1. Questioning Leibniz's Principle of Sufficient Reason

(Chapter of the forthcoming book *The Discrete Reality of Physical Space*)

Abstract.-In the formal framework defined by the Principle of Directional Evolution of the Universe (towards its maximum entropy) and by the Theorem of the Inconsistent Infinity, this chapter proves the incompleteness of Leibniz's Principle of Sufficient Reason, which renders inconclusive Leibniz's critique of Newton's absolute space, and also makes inevitable the existence of first causes that cannot be explained in terms of other causes deduced from our present knowledge of the observable universe.

1.1 Two Leibniz's Principles

As is well known, after the publication of Newton's Principia [7] a famous epistolary debate took place between S. Clarke and G.W. Leibniz, the former defending Newton's absolute space and the latter denying it, and both cases considering its consequences on the very existence of God. Without going into the details of the debate (for which the reader may consult, for instance [5, 6, 3, 8, 2, 9]), Leibniz introduced in it two of his famous principles:

> **Principle of Sufficient Reason**:*There must be a sufficient reason for things to be one way and not another.*

According to Leibniz, the above principle would turn metaphysics into a deductive science.

> **Principle of Identity of the Indiscernibles**: *There cannot exist two different things that are indistinguishable from each other.*

Since (according to Leibniz) two different and indiscernible things cannot exist, and since in Newton's absolute space things could be located in several different and indiscernible ways, Leibniz argued that God would have had to choose one of these indistinguishable ways, without any reason to choose one of them to the detriment of the others, which for Leibniz is not proper to God. Therefore, absolute space cannot exist. Clarke argued in the opposite sense, not defending the possibility of contingent events, but making God's will intervene as the only reason why things were one way and not another.

The Principle of Identity of Indiscernibles is no longer accepted by contemporary science, but the Principle of Sufficient Reason (PSR) is at least partially accepted. So here we respond to this principle. The following answer could be given without the advantages of the knowledge accumulated from Leibniz's time to ours: it would differ very little from the one given in this article. Since the important thing is the PSR answer, whether or not Leibniz is present to answer it, this advantageous knowledge will be used here, including that which has been published but not yet sufficiently accepted in contemporary science, as is the case with the inconsistency of the actual infinity. A key inconsistency for the future of mathematics and especially for the future of physics and the logical understanding of the physical world.

1.2 The formal setting of the discussion

The PSR will be discussed here within a formal scenario whose two fundamental pillars will be the Principle of Directional Evolution of the Universe and the inconsistency of the actual infinity, the latter not as a principle but as a formally demonstrable theorem. From both of them we can deduce the rest of the formal elements that constitute the formal scenario in which Leibniz's famous principle will be contested. All of these formal elements are briefly demonstrated in the appendices to this chapter. As usual, I invite the reader to jump to the proof of the inconsistency of the actual infinity (Theorem 2 of the Inconsistent Infinity). I could have chosen any of the more than forty demonstrations contained in [4, pdf]. The one included in the first appendix is a very simplified variant of one of those demonstrations, which was also one of the first I was able to develop. It contains less than 300 words that can be read in less than 3 minutes, and if the reader does not find it a correct argument, he/she can stop reading the rest of the article right there.

The formal elements to be used in the PSR discussion, which are formally proved in the final Appendix to this chapter, are the following:

1. **Principle of Directional Evolution**: *The universe evolves always in the same direction of increasing its entropy.*

2. **Theorem of the Inconsistent Infinity**: *The actual infinity subsumed in the Axiom of Infinity is inconsistent.*
3. **Theorem of the Consistent Universe**: *The universe evolves under the control of a unique set of invariant and consistent physical laws.*
4. **Theorem of Identicality**: *All particles of the same type have the same properties and behave the same way under the same conditions.*
5. **Theorem of the Formal Dependence**: *No concept defines itself; no statement proves itself; no physical object is the cause of itself; and no cause is the cause of itself.*
6. **Theorem of the First Element**: *A consistent sequence in which there is a last element and each element has an immediate predecessor is a complete totality only if it has a first arbitrary element without predecessors.*
7. **Corollary of the First Cause**: *No physical object or process can be fully explained without a first cause that cannot be explained in terms of other causes.*

The PSR could also be stated in terms of logical causes: *there is always a logical cause which explains why things are as they are and not otherwise.* In the following, both forms, Leibniz's original and the latter, will be used interchangeably.

1.3 The Principle of Sufficient Reason

The infinite regress of arguments was already considered by Aristotle [1, I.3]:

> We, on the other hand, hold that not every form of knowledge is demonstrative, but that the knowledge of ultimate principles is indemonstrable. The necessity of this fact is obvious, for if one must needs know the antecedent principles and those on which the demonstration rests, and if in this process we at last reach ultimates, these ultimates must necessarily be indemonstrable.

This, of course, is why we have always needed, and will always need, axioms and inductive laws in the foundations of all sciences. In our case, this need is demonstrated by the Theorem of Formal Dependence, a consequence of the Principle of Directional Evolution of the Universe, which is inductively based on overwhelming empirical evidence.

In fact, no one expects the shards of broken glass to spontaneously reassemble into the exact original shape of the broken glass. This typical example is often used to illustrate the Second Law of Thermodynamics, which is immediately incorporated into the Principle of the Directional Evolution of the Universe. This principle, moreover, permits the formal deduction of results that extend the Aristotelian infinite regress of arguments to definitions, and causes of objects and natural phenomena. As indicated elsewhere in this book, the case of first causes certainly goes far beyond the content of this book. And the reader can easily see why.

As noted above, contemporary science still allows the PSR to be applied, with the exception of contingent events. But both in contemporary science and in Leibniz's arguments, applying the PSR implies applying the principle of infinite regress of causes. And this is the key fact that was initially absent in Leibniz's arguments and is still absent in contemporary physics, although in Leibniz's case he came to admit a first cause of why things are as they are and not otherwise:

Because the universe had to be the best of universes.

Which is obviously an ARBITRARY cause. Indeed, since potential infinity is the only consistent infinite, it would be impossible for humans to fully explain any object or natural phenomenon without recourse to a first arbitrary cause that cannot be explained in terms of other causes (Corollary 2 of the First Cause). In the case of God (if there is one), if he is a consistent being, he could not do this either, just as he could not count the last natural number if, as we may suppose, even God cannot count a non-existent number. So the Corollary 2 of the First Cause applies to Him as well, which, as we shall see, has significant consequences for the origin of the universe itself.

One of these consequences is that Leibniz's theological objection to Newton's absolute space is inapplicable: one cannot always give a sufficient reason (a cause) for things being as they are and not otherwise, because in the end we will fall into an inevitable infinite regress of causes, from which it is only possible to get out by means of a first cause that cannot be explained in terms of other causes. Not even God could do that. But Leibniz, perhaps aware of this difficulty, proposed a first cause (the universe had to be the best of the universes) which, as we have just pointed out, is as arbitrary a cause as any other that cannot be explained by other causes. Actually, Leibniz would be quite satisfied with the Corollary 2 of the First Cause: he would only have to think of the universe as the physical object that it is. Since no object can be the cause of itself, every physical object, including the universe, must have a first cause external to the object itself.

What if the universe were eternal? Well, in that case its duration would be infinite, it would have, for example, an infinite number of seconds or any other arbitrary unit of time. That is, it would have an inconsistent duration (Theorem 2). What if the universe had arisen from a fluctuation of nothing? Well, then nothing would not be nothing, but something with the ability to fluctuate, and we would have to apply the Corollary 2 of the First Cause to that something with the ability to fluctuate. What if the present universe were a stage in a cyclic succession of universes being continuously created and destroyed? Well, in this case the number of cycles could only be finite (Theorem 2) and therefore there would be a first universe (Theorem 6 of the First Element) in the cyclic succession of universes to which the Corollary 2 of the First Cause could be applied.

The majority of contemporary physicists, all of them strictly relativistic, deny the existence of physical absolute space. According to them it is only a fiction useful to describe the evolution of the (always) relative positions of natural objects (see the final appendix to Chapter 19). At the same time, and according to these same physicists, space expands, bends, vibrates and transmits its own vibrations. And one wonders how something that does not exist can expand, deform, vibrate and transmit vibrations?

Since 2015, we have empirical evidence of gravitational waves, and this changes everything. The vibrations of space are no longer a theoretical matter, they are real, they interact with material objects (with the arms of the interferometers that detect them), and the interactions can be detected and measured. Therefore, space is real; it is a real and unique physical object; it is the same for all material objects; it is absolute; it is Newtonian. The chapters 19 and 20 discuss the physical consequences of absolute space and the nature of its substance, respectively.

Appendix A.-The actual and the potential infinity

Consider the list of the natural numbers in their natural order of precedence: 1, 2, 3, ... The Hypothesis of the Actual Infinity considers that list exists as a complete totality (i.e. one in which all elements that must be in it, are in it) even though there is no last natural number completing the list. The ellipsis ... in 1, 2, 3, ... stands for ALL natural numbers. For ALL. In contrast, the Hypothesis of the Potential Infinity defends that such a list is only endless: it is always possible to consider a number greater than any previously considered number in the list, but the complete list of ALL natural numbers does not exist. Summarized in Aristotelian terms: for the Hypothesis of the Actual Infinity the incompletable can exist as completed; for the Hypothesis of the Potential Infinity the incompletable cannot be completed, just because ir is incompletable.

Definition 1 (of Infinite Set) *A set is said infinite if it can be put into a one to one correspondence with one of its proper subsets.*

Axiom 1 (of Infinity) $\exists A : (\emptyset \in A \wedge \forall a \in A \ (a \cup \{a\} \in A))$

that reads: there exists a set A such that $\emptyset$ (the empty set) belongs to A and for every element a in A, the element $a \cup \{a\}$ also belongs to A. Although it is not explicitly declared the type of infinity involved in the set A, it can be easily proved that it is the actual infinity:

Theorem 1 (of the Actual Infinity) *The infinity in the Axiom of Infinity can only be the actual infinity.*

Proof.-Since potentially infinite sets do not exist as complete totalities, only two subsets with the same number of elements of the same potentially infinite set could be put into a one to one correspondence, and then Dedekind Definition 1 is not satisfied, because we would have a one to one correspondence between two proper subsets of a potentially infinite set, in the place of a one to one correspondence between a set and one of its proper subsets. In consequence, the infinity involved in the Axiom 1 of Infinity can only be the actual infinity. □

Appendix B.-The Formal Scenario

Theorem 2 (of the Inconsistent Infinity) *The actual infinity subsumed in the Axiom of Infinity is inconsistent.*

Proof.-The interval of rational numbers $\mathbb{Q}_{01} = (0, 1)$ is denumerable and densely ordered. So, it can be put in one-to-one correspondence f with the set $\mathbb{N}$ of natural numbers in their natural order of precedence, and $\mathbb{Q}_{01}$ can be rewritten as $\{f(1), f(2), f(3), \dots\}$. Let now x be any element of $\mathbb{Q}_{01}$ and let it be compared with the successive elements $f(1)$, $f(2)$, $f(3)\dots$ so that x is redefined as $f(i)$ if, and only if, $f(i)$ is less than the current value of x. Since all elements $f(1), f(2), f(3), \dots$ of $\mathbb{Q}_{01}$ are rational numbers which, according to the Hypothesis of the Actual Infinity subsumed in the Axiom of Infinity, exists as a complete totality, x can be successively compared with all of them:

$$\forall n \in N : \quad x \text{ is compared with } f(n) \tag{1}$$

Once compared with all elements of $\mathbb{Q}_{01}$, x is the smallest rational of that set. Indeed, if once compared with all elements of $\mathbb{Q}_{01}$, x were not the least rational of $\mathbb{Q}_{01}$, there would exist at least one element $f(v)$ in $\mathbb{Q}_{01}$ such that $f(v) < x$. But this is impossible according to (1). Therefore, it was compared with $f(v)$ and redefined as $f(v)$. So, it is impossible that $f(v) < x$. But it is also immediate to prove that: Once compared with all elements of $\mathbb{Q}_{01}$, x is not the smallest rational of that set. In effect, once compared with all elements of $\mathbb{Q}_{01}$, each element of the infinite set $\{x/2, x/3, x/4 \dots\}$ is an element of $\mathbb{Q}_{01}$ less than x. According to the above contradiction, the Axiom of Infinity, which legitimizes the existence of $\mathbb{Q}_{01}$ as a complete totality, is inconsistent. □

Principle 1 (of Directional Evolution) *The universe evolves always in the same direction of increasing its entropy.*

Definition 2 (of Consistent Set of Laws) *A set of physical laws is consistent if under the same conditions it always leads to the same results.*

Theorem 3 (of the Consistent Universe) *The universe evolves under the control of a unique set of invariant and consistent physical laws.*

Proof.-If the physical laws governing the evolution of the universe were not an invariable set of consistent laws, changes would occur with equal frequency in all directions, and no progress would be possible in any of them. Thus, directional evolution would not be possible, which violates the principle 1 of directional evolution. Thus, the universe evolves under the control of a unique set of invariant and consistent physical laws. □

Corollary 1 (of the Physical Laws) *The laws of physics apply to all regions of space and time.*

Proof.-It is an immediate consequence of Theorem 3. □

Theorem 4 (of Formal Dependence) *No concept defines itself; no statement proves itself; no physical object is the cause of itself; and no cause is the cause of itself.*

Proof.-If concepts could define themselves, their corresponding definitions would be inaccessible to our formal and experimental sciences, so the Theorem 3 would not hold, and the Principle 1 would be impossible. If propositions could prove themselves, then everything could be proved, and then sets of consistent laws would be impossible, which violates the Theorem 3. If physical objects and causes could be the cause of themselves, they would be outside Theorem 3. □

Theorem 5 (of the Incompletable Regress) *In every recursive sequence S of proofs, definitions or causes in which there is a last element to be proved (defined, caused) and each element has an immediate predecessor that proves (defines or causes) it, is incompletable.*

Proof.-If every element of the sequence has an immediate predecessor, then there is not a first element of the sequence, because this first element would have no immediate predecessor. Therefore, the sequence, if consistent, can only be potentially infinite and then incompletable (Theorem 2). □

Theorem 6 (of the First Element) *A consistent sequence in which there is a last element and each element has an immediate predecessor is a complete totality only if it has a first arbitrary element without predecessors.*

Proof.-Let $S = \ldots S_{3^*}, S_{2^*}, S_{1^*}$ be any sequence with a last element S_{1^*} and in which each element S_{n^*} has an immediate predecessor $S_{(n+1)^*}$, where n^* read last but $n-1$. If S is consistent it can only be finite or potentially infinite (Corollary 2). Therefore, if S is a complete totality it can only have a finite number n of elements. In these conditions, and taking into account that each element S_i of S has exactly one predecessor more than its immediate predecessor S_{i+1}, the element S_{1^*} has $n-1$, predecessors; the element S_{2^*} has $n-2$ predecessors; the element S_{3^*} has $n-3$ predecessors; etc. Consequently, the smallest number of predecessors that an element of S can have is $n-(n-1)=1$. That element will be $S_{(n-1)^*}$ whose predecessor can only be a first element S_{n^*} of the sequence that has no predecessor. So, S has a first element S_{n^*} with zero predecessors. □

Corollary 2 (of the First Cause) *No physical object or phenomenon can be fully explained without a first cause that cannot be explained in terms of other causes.*

Proof.-It is an immediate consequence of Theorems 4 and 6. □

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