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BORH'S DERIVATION OF BALMER-RYDBERG FORMULA THROUGH QUANTUM MECHANICS

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Abstract

According to classical electrodynamics the Rutherford's nuclear model of the hydrogen atom should be unstable. But the hydrogen atom is very stable with emission of radiation at discrete frequencies. The Balmer-Rydberg formula gives the wave number of the lines of the spectrum of the hydrogen atom in terms of the Rydberg constant *R* and quantum numbers *n* and *q*. Niels Bohr brilliantly stabilised the Rutherford's atom by invoking quantum mechanics and making two ad-hoc assumptions to derive the Balmer-Rydberg formula. The spectral lines range from the far infra-red to ultra-violet regions. A recapitulation of Bohr's derivation is given in this paper.

Keywords: Angular momentum, hydrogen spectrum, orbit, quantization, radiation, wavelength.

1. Introduction

1.1 Rutherford's Nuclear Model of the Hydrogen Atom

Lord Ernest Rutherford (1911) [1] proposed a nuclear theory of the atom consisting of a heavy positively charged central nucleus around which a cloud of negatively charged electrons revolve in circular orbits. The hydrogen atom is the simplest, consisting of one electron of charge $-e$ and mass m revolving in a circular orbit around a much heavier central nucleus of charge $+e$. This model, conceived on the basis of experimental results, has sufficed since, although with some difficulties regarding its stability and emitted radiation.

According to classical electrodynamics [2] the electron of the Rutherford's model, in being accelerated towards the nucleus of the atom, by the centripetal force, should:

(i) emit radiation over a continuous range of frequencies with power proportional to the square of its acceleration and

(ii) lose potential energy and gain kinetic energy as it spirals into the nucleus, leading to the collapse of the atom.

The second prediction is contradicted by observation as atoms are the most stable objects known in nature. The first effect is contradicted by experiments as a detailed study of the radiation from hydrogen gas, undertaken by Johann Balmer, as early as 1885 [3, 4], showed that the emitted radiation had discrete frequencies. The spectral lines of radiation from the hydrogen atom satisfy the Balmer-Rydberg formula;

$$\nu_{nq} = \frac{1}{\lambda_{nq}} = R\left(\frac{1}{n^2} - \frac{1}{q^2}\right) \tag{1}$$

where λ_{nq} is the wavelength, ν_{nq} the wave number, n and q are integers greater than 0 with $q > n$ and R is the Rydberg constant. This important formula was first obtained by Johann Balmer (1885), as a special case for $n = 2$, and then generalised by Johannes Rydberg (1888). For $n = 1$ and ($q = 2 - \infty$) we have the Lyman series in the far ultra-violet region; for $n = 2$ and ($q = 3 - \infty$) there is the Balmer (4 visible line) series and where $n = 3$ and ($q = 4 - \infty$) we get the Paschen series in the near infra-red region. Other series for n > 4 are in the far infra-red regions. R was found, by measurements, to be $1.097x10^7$ per metre [5]. Equation (1) gives the spectral series limit ($n \to \infty$) as $\nu_n = R/n^2$.

1.2 Bohr's two postulates

Niels Bohr (1885 – 1962), in a superb display of original thought, rescued the hydrogen atom from radiating and collapsing. He derived the Balmer-Rydberg formula, for the spectral lines of radiation from the hydrogen atom, by invoking the quantum theory and making two postulates [6, 7]. Bohr' postulates are as follows:

i. the electron, in the Rutherford's nuclear model of the hydrogen atom, can revolve, without radiation, round the nucleus in allowed, quantum or stable orbits for which the angular momentum L_n is quantized, such that:

$$L_n = \frac{nh}{2\pi} \tag{2}$$

where n, the quantum number, is an integer greater than zero, and h is the Plank constant.

ii. The second of Bohr's postulate is that an excited electron translates from a stable orbit of radius r_q, corresponding to quantum number q and total energy (kinetic and potential) E_q, to an inner orbit of radius r_n, corresponding to quantum number n and total energy E_n. The electron loses potential energy and gains kinetic energy and, in the process, it emits radiation of frequency f_{nq}, in accordance with de Brogilie's hypothesis, such that:

$$E_q - E_n = hf_{nq} \tag{3}$$

where n is a number greater than zero but less than q.

2. Derivation of the Balmer-Rydberg Formula

Let us apply the two postulates of Bohr to the hydrogen atom whose electron of mass m and charge $-e$ at a point P revolves with velocity v_n about a stationary nucleus of mass M and charge $+e$ in a circular orbit of radius r_n, as shown in Figure1. The angular momentum of the electron, in the nth orbit, is:

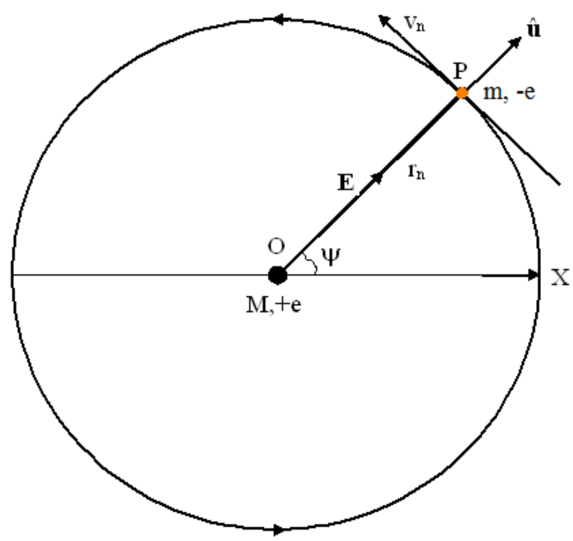

Figure 1 An electron of charge $-e$ and mass m at a point P revolving with speed v_n, through an angle ψ, in a circle of radius r_n under the attraction of a heavy nucleus of mass M and charge $+e$ at the central point O.

$$L_n = \frac{nh}{2\pi} = mv_n r_n \tag{4}$$

Equating the central forces on the electron, we get:

$$\frac{mv_n^2}{r_n} = \frac{e^2}{4\pi\varepsilon_o r_n^2} \tag{5}$$

Equations (4) and (5) give the speed and radius of revolution as:

$$v_n = \frac{e^2}{2\varepsilon_o nh} \tag{6}$$

$$r_n = \frac{\varepsilon_o n^2 h^2}{\pi me^2} \tag{7}$$

With n = 1, we get the Bohr radius $r_1 = 5.292 \times 10^{-7}\ m$.

The total energy, in the nth quantum state, is obtained as:

$$E_n = \frac{1}{2}mv_n^2 - \frac{e^2}{4\pi\varepsilon_o r_n} \tag{8}$$

Substitute for v_n and r_n from equations (6) and (7) gives:

$$E_n = -\frac{me^4}{8\varepsilon_o^2 n^2 h^2} \tag{9}$$

The total energy in the qth quantum state is:

$$E_q = -\frac{me^4}{8\varepsilon_o^2 q^2 h^2} \tag{10}$$

Equation (3) then becomes:

$$E_q - E_n = \frac{me^4}{8\varepsilon_o^2 h^2}\left(\frac{1}{n^2} - \frac{1}{q^2}\right) = hf_{nq} \tag{11}$$

$$\frac{f_{nq}}{c} = \frac{1}{\lambda_{nq}} = \frac{me^4}{8c\varepsilon_o^2 h^3}\left(\frac{1}{n^2} - \frac{1}{q^2}\right) \tag{12}$$

Equation (12) is the Balmer-Rydberg formula (with n > 0 < q) for the spectral lines of radiation from the hydrogen atom. The Rydberg constant *R*, in equation (1), is given by:

$$R = \frac{me^4}{8c\varepsilon_o^2 h^3} \tag{13}$$

Substituting the values of the physical constants in equation (13), *R* is found as 1.097×10^7 per metre, in agreement with observation [7].

3. Concluding Remarks

The stabilization of Rutherford's nuclear model of the hydrogen atom, by Niels Bohr, was recognised as a brilliant achievement of the human intellect. It gave an additional impetus to the development of quantum mechanics. However, it has some drawbacks as pointed out below:

(a) The radius of circular revolution of an electron, being proportional to n^2 (equation 7), is not quantized.
(b) The transition from one orbit to another, on one quantum jump, in zero time, as a necessary condition for radiation of energy.
(c) Failure to relate the frequency of emitted radiation to the frequency of revolution of the electron, round the positively charged nucleus.

Equation (12) is an example of Bechmann's *Correspondence Theory*, whereby the expected, desired or correct result is mathematically obtained, but on the basis of wrong underlying principles. In other the mathematics is correct but the physics is wrong and there can only be one correct physical explanation.

References

[1] E. Rutherford; "The Scattering of α and β Particles by Matter and the Structure of the Atom", *Phil. Mag.,* 21 (1911), 669.
[2] D.J. Griffith; *Introduction to Electrodynamics,* Prentice-Hall Inc., Englewood Cliff, New Jersey (1981), p 380.
[3] http://en.wikipedia.org/wiki/Balmer_series
[4] Francis Bitter; *Mathematical Aspects of Physics,* Anchor Books, Doubleday & Co. Inc., New York (1963), pp. 5–23.
[5] N. Bohr; On the Constitution of Atoms and Molecules, *Phil. Mag,,* Series 6, Vol. 26 (1913), pp. 1-25.
[6] http://en.wikipedia.org/wiki/Hydrogen_spectral_series
[7] L. De Broglie; *Ann. Phys.* (1925), 3, 22.
[8] Petr. Beckmann; *Einstein Plus Two*, The Golem Press, Boulder Co. (1987)