The Scaling Theory X: General Sagnac Effect

C P Viazminsky
kayssarv@mail2world.com

Introduction

In earlier works[4-6,9] the Sagnac effect for the case of two pulses each making a complete round at the earth's equator, but in opposite directions, was explained on the basis of the scaling theory. The similarity in nature between this particular type of Sagnac effect and that of two pulses covering equal distances in opposite direction on a line was also revealed[10]. In this work we discuss the general type of this effect from the point of the scaling theory, and proceed to highlight the translative nature underlying the "genuine" Sagnac effect whether it was rotational or translational. The translative nature of Sagnac effect is revealed through expression in terms of Sagnac elemental, which is a purely translative effect. We also show that there is no literal Sagnac effect associated with a translational motion, in the sense that there is no delay between the durations of two pulses travelling the same closed optical circuit in opposite directions. It will be also shown that Sagnac effect arises under rotation even if the optical circuit is not closed, and under translation provided that the circuit is not closed and its enclosing segment is not perpendicular to its velocity.

16. There is no Translational Closed Sagnac Effect.

Before calculating durations of return trips, it is useful to restate two basic facts of the bound scaling theory (BST). (*i*) *when the length of a rod, stationary in S or in s, is measured geometrically* (*using a common unit of length in S and in s, say similar unit bars) then the same reading is obtained,* (*ii*) *the duration of any light's trip is the same in S as well as in the frame s in which the rod is stationary*

16.1. Return Trips

Let $S \equiv OXYZ$ be a timed inertial frame and *ob* a rod of geometric length L. Imagine now that *ob* is translating along *OX* uniformly at a constant velocity $\vec{u} = u\vec{\imath}\,(u > 0)$ relative to S, and let s be a frame attached to the rod and in standard configuration with S. Suppose that at an instant of time $T = o$ corresponding to ($o \in s$ at $O \in S$) and ($b \in s$ at $B \in S$), a light's pulse is emitted from o towards the rod's end b, where it is reflected back towards o. In the frame S the pulse is seen to emanate from $(o \in s\ at\ O \in S)$, travel to $b \in s$ which occupies when light arrives, a position $b' \in S,$ and return to $o \in s$ at a position $o'' \in S$ (Fig.1).

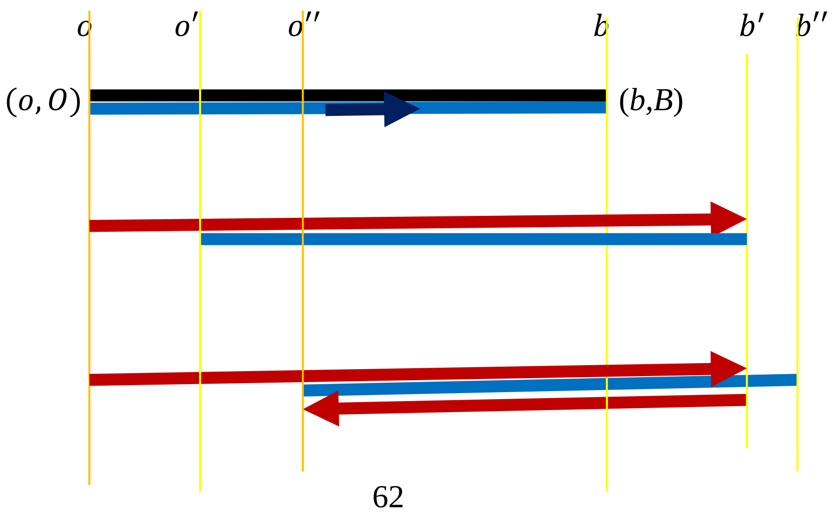

Fig.1. the first line depicts the configuration of the blue rod *ob* in S when a pulse of light is emitted from its rear *o* towards its front *b*. In the second line, the pulse in red, arrives at *b* when the later at $b' \in S$. In the third line the pulse is reflected and returns to the rod's rear which occupies when light arrives the position $o'' \in S$.

We shall calculate now the duration of the return trip

$$(o\ at\ O \to b \in s \to o \in s), \quad (16.1)$$

employing the various calculative versions of the bound scaling theory (BST) in aim to illuminate their equivalence. Since the velocity vector $\vec{u} = u\vec{\imath}$ of *s* (the rod) relative to S and the radius vector $\vec{R}_B = \overrightarrow{BO}$ are in opposite directions $(i.e.\ \theta = \pi)$, the BST (13.10) yield the pulse arriving at $b' \in S$ at an instant

$$t_+ = \frac{L}{c}\frac{1}{\Gamma(u,\pi)} = \frac{L}{c}\sqrt{\frac{1+\beta}{1-\beta}}, \quad (16.2)$$

which is the duration of the motion-wise trip $(o \in s\ \to b \in s)$. Alternatively, we could have used the formal view in part VII to say that the unit of time ts_+ in *s* is

$$ts_+ = \frac{1}{\Gamma(u,\pi)} sec = \sqrt{\frac{1+\beta}{1-\beta}}\ sec, \quad (16.3)$$

with $TS \equiv sec$ is the unit of time in *S*, whereas the same time reading $t_c = L/c$ is gotten in *s*. Hence, the duration of the trip in *s* and in *S* will be

$$t_+ = t_c.ts_+ = \frac{L}{c}\Gamma(u,0)\ \text{sec}, \quad (16.4)$$

which is the same as (16.2).

We could have instead, adopted the view of the frame *s*, in which the rod is stationary, and $\theta = \pi$ in (13.10) refers to the angle $\angle(\overrightarrow{bo} \equiv \vec{r}_b, ox)$, or to the angle between the ray $\overrightarrow{ob}$ and the velocity $(-\vec{u})$ of *B* (or *S*) in *s*, and *L* is its length as geometrically measured in *s*.

To calculate the period of the counter motion trip $(b \in s\ \to o \in s)$, or, $(b \in s\ at\ b' \in S \to o \in s\ at\ o' \in S)$, we take into account that the velocity of *s* relative to *S* and the radius vector $\vec{R}_O = \overrightarrow{Ob'}$ have the same direction, and hence the duration of the trip is

$$t_- = \frac{L}{c}\frac{1}{\Gamma(u,0)} = \frac{L}{c}\sqrt{\frac{1-\beta}{1+\beta}}. \quad (16.5)$$

Or, we could have defined in association with the given trip the unit of time ts_- in *s* by

$$ts_- = \frac{1}{\Gamma(u,0)} sec = \sqrt{\frac{1-\beta}{1+\beta}}\ sec, \quad (16.6)$$

with $TS \equiv sec$ is the unit of time in *S*. In terms of ts_- the reading of the duration of the given trip in *s* is the same as the geometric duration of the hypothetical trip in *S*, i.e. $t_c = L/c$, and hence the duration of this trip will be $t_- = t_c.ts_-$, which is the same as that obtained by (16.5).

Thus the duration of the return trip (16.1) is given by

$$t_{ret(0)} = t_+ + t_- = \frac{L}{c}\sqrt{\frac{1+\beta}{1-\beta}} + \frac{L}{c}\sqrt{\frac{1-\beta}{1+\beta}} = 2\frac{L/c}{\sqrt{1-\beta^2}}, \quad (16.7)$$

which is also the time reading when receiving the emitted pulse back at $o \in s$. The symbol (0) in $ret(0)$ denotes the angle between the rod and the X-axis.

The general case corresponds to a rod ob of length L that makes an angle $\theta = \angle(\overrightarrow{ob}, OX)$ with the X-axis of S, and translating uniformly at velocity u $(u > 0)$ in the direction of the X-axis. For the trip $(o \to b)$, which corresponds in S to the trip $(O \to b')$, the initial radius vector of the source relative to the initial position of the receiver, namely $\vec{R}_{B,O} = \overrightarrow{BO}$, makes an angle $\pi - \theta$ with the X-axis, and the BST (13.10) yields for the duration of this trip $(o \to b)$ the value:

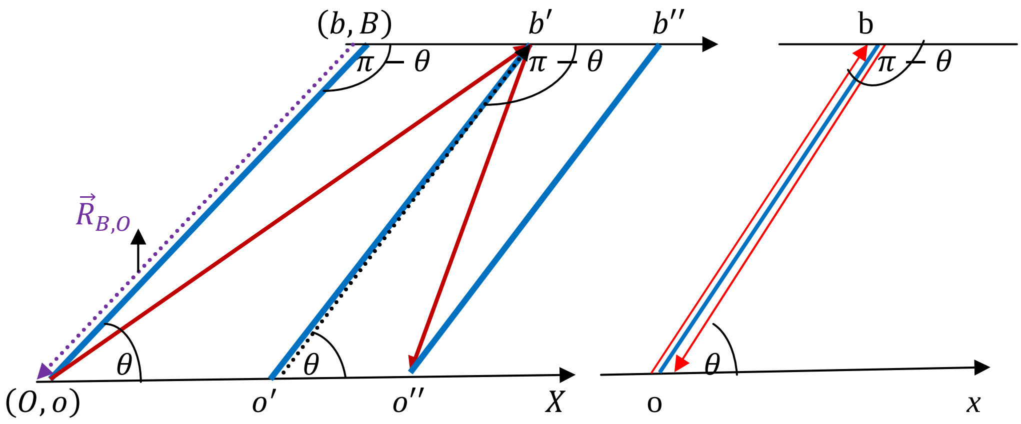

Fig.2. The view of the return trip (16.1) (left) as seen in S, (right) as observed in s

(16.8) $$t_{o\to b} = \frac{L}{c}\frac{1}{\Gamma(u, \pi - \theta)} = \frac{L}{c}\Gamma(u, \theta).$$

Now, light arrives at $b \in s$ when $b \in s$ is at $b' \in S$ and $o \in s$ is at $o' \in S$. The counter motion-wise trip $(b \to o)$ in s corresponds to the trip $(b' \in S \to o'' \in S)$. The initial radius vector of the source relative to the initial position of the receiver in this case, $\vec{R}_{o',b'} = \overrightarrow{o'b'}$, makes an angle θ with the X-axis, and hence the duration of the trip $(b \to o)$ is

(16.9) $$t_{b\to o} = \frac{L}{c}\frac{1}{\Gamma(u, \theta)} = \frac{L}{c}\Gamma(u, \pi - \theta).$$

The duration $t_{ret(\theta)}$ of the return trip (16.1) (in S and s) is simply the sum of the two consecutive forward and backwards trips:

$$t_{ret(\theta)} = t_{o\to b} + t_{b\to o} = \frac{L}{c}[\Gamma(u, \theta) + \Gamma(u, \pi - \theta)]$$

(16.10) $$= \frac{2L}{c}\frac{\sqrt{1 - \beta^2 sin^2\theta}}{\sqrt{1 - \beta^2}}.$$

Employing the formal view amounts to introduce, in conjunction with the trips $o \to b$ and $b \to o$ in s, the units of time $ts_{o\to b} = sec/\Gamma(u, \pi - \theta)$ and $ts_{b\to o} = sec/\Gamma(u, \theta)$ respectively, while the reading of the duration of each trip is the same as the geometric duration of the corresponding hypothetical trip in S, which is $t_c = L/c$. The duration of the return trip will be given by

$$t_{o\to b} + t_{b\to o} = t_c(ts_{o\to b} + ts_{b\to o}),$$

which is the same as the result (16.10).

If we prefer we may adopt the **view of the frame** s (Fig.2 (right)) in which the bar *ob* is stationary. The formula (13.10) applies here with an angle $\pi - \theta = \angle(\overrightarrow{bo} \equiv \vec{r}_b, ox)$ for the trip $o \to b$, and $\theta = \angle(\overrightarrow{ob} \equiv \vec{r}_o, ox)$ for the trip $b \to o$, and L is the length of the bar measured geometrically in s. The duration of the return trip is

$$t_{o\to b} + t_{b\to o} = \frac{L}{c}[\Gamma(u,\theta) + \Gamma(u, \pi - \theta)],$$

which is the same as in (14.10).

Now, and had the return trip started from b instead of o its duration would not change. Indeed, the duration of the return trip

(16.11) $$(b \; at \; B \to o \in s \to b \in s),$$

which is

(16.12) $$t_{b\to o} + t_{o\to b} = \frac{L}{c}[\Gamma(u, \pi - \theta) + \Gamma(u,\theta)],$$

is evidently equal to the duration of the return trip (16.1) given by (16.10) .

Now, if both return trips (16.1) and (16.12) starts at the same instant $t = t_0$, which corresponds to $(o \; at \; O \; and \; b \; at \; B)$, then each will take the same duration to be completed, and the two pulses return to their starting points o and b respectively at the same instant $t = t_0 + t_{ret(\theta)}$.

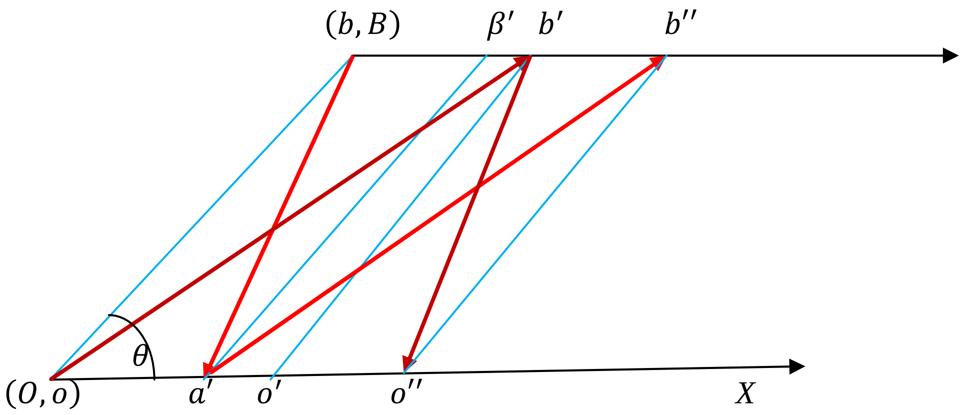

Fig.3. The view of the return trips (16.1) and (16.12) as seen in *S*. Both trips start at the same time from *o* and *b* and return simultaneously to *b* and *o* respectively.

16.2 The Michelson and Morley Experiment Revisited

Let s be an inertial frame translating at a constant velocity $u\vec{\imath}\,(u > 0)$ relative to a timed inertial frame S, with s and S are in standard configuration. Suppose that at an instant $t = 0$ in S the points $o \in s, \; p \in s \; and \; q \in s$ are at $O \in S, \; P \in S \; and \; Q \in S$ respectively, with p and q are at the same geometric distance L from o in s (which is the same in S), and that $\angle(\overrightarrow{op}, ox) = \theta$ and $\angle(\overrightarrow{oq}, ox) = \varphi$. We may imagine *op* and *oq* as two rods with the same geometric length *L* in *s* (and in *S*). Assume now that two beams of light are sent from the source of light $b \in s$ when at $B \in S$, towards the mirrors $p \in s$ and $q \in s$, where they are reflected back to $o \in s$. Each return trip covers a geometric distance *2L*, but their optical lengths depend on their directions relative to $\vec{u} = u\vec{\imath}$, which are determined by θ and φ respectively.

The durations of these trips are

$$t_\theta = \frac{2L}{c}\frac{\sqrt{1-\beta^2 sin^2\theta}}{\sqrt{1-\beta^2}}, \quad t_\varphi = \frac{2L}{c}\frac{\sqrt{1-\beta^2 sin^2\varphi}}{\sqrt{1-\beta^2}}.$$

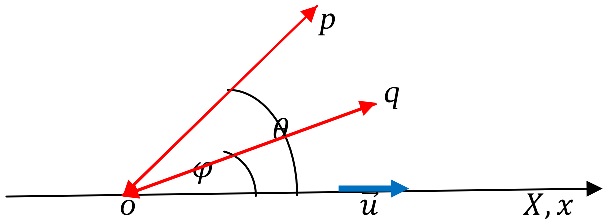

Fig.4. The return trips as viewed in *s*

The difference

(16.13) $$t_\theta - t_\varphi = \frac{2L}{c}\left(\frac{\sqrt{1-\beta^2 sin^2\theta} - \sqrt{1-\beta^2 sin^2\varphi}}{\sqrt{1-\beta^2}}\right)$$

between these durations takes a maximal value for $\theta = 0\,, \pi; \varphi = \pm\frac{\pi}{2}$. In this case

(16.14) $$t_0 - t_{\frac{\pi}{2}} = \frac{2L}{c}\left(1 - \frac{1}{\sqrt{1-\beta^2}}\right) \approx \beta^2\frac{L}{c}.$$

When *S* is the geocentric frame and *s* is an earth's rigid frame, the latter relation coincides with the result obtained in part *IX* where Michelson and Morley experiment was discussed.

16.3 No Translational Closed Sagnac's Effect

The Sagnac interferometer consists of a closed optical circuit, such as polygonal system of mirrors, or a fiber optical ring, along which two beams of light travel in opposite directions till returning to their initial point. No difference in time is registered between the durations of the two light's trips when the circuit is stationary in a geocentric frame *S* (or approximately in the earth rigid frame *s*). If however the circuit is rotating about an axis not parallel to its plane, a difference in the travel times of the two trips is observed and detected as interference fringes. The latter two sentences embodies the "literal Sagnac's effect". The word "literal" is injected here to indicate that the familiar rotational Sagnac effect which we have just described is just one aspect of a more general phenomenon which could be translational as much as rotational, and moreover, the path of light need not be a closed circuit. In this section, the phrase "Sagnac effect" will refer to the difference in travel time between two beams tracing the same path in opposite directions. When the path is closed the Sagnac effect, which may be nil, will be described as closed. It will be shown that Sagnac effect arises under rotation even if the optical circuit is not closed, and under translation provided that the circuit is not closed, and its enclosing segment is not perpendicular to its velocity.

Consider now two pulses in opposite directions, each making a full trip along a polygonal circuit. Fig.4 represents a trip (drawn in red) along one direction of a polygon **as viewed in** ***s***. The blue vectors represent the translational velocity of *s* relative to *S*. An angle, in single line, represents the angle between velocity $\vec{u}$ of *s* relative to *S* and the radius vector originating from the observer position and pointing against the direction of the corresponding trip. For example, the angle 1^+ is the angle between $\vec{u}$ and the radius vector formed by side $i = 1$ with direction opposite to that of the trip. The latter radius vector originates from the end point of the ray and ends at its starting point. The symbol $1^- = \pi - 1^+$ refers to the complement of the angle 1^+, and it is drawn in double lines. Assuming that the geometric length of the side *i* is L_i, the duration of the complete round trip, shown in red, is

$$(16.15) \qquad \sum_{i=0}^{n} \frac{L_i}{c} \frac{1}{\Gamma(u, i^+)} = \sum_{i=0}^{n} \frac{L_i}{c} \frac{\beta cosi^- + \sqrt{1 - \beta^2 sin^2 i^-}}{\sqrt{1 - \beta^2}}$$
$$= \sum_{i=0}^{n} \frac{L_i}{c} \frac{\sqrt{1 - \beta^2 sin^2 i^-}}{\sqrt{1 - \beta^2}}.$$

Similarly, the duration of the counter trip is

$$(16.16) \qquad \sum_{i=0}^{n} \frac{L_i}{c} \frac{1}{\Gamma(u, i^-)} = \sum_{i=0}^{n} \frac{L_i}{c} \frac{\sqrt{1 - \beta^2 sin^2 i^+}}{\sqrt{1 - \beta^2}},$$

which is equal to the duration of the first trip. This shows that *there is no translational closed Sagnac's effect*. It is interesting to note that the last result is valid whether the polygon was *planer or spatial*.

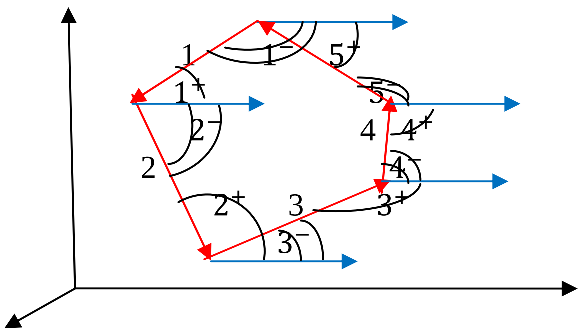

Fig.5. A polygonal interferometer translating in *S*

It follows therefore that the literal *Sagnac condition* (see part I) which states that: there is no time difference in the durations of two beams of light travelling along the circumference of a closed loop in opposite directions, holds in every inertial frame. The reader should notice that the concept of translational Sagnac effect did not appear in the Part I (section [1.2]) of this work, and that the Sagnac condition was considered erroneously equivalent to the "*strong Sagnac condition*", which states that "the travel time between the ends of a rod is the same whether we choose either end of the rod the initial point of the light's trip and the other point its end". The latter condition is valid only in a timed inertial frame. The strong Sagac condition implies the literal Sagnac condition, but the converse is not true except in a timed inertial frame. This and some other minor flaws will be rectified in subsequent section entitled "the universal space revisited". Also the interpretation of the scaling transformations as well as some other basic concepts will be simplified and sharpened.

17. The Translational Sagnac's Effect.

We show here that Sagnac's effect is intimate to non-closed optical paths translating relative to the timed inertial frame S. In what follows the translational Sagnac's effect is discussed in gradual steps starting by its simplest form.

17.1. Simple Cases

(i) The simplest case of translational Sagnac's effect corresponds to a simultaneous emission of light from each end of a rod towards the other end. If the rod is stationary in S, both pulses arrive simultaneously at their targets. If however the rod is translating along its axis then the pulse traveling against its motion arrives before the pulse travelling motion-wise.

To put our arguments in a quantitative form we consider again a rod ob of geometric length L aligning with the X-axis of the timed inertial frame S, and translating at a uniform speed $\vec{u} = u\vec{\imath}\,(u > 0)$, as shown in Fig.5. Let s be a frame attached to the rod. At an instant of time $T = o$ in S corresponding to ($o \in s$ at $O \in S$) and ($b \in s$ at $B \in S$), a light's pulse is emitted from the end o of the rod towards its end b, and at the same time another pulse is emitted from b toward o. The two events of the pulses' emission from o and b are simultaneous; both occur at $T = o$ in S. As it seen in the frame S, the pulse emitted from $(o \in s\ at\ O \in S)$ ends at $b \in s$, which occupies when light arrives at b, a position $b' \in S$, while the pulse emitted from $(b \in s\ at\ B \in S)$ ends at $o \in s$ which occupies when light arrives at it a position $b' \in S$. The durations of both trips, as calculated in section [16.1], are given by

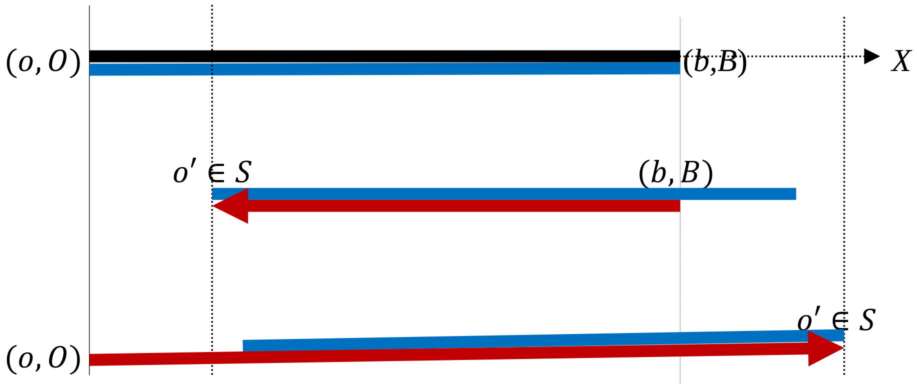

Fig.6. The paths of the pulses emitted simultaneously from the ends of the rod and received at the other end as seen in S

$$t_{o\to b} = \frac{L}{c}\frac{1}{\Gamma(u,\pi)} = \frac{L}{c}\sqrt{\frac{1+\beta}{1-\beta}}, \qquad t_{b\to o} = \frac{L}{c}\frac{1}{\Gamma(u,0)} = \frac{L}{c}\sqrt{\frac{1-\beta}{1+\beta}}.$$

The difference is

$$(17.1) \qquad \Delta t_{\theta=0} = t_{o\to b} - t_{b\to o} = \frac{L}{c}\frac{2\beta}{\sqrt{1-\beta^2}}t = T\frac{2\beta}{\sqrt{1-\beta^2}},$$

with $T = L/c$, is the geometric duration corresponding to the bar's geometric length L. The last relation shows that the pulse emitted from the front b of the rod, i.e. in opposite direction to the rod's motion, arrives at its rear o before the pulse emitted from its rear arrives at its front, by an amount $\Delta t_{\theta=0}$, given by (17.1).

(ii) If the rod is inclined by an angle θ from the X-axis (Fig.7(a)), then the difference will be

$$\Delta t_\theta = t_{o\to b} - t_{b\to o} = \frac{L}{c}\left[\frac{1}{\Gamma(u,\pi-\theta)} - \frac{1}{\Gamma(u,\theta)}\right\}$$

(17.2)
$$= \frac{L}{c}[\Gamma(u,\theta) - \Gamma(u,\pi-\theta)] = \frac{L}{c}\frac{2\beta cos\theta}{\sqrt{1-\beta^2}}.$$

which is positive for $0 \le \theta < \pi/2$, negative for $\pi/2 < \theta \le \pi$, and vanishes for $\theta = \pi/2$. The difference Δt_θ reduces to (17.1) for $\theta = 0$, and to $(-\Delta t_{\theta=0})$ for $\theta = \pi$. In both cases the pulse travelling against the rod motion arrives at the other end before the pulse travelling motion-wise.

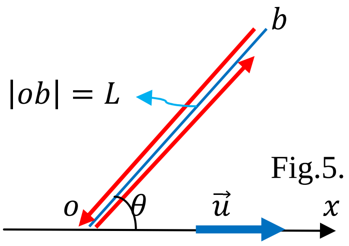

Fig.7(a).

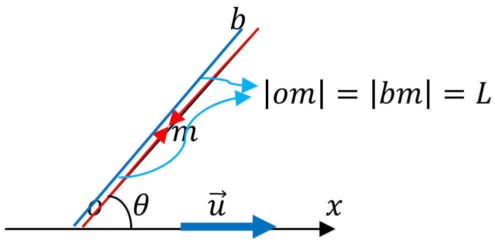

Fig.7(b).

It is useful to alter the linear Sagan'c interferometer presented in (i) and (ii) to a testable form (Fig.7(b)). This is done as follows: Let *m* be a mid point of the rod *ob* of a geometric length 2*L*, and take *m* as the end points of the two trips that start at the same time from o and b. In this version the trips $(o \to m)$ and $(b \to m)$ cover the same geometric distance *L* in *S* but in opposite directions. The difference in their optical lengths is $c\Delta t_\theta$, where Δt_θ given by (16.18). The two incident beams at *m* can be arranged to interfere and produce interference fringes.

(iii) The general case corresponds to a continuous path (a fiber optical thus shaped), consisting of straight segments, along which light travels from each end to the other end in opposite directions.

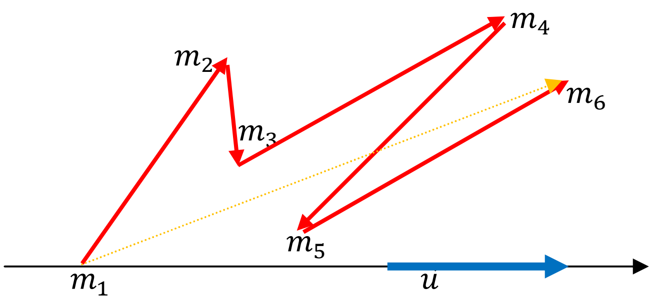

Fig.8. The s view of the light's trip along a path composed of straight segments

Viewing the trip from *s* it is easy to find that the duration of the trip shown in red, is

$$t(m_1 \to m_2 \to \cdots \to m_6) = \frac{1}{c}\sum_{i=1}^{n-1}\frac{-\overrightarrow{m_\imath m_{\imath+1}}\cdot\frac{\vec{u}}{c} + \sqrt{|\overrightarrow{m_\imath m_{\imath+1}}|^2 - \left|\frac{\vec{u}}{c}\times\overrightarrow{m_\imath m_{\imath+1}}\right|^2}}{\sqrt{1-\beta^2}}$$

where here, $n = 6$, but *n* can in general assume values as many vertices the path contains. The duration of the reverse trip is

$$t(m_6 \to \cdots \to m_1) = \frac{1}{c}\sum_{i+1=n}^{i+1=2} \frac{-\overrightarrow{m_{\imath+1}m_{\imath}}.\frac{\vec{u}}{c} + \sqrt{|\overrightarrow{m_{\imath}m_{\imath+1}}|^2 - \left|\frac{\vec{u}}{c} \times \overrightarrow{m_{\imath}m_{\imath+1}}\right|^2}}{\sqrt{1-\beta^2}}.$$

The difference between the periods of these trips is

(17.3) $$\Delta t = t(m_1 \to \cdots \to m_n) - t(m_n \to \cdots \to m_1)$$

(17.4) $$= \frac{\frac{2\vec{u}}{c^2}}{\sqrt{1-\beta^2}} . \sum_{i=1}^{i=n-1} \overrightarrow{m_{\imath}m_{\imath+1}},$$

or

(17.5) $$\Delta t = \frac{2}{c^2\sqrt{1-\beta^2}}(\vec{u}.\overrightarrow{m_1m_n}).$$

We shall call the vector $\overrightarrow{m_1m_n}$ the *circuit's vector*. The latter expression which contains (*i*) and (*ii*) as special cases, leads to a following facts:

(i)- The time difference between the durations of the mentioned opposite trips is proportional the inner product of the velocity $\vec{u}$ of s relative to S and the circuit vector $\overrightarrow{m_1m_n}$. i.e. the Sagnac effect associated with the circuit is the same as that associated with the circuit vector. The latter result is *true for both planar and spatial circuits*.

(ii)- The difference between the duration of the two trips is positive, negative or zero, according to the $\angle(\vec{u}, \overrightarrow{m_1m_n})$ being acute, obtuse, or right angle. The difference vanishes if and only if $\vec{u}.\overrightarrow{m_1m_n} = 0$, which is satisfied if and only if either the circuit's vector vanishes, in which case the path is closed (not necessarily forming a concave polygon), or if the circuit's vector is perpendicular to the relative velocity between the two frames.

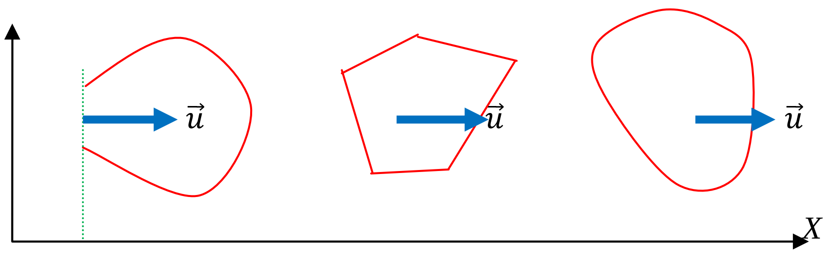

Fig.9. The Sagnac effect vanishes in all above three cases

(iii) The relation (17.4) written in the form

(17.6) $$\Delta t = \frac{2u(x_n - x_1)}{c^2\sqrt{1-\beta^2}}$$

shows that what matters in the value Δt is only the projection of the circuit's vector on the velocity of s relative to S (i.e. on OX). Thus all paths that share the same value of the projection of the circuit's vector are associated with the same delay or advance period Δt. The result (17.6) is expected since this effect should posses a translational symmetry along OX, as the BST do. The latter result is in accord with the arguments of Persson[35,36]. However, the scaling theory's approach does not require ether, while Persson's arguments rely on ether's existence.

17.2. General Treatment of the Translational Sagnac Effect

Consider now an arbitrary smooth path with length L (such as a fiber optics), and imagine that it is divided into infinitesimal line elements, with geometric length dL (which is of course the same in s and S). The difference between the periods of two opposite trips along one such element is

$$dt_\theta = \frac{dL}{c}\frac{2\beta cos\theta}{\sqrt{1-\beta^2}} = \frac{2\beta dx}{c\sqrt{1-\beta^2}},$$

and the total difference Δt is obtained through integrating the latter expression over the whole path:

$$(17.7) \qquad \Delta t = \frac{2\beta}{c\sqrt{1-\beta^2}}\int dx = \frac{2u(x_e - x_b)}{c^2\sqrt{1-\beta^2}} \equiv \frac{2\vec{u}.\overrightarrow{be}}{c^2\sqrt{1-\beta^2}},$$

where x_e and x_b denotes the abscissa of its end e and beginning b. Note that the path need not be planar.

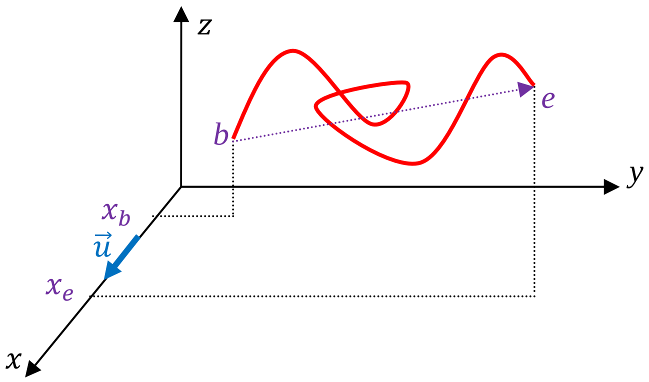

Fig.10. An arbitrary shape of the optical path depicted in *s* which is translating at velocity $\vec{u}$ relative to *S*.

18. The Sagnac Elemental Effect

18.1. Reduction of the Rotational Sagnac Effect to Infinitesimal Translational Effects

We show here that the rotational Sagnac effect can be reduced to a sum of infinitesimal translational Sagnac effects, but the way round is of course not possible. We first re-derive the Sagnac effect and show that we can always express it in a translative form. By the BST the optical length of a light's trip along the side $(m_{i+1} \to m_i)$ of an n-polygon, in S and in s, is

$$(18.1) \qquad r_{i,i+1} = \frac{-\vec{\beta}_i.\overrightarrow{R_{\iota,\iota+1}} + \sqrt{R^2_{i,i+1} - \left|\vec{\beta}_i \times \overrightarrow{R_{\iota,\iota+1}}\right|^2}}{\sqrt{1-\beta_i^2}}.$$

Note that $\overrightarrow{R_{\iota,\iota+1}} = \overrightarrow{L_{\iota,\iota+1}}$ both refer to the side $\overrightarrow{m_\iota m_{\iota+1}}$, or equivalently, to the contiguous segment $\overrightarrow{M_\iota M_{\iota+1}}$.

If the polygon is regular, and the rotation is taking place about its center, then the difference between the optical lengths of the trips $(m_i \to m_{i+1})$ and $(m_{i+1} \to m_i)$ is given by

$$(18.2) \qquad \Delta r_i \equiv r_{i+1,i} - r_{i,i+1} = \frac{\vec{\beta}_i.\overrightarrow{R_{\iota,\iota+1}} - \vec{\beta}_{i+1}.\overrightarrow{R_{\iota+1,\iota}}}{\sqrt{1-\beta^2}} = \frac{2\vec{\beta}_i.\overrightarrow{R_{\iota,\iota+1}}}{\sqrt{1-\beta^2}},$$

where $\beta = |\vec{\beta}_i|$ is the speed of any vertex (in the unit c). The last expression is the familiar translative Sagnac effect, as given by (17.2), and which corresponds to the rod making an angle $\theta = \pi/n$ with its translative velocity.

From this expression we can proceed to reach either the usual expression of the rotational Sagnac effect (14.14), or the expression of the translational sagnac effect (17.5).

The Rotational Sagnac Effect

To reach (14.14), we proceed from (18.2) as follows

(18.3) $$\Delta t = \sum_{i=1}^{i=n+1} \frac{\Delta r_i}{c} = \sum_{i=1}^{i=n+1} \frac{2\vec{\beta}_i . \overrightarrow{R_{i,i+1}}}{c\sqrt{1-\beta^2}}$$

$$= \frac{2}{c^2}\vec{\omega}. \sum_{i=1}^{i=n+1} \frac{\overrightarrow{OM_i} \times \overrightarrow{M_i M_{i+1}}}{\sqrt{1-\beta^2}} = \frac{4}{c^2\sqrt{1-\beta^2}}\vec{\omega}.\vec{A}.$$

The Translational Sagnac Effect

To reach (17.5) we proceed from (18.2) as follows

(18.4) $$\Delta t = \sum_{i=1}^{i=n+1} \frac{2\vec{\beta}_i . \overrightarrow{R_{i,i+1}}}{c\sqrt{1-\beta^2}} = \frac{2}{c^2\sqrt{1-\beta^2}} \sum_{i=1}^{i=n+1} \vec{u}_i . \overrightarrow{M_i M_{i+1}}$$

$$= \frac{2}{c^2\sqrt{1-\beta^2}} \vec{v}.\overrightarrow{circum},$$

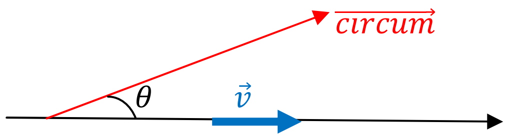

Fig.11. Geometric representation of (18.4)

where $\overrightarrow{circum}$ is any vector with length equal to the circumference nL of the polygon and $\vec{v}$ is a vector velocity with length equals to the speed $v = |\vec{u}_i|$ of any vertex of the polygon and makes an angle $\theta = \pi/n$ with $\overrightarrow{circum}$ (Fig.11). On noting that

$$v\ circum\ cos\theta = \omega a\ nL\ cos\ \theta = w\ (na^2 sin2\theta) = \omega(2A),$$

the alternative form

(18.5) $$\Delta t = \frac{2v\ circum\ \cos(\frac{\pi}{n})}{c^2\sqrt{1-\beta^2}},$$

of (18.4) is immediately reducible to (18.3).

If the polygon is regular or not, and the rotation takes place about an arbitrary axis not necessarily perpendicular to its plane, then the relation

(18.6) $$r_{i+1,i} - r_{i,i+1} = \vec{\beta}_i . \overrightarrow{R_{i,i+1}} - \vec{\beta}_{i+1} . \overrightarrow{R_{i+1,i}}$$

holds on neglecting terms in the second degree in β_i. The right hand-side of the last equation can be written as

$$\vec{\beta}_i . \overrightarrow{R_{i,i+1}} - \vec{\beta}_{i+1} . \overrightarrow{R_{i+1,i}} = \frac{\vec{\omega}}{c} . [\overrightarrow{R_i} \times \overrightarrow{R_{i,i+1}} - \overrightarrow{R_{i+1}} \times \overrightarrow{R_{i+1,i}}] =$$

$$= \frac{\vec{\omega}}{c} . [\overrightarrow{R_i} \times \overrightarrow{R_{i+1}} - \overrightarrow{R_{i+1}} \times \overrightarrow{R_i}]$$

(18.7) $$= \frac{2\vec{\omega}}{c}.\overrightarrow{R_\iota} \times \overrightarrow{R_{\iota+1}} = 2\overrightarrow{\beta_\iota}.\overrightarrow{R_{\iota+1,\iota}},$$

which represents the translational Sagnac effect corresponding to one side, with terms in second degree in β_i are neglected. The sum of all such terms yields the full Sagnac effect.

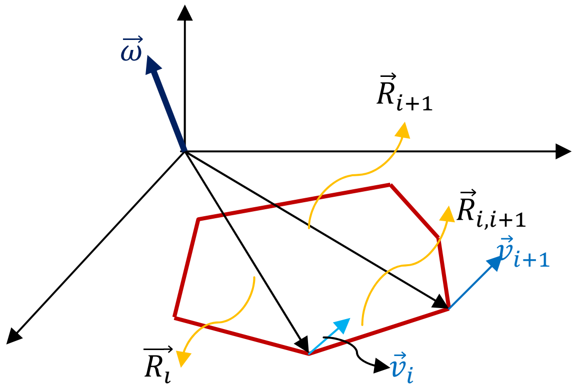

Fig.13. A non-regular polygonal optical circuit rotating about an axis not necessarily perpendicular to its plane.

18.2 General Treatment

Suppose that a closed Sagnac interferometer of any shape is rotating about an axis not necessarily perpendicular to its plane. The rotational Sagnac effect can be expressed as an integral of infinitesimal translational Sagnac effects as follows:

(18.8) $$\vec{\omega}.\vec{A} = \vec{\omega}.\iint \overrightarrow{dA} = \frac{1}{2}\vec{\omega}.\oint \vec{R} \times \overrightarrow{dL} = \frac{1}{2}\oint \vec{v}.\overrightarrow{dL},$$

where $\vec{R}$ is the position vector of an arbitrary point of the circuit relative to the point of intersection of the axis of rotation and the circuit's plane, $\vec{v}$ is the velocity of this point, and terms in the second power in β are neglected. The latter result is also valid for polygonal optical circuits not necessarily regular, in which case the line integral should be understood as Lebesgue integral, and thus reduces to a finite sum.

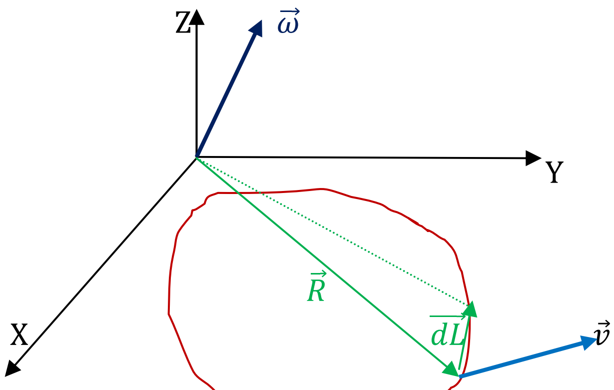

Fig.14. An optical closed circuit with an arbitrary shape rotating about an axis not perpendicular to its plane.

If the magnitude of $\vec{v}$ is constant and its direction makes always a constant angle θ with $\overrightarrow{dL}$, which corresponds to a regular polygon rotating about its center, then the last line integral reduces to the familiar form

(18.9) $$\oint \vec{v}.\overrightarrow{dl} = v\cos\theta \oint dl = v\; circum\; cos\theta.$$

Or we may get the expression

(18.10) $$\Delta t = \frac{2}{c^2} \oint \frac{\vec{v}.\overrightarrow{dL}}{\sqrt{1-\beta^2}}$$

if we make no approximations. Note that the relation (18.8) can also be obtained as follows: Choosing the axis Z along $\vec{\omega}$, then $\vec{\omega} = \omega\vec{k}$, and

$$\vec{\omega}.\vec{A} = \omega \iint \vec{k}.\overrightarrow{dA} = \frac{\omega}{2}\iint rot\big(-Y,X,f(Z)\big).\overrightarrow{dA} = \frac{\omega}{2}\oint(-Y,X,f(Z)).\overrightarrow{dL}$$

(18.11) $$= \frac{\omega}{2}\oint(-Y,X,f(Z)).(dX,dY,dZ) = \frac{\omega}{2}\oint XdY - YdX,$$

where Stokes theorem was applied to convert the surface integral to a line integral, and $f(Z)$ is an arbitrary function of Z. The latter expression can be written as

(18.12) $$\frac{\omega}{2}\oint XdY - YdX = \frac{\vec{\omega}}{2}.\oint \vec{R}\times\overrightarrow{dL} = \frac{1}{2}\oint \vec{v}.\overrightarrow{dL}.$$

18.3 The Sagnac Elemental

The relation (16.8) expressed in the form

(18.13) $$\Delta t_\theta = \frac{2\vec{u}.\overrightarrow{\Delta R}}{c^2\sqrt{1-\beta^2}},$$

with $\overrightarrow{\Delta R}$ is the circuit vector, and $\vec{u}$ is its velocity in *S*, certainly holds for infinitesimals. Consider a circuit with an arbitrary shape and not necessarily closed. This circuit can be imagined to consist of infinitesimally small segments, called *circuit's elements* and denoted by $\overrightarrow{dR}\,(\equiv \overrightarrow{dL})$. We denote the corresponding Sagnac effect pertaining to a circuit's element by dt, and call it the Sagnac elemental effect, or just, the *Sagnac elemental*. The Sagnac elemental is thus given by

(18.14) $$dt = \frac{2\vec{u}.\overrightarrow{dR}}{c^2\sqrt{1-\beta^2}},$$

with $\vec{u}$ is the velocity of the circuit's element $\overrightarrow{dR}$. Thus every circuit's element is accompanied by a Sagnac elemental that depends on the length of the circuit's element, its velocity and the orientation of these with respect to each other. The Sagnac elemental is a purely translative effect determined solely by the projection of the light's beam velocity in the circuit's element in the direction of the velocity of the latter. The full Sagnac effect corresponding to the full circuit will thus be

(18.15) $$\Delta t = \int \frac{2\vec{u}.\overrightarrow{dR}}{c^2\sqrt{1-\beta^2}},$$

with the integral is extending over the whole circuit.

The latter formula for Sagnac effect holds for an arbitrary circuit as long as the bound scaling theory (BST) is valid for light's trips in rigid frames (see section (13.4)).

Special cases are considered here.

(i). Consider two diametrically opposite points on the earth's equator $P(R = a, \theta = \pi/2, \emptyset = 0)$ and $Q(R = a, \theta = \pi/2, \emptyset = \pi)$, and assume that two pulses are sent simultaneously from each point towards the other to make the same path along the equator in the earth's rigid frame s, but in opposite direction. The frame *S* here is the geocentric frame. The resulting Sagnac effect is

$$(18.16)\quad \Delta t_{0,\pi} = \int_P^Q \frac{2\vec{u}.\overrightarrow{dR}}{c^2\sqrt{1-\beta^2}} = \int_0^\pi \frac{2uad\emptyset}{c^2\sqrt{1-\beta^2}} = \frac{2ua\pi}{c^2\sqrt{1-\beta^2}} = \frac{1}{2}\frac{2u\,circum}{c^2\sqrt{1-\beta^2}},$$

which is just half of a full round's effect:

$$(18.17)\quad \Delta t_{0,2\pi} = \frac{2u}{c^2}\frac{circum}{\sqrt{1-\beta^2}} = \frac{4\omega}{c^2}\frac{A}{\sqrt{1-\beta^2}},$$

where a is the radius of the earth, A is the area of equatorial circle and *circum* is its circumference, u the velocity of a point of the equator, and ω is the earth's angular velocity.

(ii). If $\vec{u}$ is constant then the integrated function is a total differential, and the integral depends only on the initial and final points I and F, but not on the path followed. i.e.

$$(18.18)\quad \Delta t_{I,F} = \frac{2\vec{u}.(\vec{R}_F - \vec{R}_I)}{c^2\sqrt{1-\beta^2}}.$$

This vanishes if the path is closed ($\vec{R}_F = \vec{R}_I$) signifying that there is no translative closed Sagnac effect, or if $\vec{R}_F - \vec{R}_I \perp \vec{u}$.

(iii). Suppose that an airplane is flying east-west, law over the equator, at a speed $u = \omega a$ relative to the earth rigid frame s, with ω is the angular velocity of the earth and a is its radius. The plane thus is at rest in the geocentric frame S. In this case a circuit stationary inside the plane will produce no Sagnac effect.

(iv) The motion of a circuit is a pure rotation in S if and if there exists a fixed point, say O in S, such that the velocity of every circuit's element (or vertex if the circuit is a polygon) can be written as $\vec{v} = \vec{\omega} \times \vec{R}$ with $\vec{\omega}$ is a fixed vector, and $\vec{R}$ is the position vector of an arbitrary element relative to O. In this case, the Sagnac effect (18.15) becomes

$$(18.19)\quad \Delta t = \vec{\omega}.\int \frac{2\vec{R}\times\overrightarrow{dR}}{c^2\sqrt{1-\beta^2}} = \frac{4}{c^2}\frac{\vec{\omega}.\vec{A}}{\sqrt{1-\beta^2}},$$

where $\overrightarrow{dR}$ $(= \overrightarrow{dL})$ is a circuit element, and the area's magnitude is represented by the dashed region (Fig.15), which is subtended by the circuit and the radii vectors of its ends. Note that a non-rigid circuit cannot display a purely rotational motion.

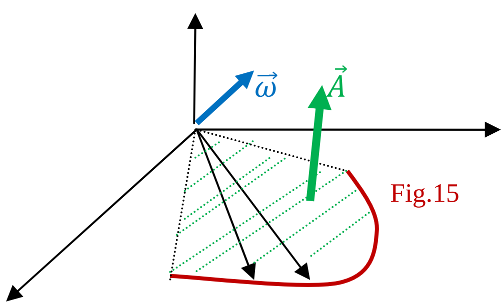

Fig.15

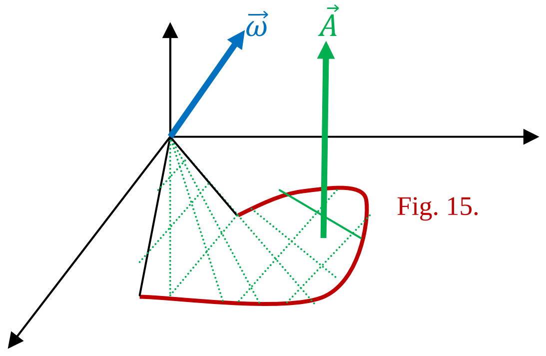

Fig. 15.

(v) For a circuit rotating about a fixed axis in s which is translating relative to S at a constant velocity $\vec{u}$, the velocity of a circuit's element will be $\vec{u} + \vec{\omega} \times \vec{r}$, where $\vec{r}$ is the position vector of the circuit's element relative to a fixed point of the rotation's axis. The Sagnac effect here is

$$\Delta t = \int \frac{2\vec{u}.\overrightarrow{dl}}{c^2\sqrt{1-\beta^2}} + \int \frac{2\vec{\omega} \times \vec{r}.\overrightarrow{dl}}{c^2\sqrt{1-\beta^2}}. \qquad (18.20)$$

where the integral extends on the circuit. If the circuit is **closed** the first term vanishes leaving

$$\Delta t = \vec{\omega}.\oint \frac{2\vec{r} \times \overrightarrow{dl}}{c^2\sqrt{1-\beta^2}} = \frac{4}{c^2}\frac{\vec{\omega}.\vec{A}}{\sqrt{1-\beta^2}}. \qquad (18.21)$$

This shows that *only the rotational motion of a closed circuit counts in Sagnac effect.* The rotational term vanishes only if the axis of rotation is parallel to the circuit's plane.