the general science journal
July 2023

ACHILLES, THE TORTOISE AND THE SPEED OF LIGHT

Antonio Leon
Retired Professor. Independent researcher in the foundations of science.
Links to other related works.

Abstract.-A photon replaces Achilles and the Tortoise in a variant of the famous Zeno's paradox discussed in this paper. The discussion takes place in the infinitist scenario of the spacetime continuum. In this theoretical scenario, and thanks to the dense order of real numbers, the paradox becomes first a dichotomy and then a contradiction, of which there are only two coherent solutions: either the speed of light is NOT finite, or the Hypothesis of the Actual Infinity is NOT consistent. An inconsistency that would change almost everything in modern mathematics and then in the formal foundation of modern physics. The article ends by pointing to a new finite and discrete scenario for space and time in which this and many other contradictions and paradoxes dissolve immediately. But it also warns of the extraordinary difficulties that will be involved in exchanging the hegemonic infinitist paradigm of our days for an alternative based on the finitist discreteness of space and time.

Keywords: actual infinity, discrete solution of Zeno's Contradiction, dense order, discrete space and time, finiteness of the speed of light, foundation of physics, power of the continuum, spacetime continuum, Zeno's Contradiction, Zeno's Dichotomy, Zeno's Paradoxes.

1. Introduction

On Zeno's paradoxes there is, for obvious reasons, an abundant literature, including the alleged solutions proposed in different areas of contemporary mathematics such as transfinite arithmetic, topology, measure theory, and internal set theory [9, 10, 34, 11, 13, 12, 26, 25], even solutions proposed within classical mechanics and quantum mechanics [20, 21]. Some of those solutions have been contested, and none of them convincingly explains where Zeno's original arguments fail. In any case, the new discussions have given rise to new problems as challenging as the paradoxes themselves. [27, 1, 29, 30, 14, 31].

The most famous of these paradoxes is undoubtedly that of Achilles and the Tortoise. In this article I discuss a formalized variant of this paradox in which instead of a tortoise and an athlete only a photon intervenes. The fastest object in the universe moving along a straight line on which certain points have been defined, which I will call Z-points in memory of Zeno of Elea (≈495 BC-≈430 BC). The scenario of the new theoretical discussion, in which a photon moves through the Z-points, will be the contemporary infinitist mathematics built on the Hypothesis of the Actual Infinity subsumed in the Axiom of Infinity. In that scenario, absolutely hegemonic in contemporary mathematics, the infinite sets exist as COMPLETE TOTALITIES. This will be the case of the densely ordered set of Z-points that our photon has to travel (Appendix A, page 4 of this paper summarizes the differences between the actual infinity and the potential infinity, and also recalls the dense order that characterizes some infinite sets, as the set of real numbers).

Once the Z-points are defined and the photon is fired, it will be proved that in this variant of Zeno's paradox, and due to the dense order of Z-points, the paradox becomes a dichotomy: Zeno Dichotomy. And being zero the only non-negative real number that is less than all real numbers greater than zero, the dichotomy becomes a contradiction: Zeno Contradiction. Since the only possible causes of Zeno Contradiction are the finiteness of the speed of light and the Hypothesis of the Actual Infinity, it seems reasonable to assume that, taking into account the overwhelming empirical confirmation of the finiteness of the speed of light, the Hypothesis of the Actual Infinity subsumed in the Axiom of Infinity must be the formal cause of Zeno Contradiction. If so, we would have an inconsistent axiom in the infinitist foundation of modern mathematics (through set theory), and then of modern physics.

In the ordinary language of the primary and secondary physics literature, most physicists ignore this infinitist formalism and describe the physical world in a reasonably correct way, although using expressions that are not compatible with the foundational infinitism of their theories (some examples are given in Section 6). Practically all the strangeness derived from the actual infinity (as, for example, that a 10^{-32}mm line segment has the same number of points as the whole three-dimensional universe) is absent from the ordinary language with which physicists describe physical phenomena. For that reason they do not find it necessary to revise the infinitist formalism that underlies their theories, in practice they act as if those foundations were something else. But they are not something else, they are what they are, and it is convenient to revise them in order to ensure that physics is built on consistent fundaments, which may not be the case if, for example, the formal proof offered in this article is well constructed and the Axiom of Infinity is inconsistent.

2. Z-points

Consider any straight line in the spacetime continuum. For example the X_o-axis of an inertial reference frame RF_o. Let us define as Z-points all points of the open interval $(0, 1)$ of the X_o-axis of RF_o, in which any metric is defined, for example, the Euclidean metric in the SI, so that the considered interval corresponds, for example, to 1 meter: $(0m, 1m)$. Although, as usual, we will not indicate neither the unit of measurement nor the coordinates that are not involved in the discussion. The

set of Z-points $(0,1)$ is an infinite non-numerable set of points containing exactly $2^{\aleph_o}$ points, the same number of points as the whole X_o-axis, or the entire observable three-dimensional universe (Dimension Problem proved by G. Cantor [2, 7, 32, 33, 8, 5, 3, 4]). Although $(0,1)$ is a complete totality, there does not exist a first Z-point following point 0, nor a last Z-point preceding point 1. Moreover, if p stands for any natural number, or even for the first infinite cardinal $\aleph_o$, there do not exist in $(0,1)$ the firsts p points following 0. Nor do the last p points preceding 1 exist. Between point 0 and ANY point within $(0,1)$ there are always the same infinite number of points, just $2^{\aleph_o}$ points. And the same applies to point 1. A very appropriate set, then, to discuss on Zeno's paradoxes.

3. Zeno's Dichotomy

Suppose that at the point $x=-1$ of the X_o-axis of RF_o, a photon γ (considered as a point particle[1]) is emitted along the X_o-axis, in the direction of its increasing values. Being c the speed of light, at the instant $1/c$ the photon γ will be exactly on the point 0, and has not yet begun to travel the Z-points of $(0,1)$. Let us represent this state of the photon by $\gamma(0)$, where 0 indicates that γ has traveled exactly 0 Z-points. At any instant after $1/c$ the photon γ will already be inside $(0,1)$, and taking into account that any subinterval of $(0,1)$ has the same number of points as the whole interval $(0,1)$, exactly $2^{\aleph_o}$ points, we can conclude that at any instant after $1/c$ the photon γ has already traveled $2^{\aleph_o}$ Z-points. Thus, from the point of view of the number of traveled Z-points, the photon γ can only have two states: the state $\gamma(0)$ at which it has traveled 0 Z-points, and the state $\gamma(2^{\aleph_o})$ at which it has already traveled $2^{\aleph_o}$ points. No intermediate state is possible for the reason given at the end of the previous section: for any cardinal $p<2^{\aleph_o}$, the first p points following the 0 point do not exist. Therefore, the infinitist dense order of real numbers turns Zeno's Paradox into Zeno's Dichotomy: the number of Z-points traveled by the photon γ can only be 0 or $2^{\aleph_o}$. Note that this is not an indeterminacy, but an impossibility: the set of positions of γ within the open real interval $(0,1)$ for which the photon γ has traversed a number of Z-points other than $2^{\aleph_o}$ is the empty set. This is infinitist mathematics!

4. Zeno's Contradiction

Let τ now be any time interval greater than zero, and suppose the photon γ takes a time τ to change from the state $\gamma(0)$ to the state $\gamma(2^{\aleph_o})$. At any instant t in the interval τ such that $0<t<\tau$, the photon γ is already within $(0,1)$; therefore it has already traveled $2^{\aleph_o}$ Z-points, and has already reached the state $\gamma(2^{\aleph_o})$. Therefore, in changing from $\gamma(0)$ to $\gamma(2^{\aleph_o})$, the photon γ takes less time than any time interval τ greater than zero. But there is only one non-negative (γ travels into the future) real number less than all real numbers greater than zero: just zero. We have to conclude, therefore, that γ takes zero time to change from the state $\gamma(0)$ in which it has not traveled any Z-point, to the state $\gamma(2^{\aleph_o})$ in which it has already traveled $2^{\aleph_o}$ Z-points. Now, since at point 0 the photon is in the state $\gamma(0)$, to change to the state $\gamma(2^{\aleph_o})$ the photon must necessarily change its position. And it has to do it in zero time, which is not possible with its finite velocity c. Therefore, the photon γ changes its position during a zero time, an does not change its position during a zero time. The fact that zero is the only non-negative real number less than any real number greater than zero causes Zeno's Dichotomy to become Zeno's Contradiction.

5. The Axiom of Infinity and Zeno's Contradiction

Obviously, no contradiction can be admitted in a scientific theory, otherwise, anything could be proved within it. And to eliminate the contradiction in a theory it is necessary to analyze the cause of that contradiction, which in a correct argument cannot be other than the inconsistency of at least one of its foundational hypotheses. In the case of Zeno Contradiction, if the above argument is correct, there are only two hypotheses that could be the reason of the contradiction: either the speed of light is NOT finite; or the Hypothesis of the Actual Infinity is NOT consistent. The finiteness of the speed of light has an overwhelming empirical confirmation, and moreover it can be confirmed experimentally in an instant, never better said. On the contrary, the inconsistency of Hypothesis of the Actual Infinity would be confirmed by the more than forty proofs that the interested reader can find in [19, Link]. It seems then reasonable to propose the inconsistency of the Actual Infinity Hypothesis (subsumed in the Axiom of Infinity) as the cause of Zeno's Contradiction.

6. A discrete solution to Zeno's Contradiction

Zeno's Contradiction, and many other contradictions and paradoxes, dissolve immediately in a discrete scenario with indivisible minimal units for space (qseats) and time (qbeats). Discreteness already empirically confirmed and universally accepted in the cases of matter, energy and all kinds of charges, electrical and non-electrical. [18]. Indeed, in our case, the photon will be in qseat 0 in a certain qbeat, and in the next contiguous qbeat it will be in the next contiguous qseat in the direction of its motion, moving at the maximum speed of one qseat per qbeat. Changing the infinitist dense order by the finitist contiguity (adjacency), Zeno's Contradiction disappears.

But the solution is not so immediate because first we have to solve the old problem of change (particularized for changes of position), a problem that has been posed for 26 centuries and has not yet been resolved. In fact, it has been completely forgotten by physics, the science of change (!), the science of the regular succession of events [24, p. 98]. It can be proved that the problem of change has no solution in the spacetime continuum [16, Link], but it can be solved in a discrete universe functioning in a similar way to CALMs (Cellular Automata Like Models [19, Link]).

The proposed discrete scenario may seem novel and ex-

[1] The following theoretical argument can be adapted immediately for the center of mass of any object moving with any finite uniform velocity.

travagant, but it is actually very old: the early pre-Socratics already considered points as indivisible units with a non-zero extension [22]. It is a pity that soon after they discovered the irrational numbers and with them the impossibility of non-zero extension points. And in the ninth and tenth centuries, the Arab philosophical and theological current known as Kalām developed a discrete cosmology that denied irrational numbers, and in which, matter, space and time were constituted by minimal indivisible units greater than zero; and motion had to occur in leaps and bounds separated by discrete units of time, less time units the faster the movement is [15, p. 62-68].

But, discrete models for space and time have at least two major drawbacks. The first is our sensory perception of the physical world as a continuous world. Although it is a deceptive perception because the human brain takes a certain amount of time ($\approx$13 milliseconds [28]) to process each image (the base of the well known α, β, γ and δ movements, and of ϕ-phenomenon). Therefore, it can process only a finite number of images per unit of time, although that time is so short that we perceive the discontinuous succession of those images as if it were a continuous succession, just as we perceive the succession of frames in a movie. So, this first drawback is not really a drawback but a suggestive warning that if the physical world were discrete, discontinuous, with sufficiently short qbeats (Planck time?) we would perceive it as continuous.

The second drawback is much more difficult to overcome. It is the existence of (absolutely) hegemonic streams of thought that leave few options for dissent. The mathematical infinitist stream of our days is one of them. It has become, moreover, the mathematical basis on which physics is formally founded. A symptom that this infinitist foundation of physics is not the most appropriate is the fact that this infinitist formalism is never consistently reflected in the ordinary language of physicists. Indeed, in many issues the ordinary language of physicists is incompatible with the infinitist formalism that underlies physical theories. The origin of this serious incoherence between what physicists say and what they should say according to the mathematical foundations of their theories, is precisely one of those foundational items: the Hypothesis of the Actual Infinity with its sequel of densely ordered continuums. Let me at this point quote the words of the renowned philosopher of physics T. Maudlin [23, p. xiv]:

> Unfortunately, physics has become infected with very low standards of clarity and precision on foundational questions, and physicists have become accustomed (and even encouraged) to just "shut up and calculate," to consciously refrain from asking for a clear understanding of the ontological import of their theories.

This is an untenable situation that physicists should consider because, among other things, it perpetuates the foundational infinitism of physical theories in exchange for a schizophrenic use of ordinary language that, on the one hand, describes physical phenomena reasonably well and, on the other, is incompatible with that foundational infinitism. Indeed, in the primary and secondary literature of physics we can find thousands of expressions such as:

... points are small rectangles of infinitesimal extension ...

... particles small enough to be considered as points ...

... it propagates through the adjacent points ...

... through each of the contiguous points ...

... in the immediately next instant ...

... at each successive point ...

... it is not only possible but absolutely certain that points will gradually coalesce.

... the more spacetime points there are in a region ...

... to an infinitesimal spacetime point ...

etc. etc.

which are incompatible with the infinitist foundation of physical theories, because the spacetime points have a null extent [18], are densely ordered, they cannot be contiguous, and the same number of them exist in any region, linear or not, of the spacetime continuum.

But, on the other hand, the infinitist formalism that underlies, for example, the relativistic spacetime continuum is so widely accepted that reputable philosophers of physics can write [23, p. xiii]:

> ... all there is to the physical world, at a fundamental level, is accounted for by the theory of space-time and the theory of matter.

But if the Hypothesis of the Actual Infinity is inconsistent, the spacetime continuum (our present and unique theory of space-time) will also be inconsistent, so that both space and time will have to be "discretized." And the same would have to be done with those theories that, as special relativity, have been built on the basis of the spacetime continuum. In this case all the inertial relativistic deformations of space and time would disappear. Deformations which, on the other hand, could be only apparent, as apparent as the refractive deformations which, however empirically confirmed Snell's Law may be, are not real. On the relativistic inertial deformations see the Appendixes B and C of this paper.

In any case, I will end this work with an urgent question: What would have to happen, and how could one collaborate, for contemporary science to begin to consider the possibility that the Actual Infinity Hypothesis is inconsistent? An inconsistency that would be anything but irrelevant to science, philosophy, and even theology.

AMM R.S.O.

Appendixes

A. The actual and the potential infinity

Consider the list of the natural numbers in their natural order of precedence: 1, 2, 3, ... The Hypothesis of the Actual Infinity considers that list exists as a complete totality, even though there is no last natural number completing the list. The ellipsis . . . in 1, 2, 3, . . . stands for ALL natural numbers. For ALL. In contrast, the Hypothesis of the Potential Infinity defends that such a list is only endless: it is always possible to consider a number greater than any previously considered number in the list, but the complete list of ALL natural numbers does not exist. Summarized in Aristotelian terms: for the Hypothesis of the Actual Infinity the incompletable can exist as completed; for the Hypothesis of the Potential Infinity the incompletable cannot be completed. It can be immediately proved, by means of Dedekind's definition of the infinite sets [6, p. 115], that the infinity in the Axiom of Infinity can only be the actual infinity [19, Link (p. 25)]. An axiom from which it is almost immediate to deduce the dense order of all continuums as the spacetime continuum: the existence of $2^{\aleph_o}$ different points BETWEEN ANY TWO OF ITS POINTS .

B. Real or apparent?

(Text taken from a relativistic mini-paper by the author)

Opinions on the real or apparent nature of the FitzGerald-Lorentz contraction are divided. Proponents of its apparent nature should decide whether this apparent nature also applies to the inertial time dilation and the inertial phase difference in synchronization. If they think not, they would have to explain why some relativistic deformations are apparent and others real, being all them deduced from the same Lorentz Transformation (LT). If they think it does, they will have to admit that LT converts between reality and a deformed reality that is not real but apparent, as apparent at refractive deformations (recall that Snell Law is also "sufficiently confirmed by empirical facts" but the corresponding deformations are not real). Proponents of the real nature of the FitzGerald-Lorentz contraction will have to answer at least five questions: How is it possible for all physical objects to have simultaneously as many different sizes and shapes as there are different relative velocity vectors at which they can be observed? How is it possible that, for a given velocity vector, this deformation is the same for all objects, regardless of their chemical composition and their crystallographic structure? How can they deform without any physical interaction, without any physical cause, except being observed at different relative velocities? How can be real a property of an object that does not depend on (nor is related to) its actual properties but on the way in which it is observed? Are there as many simultaneous and overlapping realities as there are different relative velocities at which their objects can be observed?

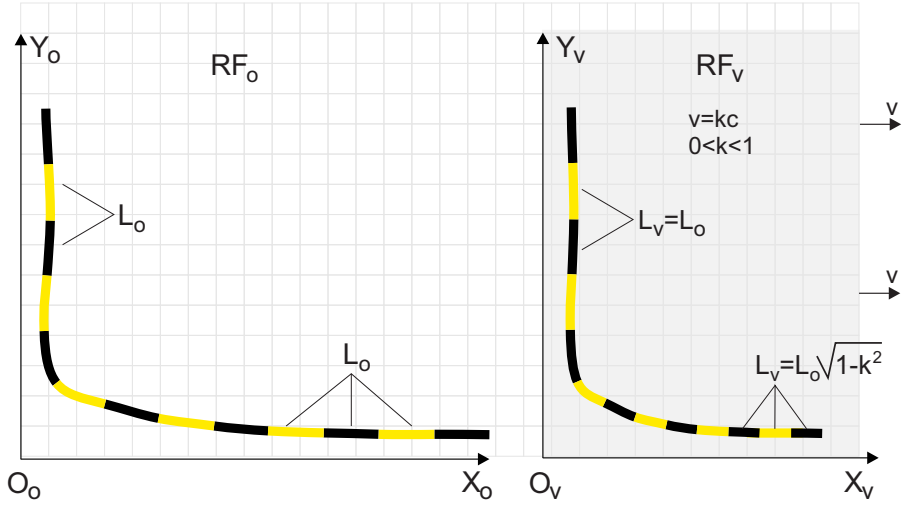

Figure 1 – The elastic cord at rest on the plane X_oY_o of its proper reference frame RF_o (left), and from the reference frame RF_v (right).

C The elastic cord

(Text and figure taken from [17, p. 69 Link]

In the reference frame RF_o, an elastic and flexible cord rests free of forces on the plane X_oY_o. The elastic cord is scaled with yellow and black marks of the same length L_o, some of which are parallel to the X_o axis, and some parallel to the Y_o axis. Since the cord is at rest and no force acts on it, all yellow and black marks have the same length, and this is in fact what is observed in RF_o (Figure 1, left). Things are quite different when this elastic cord is observed from the reference frame RF_v that, as always in this book, coincides with RF_o at a certain instant, and from whose perspective RF_o moves according to our conventions: with a uniform velocity v parallel to the increasing X_v. As Figure 1 (right) illustrates, and according to the Lorentz Transformation, all marks parallel to the X_v axis are observed with a length L_v such that $L_v = \gamma^{-1}L_o$,while all marks parallel to its Y_v axis are observed with the same length L_o, being obviously $L_v < L_o$. The observers in RF_v will, therefore, observe an elastic cord free of forces with some marks more stretched than others, which is impossible for an elastic cord free of forces. In consequence,

for all observers, except those in $RF_{_o}$ and those moving parallel to $Z_{_o}$, the elastic cord is observed with some parts more stretched than others, without any force acting on it. Obviously, this goes against the laws of mechanics governing the behaviour of elastic materials. The conclusion can only be that the FitzGerald-Lorentz contraction is apparent, as apparent as the deformation of a rod partially submerged in water.

Bibliography and References

[1] J. S. Alper and M. Bridger. Mathematics, Models and Zeno's Paradoxes. *Synthese*, 110:143 – 166, 1997.

[2] G. Arrigo and B. D'Amore. Lo veo pero no lo creo. Obstáculos epistemológicos y didácticos en el proceso de comprensión de un teorema de Cantor que involucra al infinito actual. *Educación matemática*, 11(1):5–24, 1999.

[3] Georg Cantor. *Fundamentos para una teoría general de conjuntos*. Crítica, Barcelona, 2005.

[4] Brian Clegg. *A Brief History of Infinity. The Quest to Think the Unthinkable*. Constable and Robinson Ltd, London, 2003.

[5] Josep W. Dauben. *Georg Cantor. His mathematics and Philosophy of the Infinite*. Princeton University Press, Princeton, N. J., 1990.

[6] Richard Dedekind. *Qué son y para qué sirven los números (Was sind Und was sollen die Zahlen (1888))*. Alianza, Madrid, 1998. Definición de conjunto infinito p. 115.

[7] Jean-Paul Delahaye. El carácter paradójico del Infinito. *Investigación y Ciencia (Scientifc American)*, Temas: Ideas del infinito(23):36 – 44, 2001.

[8] I. Grattan-Guiness. *Del cálculo a la teoría de conjuntos, 1630-1910*. Alianza Editorial, Madrid, 1984.

[9] Adolf Grünbaum. Modern Science and Refutation of the Paradoxes of Zeno. *The Scientific Monthly*, LXXXI:234–239, 1955.

[10] Adolf Grünbaum. *Modern Science and Zeno's Paradoxes*. George Allen And Unwin Ltd, London, 1967.

[11] Adolf Grünbaum. Modern Science and Refutation of the Paradoxes of Zeno. In Wesley C. Salmon, editor, *Zeno's Paradoxes*, pages 164 – 175. Hackett Publishing Company, Inc, Indianapolis/Cambridge, 2001.

[12] Adolf Grünbaum. Modern Science and Zeno's Paradoxes of Motion. In Wesley C. Salmon, editor, *Zeno's Paradoxes*, pages 200 – 250. Hackett Publishing Company, Inc, Indianapolis/Cambridge, 2001.

[13] Adolf Grünbaum. Zeno's Metrical Paradox of Extension. In Wesley C. Salmon, editor, *Zeno's Paradoxes*, pages 176 – 199. Hackett Publishing Company, Inc, Indianapolis/Cambridge, 2001.

[14] Nick Huggett. Zeno's Paradoxes. In Edward N. Zalta (ed.), editor, *The Stanford Encyclopaedia of Philosophy (Summer 2004 Edition)*. Stanford University, 2004.

[15] Max Jammer. *Concepts of Space: The History of Theories of Space In Physics*. Dover Publlcations Inc, New York, 1993.

[16] Antonio León. Physics and the problem of change. *Preprint version*, 2020.

[17] Antonio León. *Apparent relativity*. Self edition in KDP, 2022.

[18] Antonio León. *Towards a discrete cosmology*. Indpendently published in KDP, 2023.

[19] Antonio León. *Infinity put to the test.* Self edition in KDP. Printed at amazon.com. Free pdf link, 2023 (2021).

[20] Peter Lynds. Time and Classical and Quantum Mechanics: Indeterminacy vs. Discontinuity. *Foundations of Physics Letters*, 16:343 – 355, 2003.

[21] Peter Lynds. Zeno's Paradoxes: A Timely Solution. *philsci-archives*, pages 1 – 9, 3003. http://philsci-archives.pitt.edu/archive/00001197.

[22] Mariano Martínez. *Espacio y tiempo en la Escuela de Salamanca*, chapter Prólogo, pages 11 – 16. Ediciones Universidad de Salamanca, Salamanca, 2004.

[23] Tim Maudlin. *Philosophy of Pysics. Space and Time*. Princeton University Press, New Jersey, 2015.

[24] James Clerk Maxwell. *Materia y movimiento*. Crítica, Barcelona, 2006.

[25] William I. McLaughlin. Una resolución de las paradojas de Zenón. *Investigación y Ciencia (Scientifc American)*, (220):62 – 68, Enero 1995.

[26] William I. McLaughlin and Silvia L. Miller. An Epistemological Use of non-Standard Analysis to Answer Zeno's Objections Against Motion. *Synthese*, 92(3):371 – 384, September 1992.

[27] Alba Papa-Grimaldi. Why mathematical solutions of Zeno's paradoxes miss the point: Zeno's one and many relation and Parmenides prohibition. *The Revew of Metaphysics*, 50:299–314, December 1996.

[28] Mary Potter, Carl Hagmann, and Emily McCourt. Banana or fruit? Detection and recognition across categorical levels in RSVP. *Psychonomic Bulletin and Review*, 22(2):578–585, 2015.

[29] Jon Pérez Laraudogoitia. Supertasks. In E. N. Zaltax, editor, *The Stanford Encyclopaedia of Philosophy*. Standford University, URL = http://plato.stanford.edu, 2001.

[30] Wesley C. Salmon. Introduction. In Wesley C. Salmon, editor, *Zeno's Paradoxes*, pages 5 – 44. Hackett Publishing Company, Inc, Indianapolis, Cambridge, 2001.

[31] Z. K. Silagadze. Zeno meets modern science. *Philsci-archieve*, pages 1–40, June 2005.

[32] Hourya Sinaceur. ¿Existen los números infinitos? *Mundo Científico (La Recherche)*, Extra: El Universo de los números:24 – 31, 2001.

[33] David Foster Wallace. *Everything and more. Acompact history of infinity*. Orion Books Ltd., London, 2005.

[34] Mark Zangari. Zeno, Zero and Indeterminate Forms: Instants in the Logic of Motion. *Australasian Journal of Philosophy*, 72:187–204, 1994.