^2)/\partial t + \boldsymbol{\nabla} \cdot (\rho(\boldsymbol{p} - (e/c)\,\boldsymbol{A})) = 0, \tag{1}$$

$$(E - e\Phi)^2 = m_0^2\, c^4 + (\boldsymbol{p} - (e/c)\,\boldsymbol{A})^2\, c^2 - e\hbar c\, \boldsymbol{\sigma} \cdot \boldsymbol{B} - \hbar^2 c^2\, \rho^{-1/2} \Box \rho^{\,1/2}, \tag{2}$$

$$\oint(\boldsymbol{p} - (e/c)\,\boldsymbol{A}) \cdot d\boldsymbol{r} = 2\pi\hbar\, m, \qquad m = 0, \pm 1, \pm 2, \ldots \tag{3}$$

where are

c	the constant vacuum speed of light,
$\hbar$	the reduced Planck constant,
e	the elementary charge,
$\boldsymbol{\sigma}$	the 2x2 Pauli matrices,
m_0	the rest mass,
$\boldsymbol{p}$	the particle momentum,
ρ	the probability density,
E	the energy,
Φ, $\boldsymbol{A}$	the electromagnetic field quantities,

$\boldsymbol{B} = \boldsymbol{\nabla} x\, \boldsymbol{A}$.

Eq. (1) is the equation of continuity of the RQH, Eq. (2) the equation of motion, and Eq. (3) a quantization rule for the angular momentum going back to Bohr-Sommerfeld-Wilson.

Since the nonrelativistic approximation gives

$$m_0\, c^2\, (1 + ((\boldsymbol{p} - (e/c)\boldsymbol{A})^2 - e(\hbar/c)\, \boldsymbol{\sigma} \cdot \boldsymbol{B} - \hbar^2 \rho^{-1/2} \Box \rho^{\,1/2})/m_0^2\, c^2)^{1/2} \tag{4}$$

$$\approx m_0\, c^2 + ((\boldsymbol{p} - (e/c)\boldsymbol{A})^2 - e(\hbar/c)\, \boldsymbol{\sigma} \cdot \boldsymbol{B} - \hbar^2 \rho^{-1/2} \Box \rho^{\,1/2})/2\, m_0\,,$$

we obtain in place of the equation of motion (2)

$$E = m_0\, c^2 + e\Phi + ((\boldsymbol{p} - (e/c)\, \boldsymbol{A})^2 - e(\hbar/c)\, \boldsymbol{\sigma}\cdot\boldsymbol{B} - \hbar^2 \rho^{-1/2} \Box \rho^{\,1/2})/2\, m_0\,, \tag{5}$$

where are

$m_0\, c^2$	the rest energy,
$e\Phi$	the potential energy,
$(\boldsymbol{p} - (e/c)\, \boldsymbol{A})^2/2\, m_0$	the kinetic energy,
$-e(\hbar/c)\, \boldsymbol{\sigma}\cdot\boldsymbol{B}/2\, m_0$	the spin energy,
$-\hbar^2 \rho^{-1/2} \Box \rho^{\,1/2}/2\, m_0$	the so-called quantum potential [4].

Eq. (5) is the equation of motion for the hydrodynamic formulation of the Pauli equation [6].

3. Solution for bound photons

We start now with the systematic solution of the relativistic quantum hydrodynamic fundamental equations (1)-(3) and solve them first for bound photons. In this simplest case ($\Phi = 0,\ A = 0,\ \boldsymbol{B} = 0$), the fundamental equations (1)-(3) can be reduced to the equations

$$\partial(\rho(E/c^2)/\partial t + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{p}) = 0, \tag{6}$$

$$E^2 = m_0^2\, c^4 + \boldsymbol{p}^2\, c^2 - \hbar^2 c^2\, \rho^{-1/2} \Box\, \rho^{\,1/2} \tag{7}$$

$$\oint \boldsymbol{p}\cdot d\boldsymbol{r} = 2\pi\,\hbar\, m, \qquad m = 0, \pm 1, \pm 2, \ldots \tag{8}$$

The equation of motion (7) corresponds to the hydrodynamic formulation of the Klein-Gordon equation and provides therefore only such solutions that are equivalent to the solutions of the Klein-Gordon equation.

Since in this study we are interested only in stationary solutions and the probability density ρ must fulfill the boundary conditions $\rho\,(\boldsymbol{r}_1) = 0$ and $\rho\,(\boldsymbol{r}_2) = 0$, the equation of continuity (6) gives

$$\boldsymbol{\nabla}\cdot(\rho\boldsymbol{p}) = 0,\ \boldsymbol{p}_r\,(\boldsymbol{r}) = 0\,. \tag{9}$$

Since, furthermore, the particle momentum can only flow in closed lines and because of the azimuthal symmetry of the system, the momentum field in polar coordinates is independent of the angle φ:

$$\boldsymbol{p} = (\hbar\,/\,(r\,sin(\delta))\boldsymbol{e}_{\varphi}\,m \qquad\qquad m\ =\ 0,\ \pm 1,\ \pm 2, \ldots \tag{10}$$

Eq. (10) is in accordance with the quantization rule (8):

$$\int \boldsymbol{p}\cdot d\boldsymbol{r} = \oint (\hbar\,m\,/\,(r\,sin(\delta))\boldsymbol{e}_{\varphi}\,d\boldsymbol{r} = \hbar\,m \int d\varphi\ =\ 2\pi\,\hbar\,m\,. \tag{11}$$

Substituting Eq. (10) into the equation of motion (7) gives

$$E^2 = m_0^2\,c^4 + (\hbar\,c\,m\,/\,(r\,sin(\delta)))^2 - \hbar^2 c^2\,\rho^{-1/2}\,\boldsymbol{\nabla}^2 \rho^{\,1/2}. \tag{12}$$

If we assume

$$\rho^{\,1/2} = f(r)\ g(\varphi,\delta)\ , \tag{13}$$

Eq. (12) can be separated into two differential equations with respect to r and δ (λ = separation parameter) [3]:

$$(r^2\,/\,f)[\ (1\,/\,r^2)\partial(r^2\partial f\,/\partial r)/\partial r + (E^2\ - m_0^2\,c^4)\,f\,/\,(\hbar^2 c^2)\ ] = \lambda \tag{14}$$

and

$$\lambda = (-\,1\,/\,g\,)[\ (1\,/\,sin(\delta))\partial(sin(\delta)\partial g\,/\partial\,\delta)/\partial\,\delta - (m\ \,/\,sin(\delta))^2\,g\ ]\ . \tag{15}$$

Solutions of the differential equation (15) are the spherical harmonics

$$g_{lm}\,(\varphi,\delta) = (-1)^m\,[(l-|m|)!\,(2\,l+1)\,/\,((l+|m|)!\,4\,\pi)]^{1/2}\,e^{im\varphi}$$
$$(1\,/\,2^l\,l!\,)\,(\partial/\partial(\cos(\delta))^l\,(\cos^2(\delta)-1)^l, \qquad (16)$$

where the constants λ and m have the eigenvalues

$$\lambda = l\,(l+1), \qquad l = 0, \pm 1, \pm 2, \ldots \qquad (17)$$

and

$$-l \leq m \leq +l\,. \qquad (18)$$

Eq. (14) with (17) leads to Bessel's differential equation whose normalizable solutions are given by the spherical Bessel functions [7]:

$$f_l(\varepsilon\,r) = c\,(-\varepsilon\,r\,)^l\,(\partial/((\varepsilon\,r\,)\partial(\varepsilon\,r\,))^l\,[\sin(\varepsilon\,r)\,/\,(\varepsilon\,r)], \qquad (19)$$

where

$$\varepsilon^2 = (1\,/\,d)^2 = (E^2 - m_0^2\,c^4)\,/\,(\hbar^2 c^2) \qquad (20)$$

applies.

Substituting $l = 1,\ m = -1,\ 0,\ 1$ in (17) and (18), we obtain from (16) and (19) the eigenfunctions of the first bound photons:

$$g_{1m}\,(\varphi,\delta) = (-1)^m\,(3/4\,\pi)^{1/2}\,e^{im\varphi}\,\cos(\delta) \qquad (21)$$

and

$$f_1\,(\varepsilon\,r) = c_1\,[\sin(\varepsilon\,r)\,/\,(\varepsilon\,r)^2 - \cos(\varepsilon\,r)\,/\,(\varepsilon\,r)]. \qquad (22)$$

Thus, we obtain from (22) for the radial density distribution of the first bound photons (Fig. 1):

$$f_1^2(x) = c_1^2\,[\sin(x)\,/\,x^2 - \cos(x)\,/\,x]^2, \qquad \text{with } x = \varepsilon\,r. \qquad (23)$$

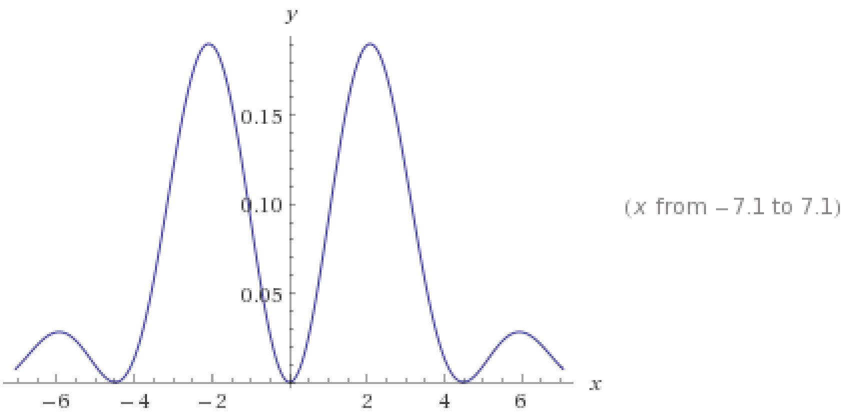

Figure 1: $y = f_1^2(x) / c_1^2 = [sin(x) / x^2 - cos(x) / x]^2$

Zero points of the density distribution (23)

$x = 0$
$x \approx \pm 4.49340945790906\ldots$
$x \approx \pm 7.72525183693771\ldots$
$x \approx \pm 10.9041216594289\ldots$
$x \approx 14.0661939128315\ldots$

are numerical solutions of the equation

$$tan(x) - x = 0\,. \qquad (24)$$

Thus, the density distribution of the very first bound photon is located in the interval

$$r_1 \le r \le r_2\,,\ \ r_1 = 0,\ \ r_2 = 4{,}49341\ d. \qquad (25)$$

Performing the integral in the equation (ε = 1 / d)

$$1 = c_1^2\ (3 / 4\ \pi) \int_0^{2\pi}\int_0^{\pi}\int_0^{4.49341\ d} r^2\ sin(\delta)\ cos^2(\delta)[sin(\varepsilon r) / (\varepsilon r)^2 - cos(\varepsilon r) / (\varepsilon r)]^2\ d\varphi\, d\delta\, dr,$$

we obtain the normalization constant

$$c_1^2 = \ 0{,}46714 / d^{\,3}. \qquad (26)$$

4. Solution for free photons

Going from bound to free photons, we must reduce the reduced relativistic quantum hydrodynamic fundamental equations (6)-(8) once again, so that no particle mass and no particle momentum are present ($m_0 = 0$, $\boldsymbol{p} = 0$):

$$\partial(\rho_0 (E_0 / c^2)/\partial t = 0 \,, \tag{27}$$

$$E_0^2 = -\hbar^2 c^2 \rho_0^{-1/2} \Delta \rho_0^{\,1/2} \tag{28}$$

$$l = 0 \,, \; m = 0 \,. \tag{29}$$

Substituting (29) in (16) and (19), we obtain the eigenfunctions of free photons in the stationary state:

$$g_{00} (\varphi, \delta) = (1 / 4\pi)^{1/2} \tag{30}$$

and

$$f_0 (k\, r) = c_0 \sin(k\, r) / (k\, r), \tag{31}$$

where

$$E_0^2 = \hbar^2 c^2 k^2 \tag{32}$$

applies.

The free photons are enclosed in the density distribution of the bound photons, i.e. in the intervals $r_1 \leq r \leq r_2$ with boundary conditions $\rho_0 (r_1) = 0$ and $\rho_0 (r_2) = 0$ – like in a box with infinite potential walls – , so that according to Eq. (31)

$$f_0 = c_0 \,(\sin(k_1 r) / (k_1 r))\, (\sin(k_2 r) / (k_2 r)), \; k_1 = n_0\, \pi / r_1 \,, \; k_2 = n_0\, \pi / r_2 \tag{33}$$

holds.

5. Derivation of the total energy equation

Since we have put the density distribution of free photons in the density distribution of bound photons, the radial amplitudes of the probability density must also be coupled together:

$$f_c = f_0\, f\,. \tag{34}$$

Substituting (34) in the energy equation (14) yields

$$E^2 = m_0^2\, c^4 + \hbar^2 c^2\, l\,(l + 1) / r^2 - \hbar^2 c^2\,(1 / r^2\,(f_c / f_0))\partial(r^2 \partial(f_c / f_0) / \partial r)/\partial r\,. \tag{35}$$

Evaluation of (35) gives ($\partial/\partial r = '$)

$$E^2 = m_0^2\, c^4 + \hbar^2 c^2\, l\,(l + 1) / r^2 - \hbar^2 c^2[f_c'' / f_c - f_0'' / f_0 + 2(1 / r - f_0' / f_0)\,(f' / f)]. \tag{36}$$

If we calculate the unknown quantities (f' / f), (f_0' / f_0), (f_0'' / f_0) occurring in (36) according to (22) and (33), we obtain

$$f' / f \quad = (1 / r)\,[\tan(\varepsilon r)(\varepsilon^2 r^2 - 2) + 2\varepsilon r] / [\tan(\varepsilon r) - \varepsilon r], \tag{37}$$

$$f_0' / f_0 \quad = k_1 \cot(k_1 r) + k_2 \cot(k_2 r) - 2 / r, \tag{38}$$

$$\begin{aligned} f_0'' / f_0 \quad = {} & 2\, k_1 k_2 \cot(k_1 r) \cot(k_2 r) - (4 / r)(k_1 \cot(k_1 r) + k_2 \cot(k_2 r)) \\ & + 6 / r^2 - k_2^{\,2} - k_1^2. \end{aligned} \tag{39}$$

Substituting (37)-(39) in (36) we get the total energy equation for free and bound photons:

$$\begin{aligned} E^2 = {} & m_0^2\, c^4 + \hbar^2 c^2\, l\,(l + 1) / r^2 + \hbar^2 c^2[2\, k_1 k_2 \cot(k_1 r) \cot(k_2 r) \\ & - 4\, k_1 \cot(k_1 r) / r - 4\, k_2 \cot(k_2 r) / r + 6 / r^2 - k_2^{\,2} - k_1^2 \\ & - 2\,(1 / r + 2 / r - k_1 \cot(k_1 r) - k_2 \cot(k_2 r)) \\ & ((1 / r)\,(\tan(\varepsilon r)(\varepsilon^2 r^2 - 2) + 2\varepsilon r) / (\tan(\varepsilon r) - \varepsilon r))] - \hbar^2 c^2 f_c'' / f_c\,. \end{aligned} \tag{40}$$

6. Casimir energy for a single bound photon

Since in Eq. (40) the term $-\hbar^2 c^2 f_c'' / f_c$ describes the derivation of the total energy eigenvalues from the radial wave function f_c and mathematically assigns an identically large wave amplitude to the foregoing forms of energy, this term can be ignored by calculating the total energy. We further reorganize Eq. (40) so that the total energies of free and bound photons are placed on two different sides of the equation separated from each other:

$$[E^2 - m_0^2 c^4 - \hbar^2 c^2 l(l+1) / r^2]^{1/2} = \tag{41}$$
$$\pm \hbar c [-k_1^2 - k_2^2 + 2 k_1 k_2 \cot(k_1 r) \cot(k_2 r)$$
$$- 4 k_1 \cot(k_1 r) / r - 4 k_2 \cot(k_2 r) / r + 6 / r^2$$
$$+ 2 (3 / r - k_1 \cot(k_1 r) - k_2 \cot(k_2 r)) h]^{1/2},$$

$$\text{with} \quad h = -(1/r)(\tan(\varepsilon r)(\varepsilon^2 r^2 - 2) + 2\varepsilon r) / (\tan(\varepsilon r) - \varepsilon r). \tag{42}$$

The total energy of free photons in the interior of the bound photon must now interact at the point $\rho_0(\mathbf{r_2}) = 0$ with free photons in vacuum and give, by means of summation, the Casimir energy of the bound photon. For this purpose we proceed as in the famous work of Casimir [8] and carry out the summation over the quantum number n_0 using the Euler-MacLaurin sum formula:

$$E_{Cas}(r) = \hbar c \left(\sum^{\infty}_{n0=0} F(n_0) - \int^{\infty}_{0} F(n_0)\, dn_0\right) \tag{43}$$
$$= \hbar c [-(1/2) F(0) - (1/12) F'(0) + (1/720) F'''(0)],$$

where according to (41)

$$F(x) = [-(k_2^2 - k_1^2) x^2 + 2 k_1 k_2 x^2 \cot(k_1 r x) \cot(k_2 r x) \tag{44}$$
$$- (4/r + 2h)(k_1 x \cot(k_1 r x) + k_2 x \cot(k_2 r x)) + 6/r^2 + 6h/r]^{1/2}$$

$$\text{with} \quad x = n_0, \; k_1 = \pi / r_1, k_2 = \pi / r_2 \tag{45}$$

holds.

Eq. (44) gives

$$F(0) = [\, 2 / r^2 - 4\, h / r\; - 8 / r^2\; + 6 / r^2 + 6\, h / r]^{1/2} = (2\, h / r)^{1/2} \qquad (46)$$
$$= [(2 / r^2)(\tan(\varepsilon\, r)(\varepsilon^2 r^2 - 2) + 2\varepsilon\, r) / (\varepsilon\, r - \tan(\varepsilon\, r))]^{1/2}.$$

Formation of derivatives $F'(x)$, $F'''(x)$ (see **Appendix**) and examination of the limit $x \to 0$ yield

$$F'(0) = 0 \qquad (47)$$

and

$$F'''(0) = 0. \qquad (48)$$

Hence, we obtain according to (43) with (46)-(48) the Casimir energy for a single bound photon:

$$E_{Cas}\,(r) = -(1 / 2\,)\, \hbar\, c\, [(2 / r^2)(\tan(\varepsilon\, r)(\varepsilon^2 r^2 - 2) + 2\varepsilon\, r) / (\varepsilon\, r - \tan(\varepsilon\, r))]^{1/2}. \qquad (49)$$

7. The total energy of a single bound photon

Substituting the Casimir energy (49) in the total energy equation (41) we obtain

$$E^2 = m_0^2\, c^4\; + \hbar^2 c^2\, l\,(l + 1) / r^2 \qquad (50)$$
$$+ \hbar^2 c^2\, (1 / 2\, r^2)\, [\, (\tan(\varepsilon\, r)(\varepsilon^2 r^2 - 2) + 2\varepsilon\, r) / (\varepsilon\, r - \tan(\varepsilon\, r))].$$

In order to determine the total energy of a single bound photon, we must calculate mean values in Eq. (50) with the help of the probability density:

$$\langle E^2\rangle = m_0^2\, c^4\; + \hbar^2 c^2\, l\,(l + 1)\, \langle 1 / r^2\rangle \qquad (51)$$
$$+ \hbar^2 c^2\, (1 / 2)\, \langle [(1 / r^2)\, (\tan(\varepsilon\, r)(\varepsilon^2 r^2 - 2) + 2\varepsilon\, r) / (\varepsilon\, r - \tan(\varepsilon\, r))]\rangle\, .$$

The mean values in Eq. (51) can be determined by the integrals

$$<1 / r^2> = c_1^2\ (3/4\ \pi)\ \int_0^{2\pi}\int_0^{\pi}\int_0^{4.49341} r^2 \sin(\delta)(1 / r^2)\cos^2(\delta)$$ (52)

$$[\sin(\varepsilon r) / (\varepsilon r)^2 - \cos(\varepsilon r) / (\varepsilon r)]^2\ d\varphi\, d\delta\, d\, r$$

$$= 0{,}19314 / d^{\,2}$$

and

$$<[(1 / r^2)\ (\tan(\varepsilon r)(\varepsilon^2 r^2 - 2) + 2\varepsilon r) / (\varepsilon r - \tan(\varepsilon r))]>$$ (53)

$$= c_1^2\ (3/4\ \pi)\ \int_0^{2\pi}\int_0^{\pi}\int_0^{4.49341} r^2 \sin(\delta)\ (1 / r^2)$$

$$(\tan(\varepsilon r)(\varepsilon^2 r^2 - 2) + 2\varepsilon r) / (\varepsilon r - \tan(\varepsilon r))\cos^2(\delta))$$

$$[\sin(\varepsilon r) / (\varepsilon r)^2 - \cos(\varepsilon r) / (\varepsilon r)]^2\ d\varphi\, d\delta\, d\, r$$

$$= 0{,}096569 / d^{\,2}\ .$$

Substituting (52) and (53) in the Eq. (51) yields

$$<E^2> = m_0^2\, c^4 + 0{,}38628\ \hbar^2 c^2 / d^{\,2} + 0{,}0482845\ \hbar^2 c^2 / d^{\,2}.$$ (54)

Reorganizing Eq. (54) we obtain

$$(E + m_0 c^2\ [1 + 0{,}38628 / x^2]^{1/2})\ \cdot$$ (55)

$$(E - m_0 c^2\ [1 + 0{,}38628 / x^2]^{1/2})$$

$$= (\ m_0 c^2\ 0{,}21974 / x\ )^{2}$$

with $x = m_0 c^2 d / (\hbar\, c).$ (56)

Eq. (55) is quadratic in E . But it can be split into two independent energy equations:

$$E_1 = m_0 c^2 [1 + 0{,}38628 / x^2]^{1/2} - m_0 c^2\ 0{,}21974 / x$$ (57)

and

$$E_2 = - m_0 c^2 [1 + 0{,}38628 / x^2]^{1/2} + m_0 c^2\ 0{,}21974 / x\ ,$$ (58)

with $E_1 + E_2 = 0$.

At the beginning of this work we assumed without any explanation that free and bound photons are the simplest case for the solution of the relativistic quantum hydrodynamic fundamental equations. However, it turns out by the energy equations (57) and (58), that the single bound photon is a binding state of a particle and its antiparticle. The particles have according to (57) the effective potential (Fig. 2)

$$E_{1eff}(x) = E_1 / m_0 c^2 = [1 + 0{,}38628 / x^2]^{1/2} - 0{,}21974 / x \,. \tag{59}$$

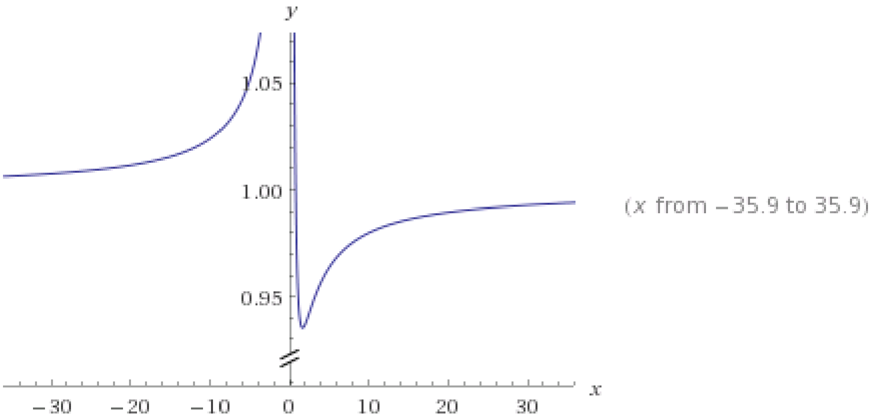

Figure 2: y = $E_1 / m_0 c^2 = [1 + 0{,}38628 / x^2]^{1/2} - 0{,}21974 / x$

Hence, centrifugal and Casimir forces keep each other in balance. The equilibrium state is given by the lowest point of the potential well:

$$d\,([1 + 0{,}38628 / x^2]^{1/2} - 0{,}21974 / x) / dx = \tag{60}$$
$$(0{,}21974\, x - 0{,}38628 / [1 + 0{,}38628 / x^2]^{1/2}) / x^3 = 0.$$

Solving Eq. (60) , we obtain $x_{Min} = 1{,}64436$, i.e. $d_{Min} = 1{,}64436\, \hbar c / (m_0 c^2)$. Accordingly, the total energy of a single particle in equilibrium is given by

$$E_1 / m_0 c^2 = [1 + 0{,}38628 / 1{,}64436^{\,2}]^{1/2} - 0{,}21974 / 1{,}64436 \tag{61}$$
$$= 1 + 0{,}06905 - 0{,}13363 = 0{,}93542 \,,$$

where are

$1 \cdot m_0 c^2$	the naked, unmeasurable mass energy,
$0{,}06905\ m_0 c^2$	the kinetic energy,
$-\,0{,}13363\ m_0 c^2$	the Casimir energy,
$0{,}93542 \cdot m_0 c^2$	the resulting, effective mass energy.

However, it cannot be specified that antiparticles have corresponding stability conditions like particles. A single antiparticle has according to (58) the effective potential

$$E_{2eff}(x) = E_2 / m_0 c^2 = -[1 + 0{,}38628 / x^2]^{1/2} + 0{,}21974 / x\,. \qquad (62)$$

Based on the effective potential (62) it is not possible to state any equilibrium, because no potential well, but a potential barrier exists, i.e. a single antiparticle, looking for stability, breaks apart as soon as it is created. Only the bound state of a particle and its antiparticle fulfills the stability condition $E_1 + E_2 = 0$ and has energetic neutrality towards the vacuum. But the bound state of a particle and its antiparticle cannot be calculated using the reduced fundamental equations (6)-(8). For this purpose, the full force of the fundamental equations of the RQH (1)-(3) is required; but this is a task beyond the scope of the present paper.

8. Summary and conclusions

In this study we started with the systematic solution of the fundamental equations of the RQH and solved them in the simplest case for free and bound photons. The resulting solutions are spherical harmonics and spherical Bessel functions. In order to calculate the Casimir energy, we put the density distribution of free photons in the density distribution of bound photons and derived thereby the total energy equation for free and bound photons. For a single bound photon we calculated the Casimir energy using the Euler-MacLaurin sum formula. The occurrence of the Casimir energy in the quantum hydrodynamic picture of elementary particles is inevitable, because free photons are enclosed in the interior of elementary particles like in a box with infinite potential walls and on the border to vacuum, always takes place pressure compensation with vacuum photons.

The question of the stability of a single bound photon is solved by centrifugal and Casimir forces keeping each other in balance. The fact that the total energy equation can be split into two independent energy equations is interpreted as the effect that the single bound photon is a binding state of a particle and its antiparticle. In the case of antiparticles, centrifugal and Casimir forces responsible for stability act in reverse direction, so that antiparticles break apart as soon as they are created. From this it follows that the antiparticle persists only in a bound state with the particle keeping energetic neutrality towards the vacuum. But whether the binding state of a particle and its antiparticle is quantum hydrodynamically possible or not, can only be decided if the fundamental equations of the RQH (1)-(3) have been entirely solved.

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Appendix

The Derivatives ***F′(x)*** **and** ***F‴(x)***

$$\frac{\partial}{\partial x}\left(\sqrt{\left(-\left(\left(\mathrm{k1}^2+\mathrm{k2}^2\right)x^2\right)+2\,\mathrm{k1}\,\mathrm{k2}\,x^2\cot(\mathrm{k1}\,r\,x)\cot(\mathrm{k2}\,r\,x)-\left(\frac{4}{r}+2\,h\right)(\mathrm{k1}\,x\cot(\mathrm{k1}\,r\,x)+\mathrm{k2}\,x\cot(\mathrm{k2}\,r\,x))+\frac{6}{r^2}+\frac{6\,h}{r}\right)}\right)=$$

$$\left(-\left(2\,h+\frac{4}{r}\right)\left(-\mathrm{k1}^2\,r\,x\csc^2(\mathrm{k1}\,r\,x)+\mathrm{k1}\cot(\mathrm{k1}\,r\,x)-\mathrm{k2}^2\,r\,x\csc^2(\mathrm{k2}\,r\,x)+\mathrm{k2}\cot(\mathrm{k2}\,r\,x)\right)-2\,x\left(\mathrm{k1}^2+\mathrm{k2}^2\right)-2\,\mathrm{k1}^2\,\mathrm{k2}\,r\,x^2\csc^2(\mathrm{k1}\,r\,x)\cot(\mathrm{k2}\,r\,x)-2\,\mathrm{k1}\,\mathrm{k2}^2\,r\,x^2\cot(\mathrm{k1}\,r\,x)\csc^2(\mathrm{k2}\,r\,x)+4\,\mathrm{k1}\,\mathrm{k2}\,x\cot(\mathrm{k1}\,r\,x)\cot(\mathrm{k2}\,r\,x)\right)\Big/$$

$$\left(2\sqrt{\left(-\left(2\,h+\frac{4}{r}\right)(\mathrm{k1}\,x\cot(\mathrm{k1}\,r\,x)+\mathrm{k2}\,x\cot(\mathrm{k2}\,r\,x))+\frac{6\,h}{r}+x^2\left(-\left(\mathrm{k1}^2+\mathrm{k2}^2\right)\right)+2\,\mathrm{k1}\,\mathrm{k2}\,x^2\cot(\mathrm{k1}\,r\,x)\cot(\mathrm{k2}\,r\,x)+\frac{6}{r^2}\right)}\right)$$

$$
\begin{aligned}
&\frac{\partial^3}{\partial x^3}\Bigg(\sqrt{\Big(-\big((\mathrm{k1}^2+\mathrm{k2}^2)\,x^2\big)+2\,\mathrm{k1}\,\mathrm{k2}\,x^2\cot(\mathrm{k1}\,r\,x)\cot(\mathrm{k2}\,r\,x)-}\\
&\qquad\sqrt{\left(\frac{4}{r}+2\,h\right)(\mathrm{k1}\,x\cot(\mathrm{k1}\,r\,x)+\mathrm{k2}\,x\cot(\mathrm{k2}\,r\,x))+\frac{6}{r^2}+\frac{6\,h}{r}\Big)}\Bigg)=\\
&\Big(3\big(-2\,\mathrm{k1}^2\,\mathrm{k2}\,r\,x^2\cot(\mathrm{k2}\,r\,x)\csc^2(\mathrm{k1}\,r\,x)-2\,\mathrm{k1}\,\mathrm{k2}^2\,r\,x^2\cot(\mathrm{k1}\,r\,x)\csc^2(\mathrm{k2}\,r\,x)-\\
&\qquad 2\,(\mathrm{k1}^2+\mathrm{k2}^2)\,x+4\,\mathrm{k1}\,\mathrm{k2}\,x\cot(\mathrm{k1}\,r\,x)\cot(\mathrm{k2}\,r\,x)-\\
&\qquad\left(2\,h+\frac{4}{r}\right)(-\mathrm{k1}^2\,r\,x\csc^2(\mathrm{k1}\,r\,x)-\mathrm{k2}^2\,r\,x\csc^2(\mathrm{k2}\,r\,x)+\\
&\qquad\quad \mathrm{k1}\cot(\mathrm{k1}\,r\,x)+\mathrm{k2}\cot(\mathrm{k2}\,r\,x))\big)^3\Big)\Big/\\
&\quad\Big(8\big(-(\mathrm{k1}^2+\mathrm{k2}^2)\,x^2+2\,\mathrm{k1}\,\mathrm{k2}\cot(\mathrm{k1}\,r\,x)\cot(\mathrm{k2}\,r\,x)\,x^2-\\
&\qquad\left(2\,h+\frac{4}{r}\right)(\mathrm{k1}\,x\cot(\mathrm{k1}\,r\,x)+\mathrm{k2}\,x\cot(\mathrm{k2}\,r\,x))+\frac{6\,h}{r}+\frac{6}{r^2}\big)^{5/2}\Big)-\\
&\quad\Big(3\big(4\,\mathrm{k2}\,r^2\,x^2\cot(\mathrm{k1}\,r\,x)\cot(\mathrm{k2}\,r\,x)\csc^2(\mathrm{k1}\,r\,x)\,\mathrm{k1}^3-\\
&\qquad 8\,\mathrm{k2}\,r\,x\cot(\mathrm{k2}\,r\,x)\csc^2(\mathrm{k1}\,r\,x)\,\mathrm{k1}^2+4\,\mathrm{k2}^2\,r^2\,x^2\csc^2(\mathrm{k1}\,r\,x)\\
&\qquad \csc^2(\mathrm{k2}\,r\,x)\,\mathrm{k1}^2-8\,\mathrm{k2}^2\,r\,x\cot(\mathrm{k1}\,r\,x)\csc^2(\mathrm{k2}\,r\,x)\,\mathrm{k1}+\\
&\qquad 4\,\mathrm{k2}^3\,r^2\,x^2\cot(\mathrm{k1}\,r\,x)\cot(\mathrm{k2}\,r\,x)\csc^2(\mathrm{k2}\,r\,x)\,\mathrm{k1}+\\
&\qquad 4\,\mathrm{k2}\cot(\mathrm{k1}\,r\,x)\cot(\mathrm{k2}\,r\,x)\,\mathrm{k1}-2\,(\mathrm{k1}^2+\mathrm{k2}^2)-\\
&\qquad\left(2\,h+\frac{4}{r}\right)(2\,r^2\,x\cot(\mathrm{k1}\,r\,x)\csc^2(\mathrm{k1}\,r\,x)\,\mathrm{k1}^3-2\,r\csc^2(\mathrm{k1}\,r\,x)\,\mathrm{k1}^2-\\
&\qquad\quad 2\,\mathrm{k2}^2\,r\csc^2(\mathrm{k2}\,r\,x)+2\,\mathrm{k2}^3\,r^2\,x\cot(\mathrm{k2}\,r\,x)\csc^2(\mathrm{k2}\,r\,x))\big)\\
&\qquad\big(-2\,\mathrm{k1}^2\,\mathrm{k2}\,r\,x^2\cot(\mathrm{k2}\,r\,x)\csc^2(\mathrm{k1}\,r\,x)-2\,\mathrm{k1}\,\mathrm{k2}^2\,r\,x^2\cot(\mathrm{k1}\,r\,x)\\
&\qquad\quad \csc^2(\mathrm{k2}\,r\,x)-2\,(\mathrm{k1}^2+\mathrm{k2}^2)\,x+4\,\mathrm{k1}\,\mathrm{k2}\,x\cot(\mathrm{k1}\,r\,x)\cot(\mathrm{k2}\,r\,x)-\\
&\qquad\left(2\,h+\frac{4}{r}\right)(-\mathrm{k1}^2\,r\,x\csc^2(\mathrm{k1}\,r\,x)-\mathrm{k2}^2\,r\,x\csc^2(\mathrm{k2}\,r\,x)+\\
&\qquad\quad \mathrm{k1}\cot(\mathrm{k1}\,r\,x)+\mathrm{k2}\cot(\mathrm{k2}\,r\,x))\big)\Big)\Big/\\
&\quad\Big(4\big(-(\mathrm{k1}^2+\mathrm{k2}^2)\,x^2+2\,\mathrm{k1}\,\mathrm{k2}\cot(\mathrm{k1}\,r\,x)\cot(\mathrm{k2}\,r\,x)\,x^2-\\
&\qquad\left(2\,h+\frac{4}{r}\right)(\mathrm{k1}\,x\cot(\mathrm{k1}\,r\,x)+\mathrm{k2}\,x\cot(\mathrm{k2}\,r\,x))+\frac{6\,h}{r}+\frac{6}{r^2}\big)^{3/2}\Big)+\\
&\quad\Big(-4\,\mathrm{k2}\,r^3\,x^2\cot(\mathrm{k2}\,r\,x)\csc^4(\mathrm{k1}\,r\,x)\,\mathrm{k1}^4-8\,\mathrm{k2}\,r^3\,x^2\cot^2(\mathrm{k1}\,r\,x)\\
&\qquad \cot(\mathrm{k2}\,r\,x)\csc^2(\mathrm{k1}\,r\,x)\,\mathrm{k1}^4+\\
&\qquad 24\,\mathrm{k2}\,r^2\,x\cot(\mathrm{k1}\,r\,x)\cot(\mathrm{k2}\,r\,x)\csc^2(\mathrm{k1}\,r\,x)\,\mathrm{k1}^3-\\
&\qquad 12\,\mathrm{k2}^2\,r^3\,x^2\cot(\mathrm{k1}\,r\,x)\csc^2(\mathrm{k1}\,r\,x)\csc^2(\mathrm{k2}\,r\,x)\,\mathrm{k1}^3-\\
&\qquad 12\,\mathrm{k2}\,r\cot(\mathrm{k2}\,r\,x)\csc^2(\mathrm{k1}\,r\,x)\,\mathrm{k1}^2+\\
&\qquad 24\,\mathrm{k2}^2\,r^2\,x\csc^2(\mathrm{k1}\,r\,x)\csc^2(\mathrm{k2}\,r\,x)\,\mathrm{k1}^2-\\
&\qquad 12\,\mathrm{k2}^3\,r^3\,x^2\cot(\mathrm{k2}\,r\,x)\csc^2(\mathrm{k1}\,r\,x)\csc^2(\mathrm{k2}\,r\,x)\,\mathrm{k1}^2-\\
&\qquad 4\,\mathrm{k2}^4\,r^3\,x^2\cot(\mathrm{k1}\,r\,x)\csc^4(\mathrm{k2}\,r\,x)\,\mathrm{k1}-\\
&\qquad 8\,\mathrm{k2}^4\,r^3\,x^2\cot(\mathrm{k1}\,r\,x)\cot^2(\mathrm{k2}\,r\,x)\csc^2(\mathrm{k2}\,r\,x)\,\mathrm{k1}-\\
&\qquad 12\,\mathrm{k2}^2\,r\cot(\mathrm{k1}\,r\,x)\csc^2(\mathrm{k2}\,r\,x)\,\mathrm{k1}+\\
&\qquad 24\,\mathrm{k2}^3\,r^2\,x\cot(\mathrm{k1}\,r\,x)\cot(\mathrm{k2}\,r\,x)\csc^2(\mathrm{k2}\,r\,x)\,\mathrm{k1}-\\
&\qquad\left(2\,h+\frac{4}{r}\right)(-2\,r^3\,x\csc^4(\mathrm{k1}\,r\,x)\,\mathrm{k1}^4-4\,r^3\,x\cot^2(\mathrm{k1}\,r\,x)\csc^2(\mathrm{k1}\,r\,x)\,\mathrm{k1}^4+\\
&\qquad\quad 6\,r^2\cot(\mathrm{k1}\,r\,x)\csc^2(\mathrm{k1}\,r\,x)\,\mathrm{k1}^3-2\,\mathrm{k2}^4\,r^3\,x\csc^4(\mathrm{k2}\,r\,x)-4\,\mathrm{k2}^4\,r^3\\
&\qquad\quad x\cot^2(\mathrm{k2}\,r\,x)\csc^2(\mathrm{k2}\,r\,x)+6\,\mathrm{k2}^3\,r^2\cot(\mathrm{k2}\,r\,x)\csc^2(\mathrm{k2}\,r\,x))\Big)\Big/\\
&\quad\Big(2\sqrt{\big(-(\mathrm{k1}^2+\mathrm{k2}^2)\,x^2+2\,\mathrm{k1}\,\mathrm{k2}\cot(\mathrm{k1}\,r\,x)\cot(\mathrm{k2}\,r\,x)\,x^2-}\\
&\qquad\sqrt{\left(2\,h+\frac{4}{r}\right)(\mathrm{k1}\,x\cot(\mathrm{k1}\,r\,x)+\mathrm{k2}\,x\cot(\mathrm{k2}\,r\,x))+\frac{6\,h}{r}+\frac{6}{r^2}\big)}\Big)
\end{aligned}
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