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Master equations corresponding to systems with strong fluctuating forces

Patrick Linker[1]

[1]Technische Universität Darmstadt, Schloßgartenstraße 7, 64289 Darmstadt, Germany

Abstract

This paper deals with the time evolution of probability distributions in systems that are driven by small fluctuating forces. With the assumption that the strength of the fluctuating forces are small in comparison to the deterministic forces, which are varying very slow with time, the quantum-mechanical propagation function and its change with time can be derived. This will lead to a system of integro-differential-equations, which contains propagation functions with different initial and final time points.

Keywords

Master equation, Quantum physics, Statistical mechanics

Introduction

In Statistical Mechanics, the density for any particle state is given by the diagonals of the quantum-mechanical density matrix, given by

$$\rho_{nn} = < n|e^{-\frac{H}{kT}}|n > \quad (1)$$

with the quantum-mechanical momentum state $|n>$, the Hamilton operator H , Boltzmann's constant k and the system temperature T. The propagation function (Feynman's path integral) for the Lagrangian $L(t)$ has the form [1]:

$$G(t_i, q_i, t_f, q_f) := \int d\Gamma e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dtL(t)} \quad . \quad (2)$$

Here, Γ is the integration space of all time points lying between t_i and t_f. Furthermore, the density matrix is equivalent to Feynman's path integral, if $t_f = t_i + \frac{\hbar}{kT}$ and the Wick rotation $t \rightarrow -it$ is performed. For transforming the path integral (2) into the density matrix depending on $|n>$, the path integral must be Fourier transformed over the coordinate difference $q_f - q_i$ like this:

$$\rho_{nn} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dq_f \, e^{-in(q_f - q_i)} G(-it_i, q_i, -it_f, q_f). \quad (3)$$

The time integral in (2) has the characteristic time $t_f - t_i = \tau = \frac{\hbar}{kT}$. Since the Lagrangian is given by

$$L(t) = T - V(t) - W(t) \quad (4)$$

with the time-independent kinetic energy T (T even depend on coordinates), the potential $V(t)$ varying slow with time and the stochastic potential $W(t)$, which varies at time differences, that are much smaller than τ. Due to the fast varying of $W(t)$, the path integral (2) must be split into time points between t_i and t_f and the integration space for this propagator must be extended into integrations about state variables, that occur between this spatial time. By time derivation of a path integral, it can be derived an expression of other path integral types (i.e. path integrals with other spatial time). The consequence is a system of integro-differential-equations that become larger, when the spatial time of the path integrals increase.

Theory

For the stochastic potential $W(t)$ are given the following simplifying assumptions:

- The maximum value of $|W(t)|$ is small in comparison with the value of $V(t)$.
- The variability in values of $W(t)$ can be assumed as small.

From the time t_i until t_f, there are given N significant changes in values of $W(t)$. There holds the relation:

$$\tau = \sum_{i=1}^{N} \tau_i. \qquad (5)$$

Here, τ_i is a spatial time, in which the value of $W(t)$ is changing. These spatial times have only small deviations on the mean value $\frac{\tau}{N}$, because of the assumptions of small variability. With (5) the time integral about $W(t)$ can be computed as follows:

$$\int_{t_i}^{t_f} dt\, W(t) = \sum_{i=1}^{N} \tau_i W\left(t_i + \sum_{j=1}^{i} \tau_j\right). \qquad (6)$$

The time derivative of (6) has the form:

$$\frac{d}{dt_i}\int_{t_i}^{t_f} dt\, W(t) = \sum_{i=1}^{N} \frac{\tau_i}{\tau_{i+1}}\left(W\left(t_i + \sum_{j=1}^{i+1}\tau_j\right) - W\left(t_i + \sum_{j=1}^{i}\tau_j\right)\right). \qquad (7)$$

Because $V(t)$ varies very slow in comparison to $W(t)$, the values of $V(t)$ remain constant for $t \in [t_i, t_i + \tau]$. By using the Leibnitz rule, following relationship holds:

$$\frac{d}{dt_i}\int_{t_i}^{t_f} dt\, (T + V(t)) = V(t_i + \tau) - V(t_i). \qquad (8)$$

Derivation of the Wick rotation of (2) by t_i and using (7) and (8), it follows:

$$\frac{\partial}{\partial t_i} G(-it_i, q_i, -it_f, q_f) = -\frac{1}{\hbar}\int d\Gamma \frac{d}{dt_i}\int_{t_i}^{t_f} dt\, (T + V(t) + W(t)) e^{\frac{-1}{\hbar}\int_{t_i}^{t_f} dtH(t)} =$$

$$= -\frac{1}{\hbar}\int d\Gamma (V(t_i+\tau) - V(t_i) + \sum_{i=1}^{N} \frac{\tau_i}{\tau_{i+1}}\left(W\left(t_i + \sum_{j=1}^{i+1}\tau_j\right) - W\left(t_i + \sum_{j=1}^{i}\tau_j\right)\right)) e^{\frac{-1}{\hbar}\int_{t_i}^{t_f} dtH(t)}. \qquad (9)$$

May be $[A]$ the Fourier transform of A over $q_f - q_i$, i.e. $[A] = \int_{-\infty}^{\infty} dq_f \, e^{-in(q_f - q_i)} A$ and $*$ the convolution product respective to the state variable n. The Fourier transform of (9) with the remaining space coordinate q_i yields:

$$\frac{\partial}{\partial t_i}[G]\left(-it_i, q_i, -it_f, q_f\right) =$$
$$= \frac{1}{\sqrt{2\pi}}[V]e^{inq_i} * [G]\left(-it_i, q_i, -it_f, q_f\right) - V[G]\left(-it_i, q_i, -it_f, q_f\right)$$
$$+ \sum_{i=1}^{N} \frac{\tau_i}{\sqrt{2\pi}\tau_{i+1}} \left([G]\left(-it_i, q_i, -i(t_i + \sum_{j=1}^{i+1}\tau_j), q(t_i + \sum_{j=1}^{i+1}\tau_j)\right) * [W]e^{inq_i} \right.$$
$$* [G]\left(-i(t_i + \sum_{j=1}^{i+1}\tau_j), q(t_i + \sum_{j=1}^{i+1}\tau_j), -it_f, q_f\right)$$
$$- [G]\left(-it_i, q_i, -i(t_i + \sum_{j=1}^{i}\tau_j), q(t_i + \sum_{j=1}^{i}\tau_j)\right) * [W]e^{inq_i}$$
$$\left. * [G]\left(-i(t_i + \sum_{j=1}^{i}\tau_j), q(t_i + \sum_{j=1}^{i}\tau_j), -it_f, q_f\right)\right). \quad (10)$$

With the abbreviation $[G]\left(-i(t_i + \sum_{j=1}^{i}\tau_j), q(t_i + \sum_{j=1}^{i}\tau_j), -it_f, q_f\right) \equiv [G](t, t_f)$ and the continuous limit $\sum_{i=1}^{N} \frac{\tau_i}{\tau_{i+1}} \mapsto \int_{t_i}^{t_f} dt \nu(t)$ (with the characteristic time-dependent jump rate of the stochastic potential $\nu(t)$) equation (10) can be written as:

$$\frac{\partial}{\partial t_i}[G]\left(t_i, t_f\right) =$$
$$= \frac{1}{\sqrt{2\pi}}[V]e^{inq_i} * [G]\left(t_i, t_f\right) - V[G]\left(t_i, t_f\right)$$
$$+ \frac{1}{\sqrt{2\pi}} \int_{t_i}^{t_f} dt \nu(t) \left([G]\left(t_i, t + \frac{1}{\nu(t)}\right) * [W]e^{inq_i} * [G]\left(t + \frac{1}{\nu(t)}, t_f\right) \right.$$
$$\left. - [G](t_i, t) * [W]e^{inq_i} * [G]\left(t, t_f\right)\right). \quad (11)$$

Equation (11) is a system of integro-differential-equations with the parameter $t_f > t_i$, which can be chosen arbitrary. The density matrix (3) is given by

$$\rho_{nn} = [G]\left(t_i, t_i + \frac{\hbar}{kT}\right) \quad (12)$$

For determination of (12), the functions $[G](t_i, t_i + t_+)$ with $0 < t_+ < \frac{\hbar}{kT}$ must be determined by solving the Master equation (11) with different choices of t_f. Because of Heisenberg's uncertainty principle

$$\Delta t \Delta E \geq \frac{\hbar}{2}, \quad (13)$$

an energy uncertainty ΔE become greater as Δt become smaller. Hence, the energy uncertainty increases, since $\nu(t)$ is increasing. Because of that uncertainty effects, the energy conservation for the Master equation (11) must not necessary hold.

Conclusions

With the principle of quantum propagators and the assumption of fast-varying potentials, non-markovian models for stochastical processes can be derived. The fast varying of potentials with time has the consequence of uncertainity in energy and so, the energy in (11) must not be necessarily conserved. Hence, very short laser pulses can generate energy uncertainities as well as quantum memory effects.

References

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