Euclidation of the Physical Space

8. The Euclidation Process

In a timed inertial frame S a force-free particle b exhibits a uniform straight motion

$$\vec{R}(t') = \vec{R}_0 + \vec{v}t',$$

with $\vec{R}_0 = \vec{R}(0)$. Time here is read locally by timers contiguous to the particle's current positions. Given $\vec{R}_0$ and $\vec{v}$ the observer O, or any other S observer, can determine the position of the particle at any instant *t*. The calculation process does not require light's signals; it is instead a purely rational geometric process, which uses the rules of Euclidean trigonometry. To see what difference, observation through light's signals makes, we consider an *S*-observer B that sends, when contiguous to the particle b, a light's message informing another *S*-observer, say O, of his own position and time reading $(\vec{R}, t')$. At receiving the message, O reads on his own clock the instant $t'' = t' + R(t')/c$, but the particle is no more at its earlier position *B*; it is instead at a new position

$$\vec{R}(t'') = \vec{R}_0 + \vec{v}t''.$$

Note that when O receive the message, every *S* observer read at his own clock the same instant t''.

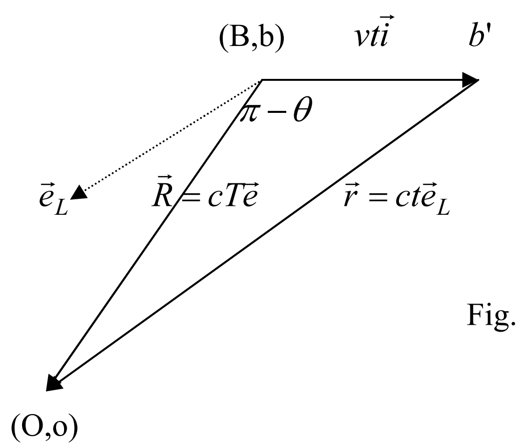


Fig.1

We aim here to endow the scaling transformations with a similar picture to that we have just presented, however, with path of the light's signal when reaching the observer $O \in S$ is represented by a straight segment joining the *present* position of b and O. If this can be done, then the observer O can tell at any instant of actual observation (i.e. when he receives the signal) the present position of b, through employing the rules of Euclidean trigonometry. Also, graphical representation of sum of displacements will have the same status that enjoys in mechanics, however, with one revision that will be explained in the sequel. The active view is appropriate here because one frame only is involved in the observation process. To explain the meaning of the latter statements we revert to our source of light b which is moving in S at velocity $\vec{v} = v\vec{\imath}$. In the frame S, light emitted from (b when at B) at $T = 0$ reaches O (and o) at an instant $t = r/c$, where $r = \Gamma(v, \theta)R$. i.e., the pulse emanating from b takes a duration *t* equal to the duration taken by a pulse emanating from a stationary source B times $\Gamma(v,\theta)$. During this period t, the source b moves to a new location $b' \in S$, with $\overrightarrow{Bb'} = vt\vec{\imath}$.

From aesthetical and pragmatic points of view, it is desirable to visualize the familiar displacement law in Euclidean geometry

$$\overrightarrow{BO} = \overrightarrow{Bb'} + \overrightarrow{b'O},$$

holds, with the lengths assigned to the sides of the triangle $Bb'O$ are those given by the scaling theory, i.e.

$$\left|\overrightarrow{BO}\right| = R \equiv cT, \qquad \left|\overrightarrow{Bb'}\right| = vt, \qquad \left|\overrightarrow{b'O}\right| = r = ct.$$

Equivalently, the law of displacements' addition, should be valid for

(8.1) $$cT\vec{e} = vt\vec{\imath} + ct\vec{e}_L.$$

where $T = R/c$, by definition of the geometric time.

This target is fortunately attainable, but only on the expense of the time and distance assigned classically in S to the displacement of the source and to the length and duration of a pictorial light trip $(b' \to O)$.

To implement our target, we calculate t using the figure above, or using the relation (8.1), to find

(8.2) $$t = \frac{\beta cos\theta + \sqrt{1-\beta^2 sin^2\theta}}{1-\beta^2} T,$$

with $T = R/c$. Introducing the *Euclidean factor*

(8.3) $$G(\beta,\theta) \equiv \frac{-\beta cos\theta + \sqrt{1-\beta^2 sin^2\theta}}{1-\beta^2} = \frac{\Gamma(\beta,\pi-\theta)}{\sqrt{1-\beta^2}},$$

we write the equation (8.2) in the form

(8.4) $$t = G(\beta,\pi-\theta)T = \frac{\Gamma(\beta,\theta)}{\sqrt{1-\beta^2}}\frac{R}{c}.$$

This however is at odd with the scaling transformations which we have strong conviction of its validity, since it proved compatible with experiment.

The alternative approach is to write the scaling transformations $t = \Gamma(\beta,\theta)T$ in terms of the Euclidean factor to obtain the scaling transformation in the form

(8.5) $$t = G(\beta,\pi-\theta)\left(\sqrt{1-\beta^2}\,\frac{R}{c}\right).$$

The last form shows that the rules of Euclidean geometry are applicable to the depiction above, provided the geometric distance R (or the geometric time $T = R/c$) is contracted by the factor $\gamma^{-1} = \sqrt{1-\beta^2}$. Although true, the last request is not a convenient choice since the frame S is the site of geometric measurements, and it is not handy to scale geometric distance (and time) for each moving object separately. Instead, it is better to expand the proper time t (in fig.1) associated with the moving object by the factor γ. Writing the latter equation in the form

(8.6a) $$t' \equiv \frac{t}{\sqrt{1-\beta^2}} = G(\beta,\pi-\theta)\,T,$$

yields the desired result. Multiplying both sides of the last equation by c we get

(8.6b) $$r' \equiv \frac{r}{\sqrt{1-\beta^2}} = G(\beta,\pi-\theta)\,R,$$

where we have set

(8.7) $$r' \equiv ct'.$$

Thus the familiar rules of Euclidian geometry are applicable to the triangle $Bb'O$ in (fig.1) only after rescaling its two sides attached to the source in its present position b' through multiplication by γ, or simply after dividing *t* (and *r*) by $\sqrt{1-\beta^2}$. On making the replacements

(8.8) $$t \longrightarrow t' \equiv \frac{t}{\sqrt{1-\beta^2}}, \quad r \longrightarrow r' \equiv \frac{r}{\sqrt{1-\beta^2}}$$

(in fig.1), the triangle $Bb'O$ with sides

(8.9) $$|\overrightarrow{BO}| = cT, \qquad |\overrightarrow{Bb'}| = vt' = \frac{vt}{\sqrt{1-\beta^2}}, \qquad |\overrightarrow{b'O}| = ct' = \frac{ct}{\sqrt{1-\beta^2}},$$

conforms to the rules of Euclidean geometry, and the scaling transformations still hold. It follows therefore that *when observations of stationary sources in a frame s which are moving at velocity $\vec{v}$ in S are made from S, then the rules of Euclidean geometry are applicable to the characters of the displacements of these sources and the associated light's trips, provided these characters are expanded by the Lorentz factor* $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

The *Euclidized form* (8.6) of the scaling transformation enable us to use the rules of the Euclidean trigonometry, and visualize the displacement of the source from its initial position to its final one (during the period of the light's trip) in the traditional way. The concepts of universal space still dominates, and the pulse emitted from *b* follows the universal path $(B\ when\ b) \rightarrow (O\ (and\ o))$ in *S*. If $t = 0$ is read at *B* when light is emitted, then *t* given by (8.5), or (8.6a), is what read by the clocks at O, b', and B, when light is received. However, and if *t* is calculated erroneously from (8.2), i.e. without expanding *t* by γ, then the value obtained corresponds to $t' = t/\sqrt{1-\beta^2}$, which is greater than the proper time *t* which is read on the clocks at *O*, b', and *B*, and it must be contracted by $\sqrt{1-\beta^2}$ of its value.

The Change of a Ray's Direction Between Two Frames

In the frame *S*, the light's trip takes place along the segment *BO,* while no ray exist along $b'O$ (figure 1). In the frame *s*, the pulse follows the segment *bo*, while no ray exists along $B'o$ (figure 2). In spite of the last two statements, we shall show that the angle $\delta = \angle(\overrightarrow{OB}, \overrightarrow{Ob'})$ ($\delta' = \angle(\overrightarrow{ob}, \overrightarrow{oB'})$) can be identified in some specific sense with the "so called" aberration angle in *S* (in *s*). On replacing *t* by t', the triangle OBb' in figure 1, becomes Euclidean. Applying the law of sines in this triangle yields:

$$\frac{sin\,\delta}{Bb'} = \frac{sin\,(\pi-\theta)}{b'O} \Longrightarrow \frac{sin\,\delta}{vt'} = \frac{sin\ \theta}{ct'},$$

or

(8.10a) $$sin\,\delta = \frac{v}{c}\,sin\,\theta$$

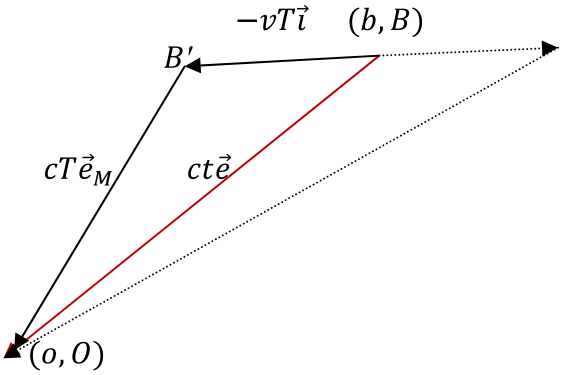
Similar calculation applied to the triangle obB' yield

(8.10b) $$sin\,\delta' = \frac{v}{c} sin\ \theta$$

where $\delta' = \angle(\overrightarrow{ob}, \overrightarrow{oB'})$.

The identification of the last expressions with that of the aberration angle requires special attention. Indeed, and when measured within *S* (within *s*), the negative direction of the ray is found to make the same angle θ with the direction of the *X*-axis (*x*-axis). However, if this direction is measured in the other frame the result will be different. In the S frame for example, the direction of the ray within S is determined by the angle between OB and OX, with B and O are two S observers. The measurement from S of the direction of the ray $(b \rightarrow o)$ when light is considered to emanate from b requires two S observers located at $b^{'}$ and O, which yields an angle $\theta + \delta$. In spite of this, light propagates in S along $\overrightarrow{BO}$, and there is no light coming along $\overrightarrow{b'O}$. Thus δ as observed S is the angle between $\overrightarrow{OB}$ and a fixed direction $\overrightarrow{Ob'}$, which is identical to $\overrightarrow{ob}$ in s when light is received. Still there is an important issue to mention. The velocity of the source has no effect at all on the aberration angle, and the velocity appearing in (8.10) is the relative velocity between S and s, *which happened to be the velocity of the source in this case, because b is stationary in s.*

Fig.2



To elaborate on the issue of direction's change of a ray, we mention first that the values of the directional angles of the received ray have the same values when measured exclusively within each frame. The observation from S of the direction of the ray $\overrightarrow{bo}$ in s, requires two S observers residing simultaneously at its beginning and end. More general, and if $p \in s$ is point on $\overrightarrow{bo}$ through which light passes in its course to $o \in s$ (at $O \in S$), then p occupies when light reaches o a position $p' \in S$. *The S observers judge that the direction* $\overrightarrow{bo}$ *specified in s corresponds to the direction* $\overrightarrow{p'O} \in S$. When the source b is stationary in s, as it is the case here, the point $p \in s$ can be chosen $b \in s$. The latter basic issue, which we call "*direction's change of a ray between two frames*", will be discussed in detail when aberration is studied in a subsequent section.

Euclidation of the physical space for the frame s can be carried out similarly. The result is that *when observations of stationary sources in the frame S are made from s, then the rules of Euclidean geometry are applicable to the characters of the displacements of these sources and the associated light's trips, provided these characters are expanded by the Lorentz factor* $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. The necessary replacements for Euclidation in this case are

(8.11) $$T \longrightarrow T^{'} \equiv \frac{T}{\sqrt{1-\beta^2}}, \quad R \longrightarrow R^{'} \equiv \frac{R}{\sqrt{1-\beta^2}}.$$

Moreover, and when observed from s, there corresponds to the direction of the ray $\overrightarrow{BO}$ in S, the direction $\overrightarrow{B'o}$ in s.

9. Combination of the Velocity of Light with the Velocity of its Emitter.
The Euclidean Factor

We start by reviewing some relevant properties of the Euclidean Factor. We have found in the previous section that the rules of the Euclidean trigonometry applied to the triangle OBb' yields

(9.1) $$r = G(\beta, \pi - \theta)R = \frac{\Gamma(\beta, \theta)}{\sqrt{1-\beta^2}}R,$$

with $= ct$, $R = cT$, and

(9.2) $$G(\beta, \theta) = \frac{-\beta cos\theta + \sqrt{1-\beta^2 sin^2\theta}}{1-\beta^2} = \frac{\Gamma(\beta, \pi - \theta)}{\sqrt{1-\beta^2}}$$

is the Euclidean factor. The Euclidean factor has the properties

(9.3a) $$G(0, \theta) = 1,$$

(9.3b) $$G(\beta, 0) = \frac{1}{1+\beta}, \quad G(\beta, \pi) = \frac{1}{1-\beta}, \quad G\left(\beta, \tfrac{1}{2}\pi\right) = \frac{1}{\sqrt{1-\beta^2}}$$

(9.3c) $$G(-\beta, \theta) = G(\beta, \pi - \theta) = \frac{\beta cos\theta + \sqrt{1-\beta^2 sin^2\theta}}{1-\beta^2} = \frac{\Gamma(\beta, \theta)}{\sqrt{1-\beta^2}}$$

(9.3d) $$G(\beta, \theta)G(-\beta, \theta) = \frac{1}{1-\beta^2}$$

(9.3d') $$G(\beta, \theta)G(\beta, \pi - \theta) = \frac{1}{1-\beta^2} = (G(\beta, \pi/2))^2$$

(9.3e) $$G^{-1}(\beta, \theta) = (1-\beta^2)G(-\beta, \theta)$$

Velocity Addition

Let's return to the functionally dependent dual constraints

(9.4a) $$cT\vec{e} = (c\vec{e}_L + v\vec{\imath})t,$$

(9.4b) $$ct\vec{e} = (c\vec{e}_M - v\vec{\imath})T,$$

in section 4, which express the Galilean law of velocity addition applied provisionally in *S* and *s* respectively to the velocity of a pulse and its supposed emitter. These formulae do not conform to the rules of Euclidean geometry unless *t* in the first and *T* in the second are scaled by the factor γ. On carrying out this scaling we obtain the formulae

(9.5a) $$cT\vec{e} = (c\vec{e}_L + v\vec{\imath})\frac{t}{\sqrt{1-\beta^2}},$$

(9.5b) $$ct\vec{e} = (c\vec{e}_M - v\vec{\imath})\frac{cT}{\sqrt{1-\beta^2}}.$$

Substituting for *t* from $t = \Gamma(\beta, \theta)T$ yields

(9.6a) $$c\vec{e} = (c\vec{e}_L + v\vec{\imath})\frac{\Gamma(\beta,\theta)}{\sqrt{1-\beta^2}} = G(-\beta, \theta)(c\vec{e}_L + v\vec{\imath})$$

(9.6b) $$c\vec{e} = (c\vec{e}_M - v\vec{\imath})\frac{\Gamma(-\beta, \theta)}{\sqrt{1-\beta^2}} = G(\beta, \theta)(c\vec{e}_M - v\vec{\imath})$$

Taking the cross product of both sides in each relation (9.b) by $\vec{e}$ yields

(9.7a) $$\vec{e} \times \vec{e}_L = -\vec{e} \times \beta\vec{\imath},$$

(9.7a) $$\vec{e} \times \vec{e}_M = \vec{e} \times \beta\vec{\imath},$$

which gives

(9.8) $$\sin\delta = \beta\sin\theta,$$

where $\delta = \angle(\vec{e}_L, \vec{e}) = \angle(\vec{e}, \vec{e}_M)$. Either relation (9.6) gives the *law of combination of the velocity of light signal, which is c, with the velocity of its emitter*. The resulting velocity is *c*, and the direction of the resulting pulse is tilted from that of the original one by the aberration angle δ. The relation (9.6a) ((9.6b)) specifies the composition of the velocity $v\vec{\imath}$ $(-v\vec{\imath})$ of a source and the velocity $c\vec{e}_L(c\vec{e}_M)$ of the pulse it emits. The resulting velocity $c\vec{e}$ is along a vector $\vec{e}$ in between the vectors $v\vec{\imath}$ and $\vec{e}_L$ $(\vec{e}_M)$ and makes an angle δ with the latter. In fact both relations (9.6) are the same, and can be merged in one relation

(9.9) $$c\vec{e} = G(-\beta,\theta)\left(c\vec{f} + v\vec{\imath}\right),$$

where $\vec{f}$ is a unit vector, *v* is the algebraic magnitude of the source's velocity, and $\beta = v/c$. The resultant velocity is in between the summed vectors and tilted from $\vec{f}$ by the aberration angle $\delta = \angle\left(\vec{f}, \vec{e}\right)$, where $\sin\delta = \beta\sin\theta$.

Each of the equivalent forms

(9.10a) $$c\vec{e} = G(-\beta,\theta)\left(c\vec{f} + v\vec{\imath}\right)$$

(9.10b) $$c\vec{f} = (1-\beta^2)G(\beta,\theta)c\vec{e} - v\vec{\imath},$$

can be used to sum the velocities of a pulse and its source, with the velocity of the source is $v\vec{\imath}$ in each relation.

Some special cases of addition of the velocities of a light signal and its emitter are studied here. In all cases the resulting velocity is *c* of course. We assume in all cases that $v > 0$.

-The value $\theta = \pi$, which corresponds to the same direction for the emitter's and the pulse's velocities, gives

(9.11a) $$c\vec{f} = (1-\beta^2)G(\beta,\pi)c\vec{e} - v\vec{\imath} = (1+\beta)c\vec{\imath} - v\vec{\imath} = c\vec{\imath},$$

(9.11b) $$\delta = 0.$$

-The value $\theta = 0$ corresponds to opposite directions for the velocities of the emitter and the pulse. It gives

(9.12a) $$c\vec{f} = (1-\beta^2)G(\beta,0)c\vec{e} - v\vec{\imath} = (1-\beta)c(-\vec{\imath}) - v\vec{\imath} = -c\vec{\imath},$$

(9.12b) $$\delta = 0.$$

-The value $\theta = \pi/2$, which corresponds to perpendicular directions of the emitter and the pulse, gives

(9.13a) $$c\vec{f} = (1-\beta^2)G(\beta,\pi/2)c\vec{e} - v\vec{\imath} = \sqrt{1-\beta^2}c\vec{e} - v\vec{\imath},$$

(9.13b) $$\delta = sin^{-1}\beta.$$

The last relation affirms also that

$$c = \left|c\vec{f}\right| = \left|\sqrt{1-\beta^2}c\vec{e} - v\vec{\imath}\right| = c.$$

The Scaling Transformations in Terms of $\boldsymbol{\delta}$

From (9.8) we obtain

(9.14) $$cos\delta = \sqrt{1-\beta^2 sin^2\theta},$$

(9.15) $$cos\theta = \sqrt{1-\beta^{-2}sin^2\delta}.$$

And the scaling factor assumes the form

(9.16) $$\Gamma(\beta,\theta(\delta)) = \Gamma'(\beta,\delta) = \frac{\sqrt{\beta^2 - sin^2\delta} + cos\delta}{\sqrt{1-\beta^2}}.$$

Or

(9.17) $$\Gamma'(\beta,\delta) = \frac{\sqrt{1-\beta^{-2}sin^2\delta} + \beta^{-1}cos\delta}{\sqrt{\beta^{-2}-1}}.$$

It may be interesting to investigate the correspondence

(9.18) $$\theta \leftrightarrow \delta, \quad \beta \leftrightarrow \beta^{-1}, \quad r \leftrightarrow ir \quad (or\ R \leftrightarrow iR),$$

in the scaling transformations associated with the change of representations

(9.19) $$(R,\theta,\emptyset) \rightarrow (R,\delta,\emptyset), \qquad (r,\theta,\emptyset) \rightarrow (ir,\delta,\emptyset).$$