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Logical Analysis of the Foundations of Differential and Integral Calculus

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Abstract. Critical analysis of the generally accepted (standard) foundations of differential and integral calculus is proposed. Methodological basis of the analysis is the unity of formal logic and of rational dialectics. It is shown that the generally accepted foundations are based on the logically erroneous concept of "infinitesimal quantity (uninterruptedly diminishing quantity)" and contain logical errors.
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"**Physics, in the first place, gives us a presentiment of the solution; secondly, prompts to us a train of reasoning**" (Henri Poincaré).

Introduction

As is known, the formalism of differential and integral calculus is widely and successfully used in natural sciences. However, this does not mean that the problem of substantiation of differential and integral calculus is completely solved in 20-21 centuries, and the foundations of differential and integral calculus are not in need of formal-logical analysis now. Recently, necessity of critical analysis of the foundations of differential and integral calculus within the framework of the correct methodological basis – unity of formal logic and of rational dialectics – arises.

Critical analysis is impossible without plausible reasoning. "We fasten our mathematical knowledge with the help of demonstrative reasoning, but we reinforce our assumptions with the help of plausible reasoning. Everything new that we learn about the world is bound up with plausible reasoning. Plausible reasoning is the only type of reasoning that we are interested in everyday affairs. Mathematics in the making resembles any other human knowledge which is in the creation process. You have to guess mathematical theorem before you prove it; you must guess the idea of the proof before you carry out it in detail. The result of the creative work of mathematician is demonstrative reasoning, proof; but proof open up with the help of plausible reasoning, with the help of guess. Demonstrative reasoning and plausible reasoning supplement each other. The solution of the mathematical problem can be also suggested by the Nature; physics provides us with such keys. A mathematical picture would be too narrow without solution with the help of a physical interpretation" [1].

The purpose of this work is to propose a critical analysis of the foundations of differential and integral calculus. The critical analysis is based on plausible reasoning within the framework of methodological basis – unity of formal logic and of rational dialectics.

Plausible Foundations of Differential and Integral Calculus

There is a continuous function y of one argument x:

$$y = f(x).$$

1. Let the argument x take the increment Δx. New (accrued) value of the argument is $x+\Delta x$. Then the quantity of function y takes increment Δy, and the new (accrued) value of the function will be

$$y+\Delta y=f\left(x+\Delta x\right).$$

The increment Δy of the function has form:

$$\Delta y=f\left(x+\Delta x\right)-f\left(x\right) .$$

2. If the increment Δx of the argument tends to zero (i.e. $\Delta x\to 0$), then Δx becomes infinitesimal quantity (i.e. uninterruptedly diminishing quantity). The limit of this tendency is described as follows:

$$\lim_{\Delta x\to 0}\Delta x=0.$$

3. The concepts of "variable quantity Δx tends to the limit 0", "variable quantity Δx tending to the limit 0", and "process of tendency of variable quantity Δx to the limit 0" are not identical to the concept "limit of variable quantity Δx is equal to 0", i.e. expression $\Delta x\to 0$ is not identical to the expression $\lim_{\Delta x\to 0}\Delta x=0$:

$$\left(\Delta x\to 0\right)\neq\lim_{\Delta x\to 0}\Delta x.$$

4. The process $\Delta x\to 0$ is briefly designated by the symbol $d\,x$:

$$d\,x\equiv\left(\Delta x\to 0\right).$$

The "variable quantity" $d\,x$ is uninterruptedly diminishing quantity, and it is called the differential of variable quantity x. The variable quantity Δx in expression $\Delta x\to 0$ and the "variable quantity" $d\,x$ run the set of permissible values not stopping at one of them.

5. If $\Delta x\to 0$, then the increment Δy of the function is infinitely small (infinitesimal): $\Delta y\to 0$. The limit of this tendency is:

$$\lim_{\Delta x\to 0}\Delta y=0.$$

6. The concepts "variable quantity Δy tends to the limit 0", "process of tendency of variable quantity Δy to the limit 0", and "variable quantity Δy tending to the limit 0" are not identical to the concept of "limit of variable quantity Δy is equal to 0", i.e. expression $\Delta y\to 0$ is not identical to the expression $\lim_{\Delta x\to 0}\Delta y=0$:

$$\left(\Delta y\to 0\right)\neq\lim_{\Delta x\to 0}\Delta y.$$

7. The expression $\Delta y \to 0$ is briefly designated by the symbol dy. The “variable quantity” dy is uninterruptedly diminishing quantity, and it is called the differential of variable quantity y. Obviously,

$$d\,y \equiv (\Delta y \to 0).$$

The variable quantity Δy in the expression $\Delta y \to 0$ and the “variable quantity” dy run the set of permissible values not stopping at one of them.

8. The ratio of the increments and limit of this ratio have the following form:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} ,$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

9. The ratio of the increments

$$\frac{\Delta y}{\Delta x}$$

before passing to the limit depends on two variable quantities:
(a) on initial value of the argument x;
(b) on the quantity of the increment Δx of argument.
But the limit of this ratio under $\Delta x \to 0$ no longer depend on vanishing Δx because the initial value x of the argument under finding of the mentioned limit is assumed constant (any limit of a variable quantity is constant). Therefore, the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

being constant quantity may be dependent only on the initial value of the argument x. This limit will be an expression containing only the letter x, and, consequently, it will be a new function y' (or $f'(x)$) of the argument x.

10. A new function y' (or $f'(x)$) of the argument x made by mentioned function $y = f(x)$ is called the derivative of this function $y = f(x)$. Emphasizing the fact that this new function made by the function $y = f(x)$ with the help of some process, one designates the derivative with such symbols: y' or $f'(x)$.

11. The ratio of the differentials

$$\frac{dy}{dx}$$

has the following sense:

$$\frac{dy}{dx} \equiv \frac{(\Delta y \to 0)}{(\Delta x \to 0)} .$$

Obviously,

$$\frac{d\,y}{d\,x} \neq \lim_{\Delta x \to 0} \frac{\Delta\,y}{\Delta\,x}.$$

12. If relation between $\frac{d\,y}{d\,x}$ and $\lim\limits_{\Delta x \to 0} \frac{\Delta\,y}{\Delta\,x}$ has form

$$\frac{d\,y}{d\,x} \approx \lim_{\Delta x \to 0} \frac{\Delta\,y}{\Delta\,x}, \quad \text{i.e.} \quad \frac{d\,y}{d\,x} \approx y'$$

or form of strict equality

$$\frac{d\,y}{d\,x} = y', \ \text{i.e.} \quad d\,y = y'\,d\,x,$$

then this relation represents postulate based on intuition.

13. Stopping of the process $\Delta\,x \to 0$, $\Delta\,y \to 0$ and the return from infinitesimal variables (i.e. uninterruptedly diminishing quantity) $d\,x$, $d\,y$ to the finite variable quantities $\Delta\,x$, $\Delta\,y$ not tending to 0 are carried out with operation of integration which is designated by integral $\int$:

$$d\,x = d\,x, \ \int d\,x = \int d\,x, \ \Delta\,x = \int d\,x. \ \ x = \int d\,x + c, \text{ where } \Delta\,x = x - c, \ \ c = const;$$

$$d\,y = d\,y, \ \ \int d\,y = \int d\,y, \ \Delta\,y = \int f'(x)\,d\,x, \ \ y = \int f'(x)\,d\,x + c, \text{ where } \Delta\,y = y - c, \ \ c = const.$$

The above formulae satisfy formal logic law – the law of identity since the left and right sides of formulas have the same sense, belong to the same qualitative determinacy:

(infinitely diminishing quantity) = *(infinitely diminishing quantity)*

and

(finite quantity) = *(finite quantity).*

Discussion

1. The main difference between these formulae and the standard (generally accepted) formulae of differential calculus is that the standard formulae

$$d\,x = \Delta\,x, \ \ d\,y = \Delta\,y$$

not satisfy formal logic law – the law of identity since the left and right sides of formula do not have the same meaning, do not belong to the same qualitative determinacy. Really, the variable quantities $d\,x$ and $d\,y$ are infinitely small quantities (i.e. infinitely diminishing quantities), and variables quantities $\Delta\,x$ and $\Delta\,y$ are finite quantities (i.e. not infinitely diminishing quantities). From point of view of formal logic (i.e. the law of identity), the relation of identity between the quantities must be satisfied:

$$(infinitely\ diminished\ quantity) = (infinitely\ diminished\ quantity)$$

and

$$(finite\ quantity) = (finite\ quantity).$$

In addition, in accordance with the law of contradiction, infinitely small quantities (i.e. infinitely diminishing quantities) and the finite quantities (i.e. not infinitely diminishing quantities) must be connected by the logical relation of negation:

$$(infinitely\ diminished\ quantity) \neq (not\ infinitely\ diminished\ quantity).$$

But the standard mathematical relations

$$d\,x = \Delta\,x, \quad d\,y = \Delta\,y$$

contrary to the law of identity and, consequently, represent a logical error.

2. In classical mechanics, use of the definition of derivative leads to a logical error. Really, let point M be moved in the positive direction of the axis Ox. Motion is characterized by a change of coordinate $x(t)$ – continuous function of time t (because motion is a change in general). If $\lim_{\Delta t \to 0} \Delta t = 0$, then $\lim_{\Delta t \to 0} \Delta x = 0$, i.e., according to practice and formal logic, value of coordinate does not change and, therefore, there is no movement. But, contrary to practice and formal logic, differential calculus and classical mechanics contain the assertion that velocity $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$ exists without motion. Then velocity $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$ is not real (i.e. not physical) quantity, but fictitious quantity. Consequently, use of non-physical (unreal) quantity (i.e. the first and second derivatives of function) in classical mechanics is a logic error.

3. According to formal logic (i.e. the law of identity and the law of contradiction) the following logical relations between the variable quantities must be fulfilled:

$$(real\ quantity) = (real\ quantity),$$

$$(unreal\ quantity) = (unreal\ quantity),$$

$$(real\ quantity) \neq (unreal\ quantity).$$

But

$$\frac{d\,y}{d\,x}$$

in the relation

$$\frac{d\,y}{d\,x} = \lim_{\Delta x \to 0} \frac{\Delta\,y}{\Delta\,x}$$

is unreal quantity (mathematical fiction) since

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

is unreal quantity. Consequently, the quantity

$$\frac{d\,y}{d\,x}$$

is a logical error.

4. Infinitesimal quantity (i.e. uninterruptedly diminishing quantity) can not take numerical values. Really, if one will substitute, for example, value $\Delta x = 0{,}1$ in relation

$$d\,x \equiv (\Delta x \to 0),$$

then one will obtain meaningless relation:

$$0{,}1 \to 0\,.$$

Variable quantities $d\,x \equiv (\Delta x \to 0)$ and $d\,y \equiv (\Delta y \to 0)$ tend to zero without taking a single numerical value. But such behavior of variable quantities contraries to the experience. Consequently, the infinitesimal quantities $d\,x$, $d\,y$ are fictitious quantities, and the concept "infinitesimal quantity (uninterruptedly diminishing quantity)" represents a logical error.

Conclusion

Thus, the generally accepted (standard) foundations of differential and integral calculus are based on the logically erroneous concept of "infinitesimal quantity (uninterruptedly diminishing quantity)" and contain logical errors.

Reference

[1] G. Polya. Mathematics and plausible reasoning. Princeton, 1954.