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Physicists' Blind Faith in Infinity

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Abstract.- Physics never questions the hypothesis of the actual infinity that underlies its mathematical language. This hypothesis, debated for more than 25 centuries, suddenly ceased to be discussed when it was axiomatically accepted (Axiom of Infinity) more than a century ago. Ironically, it was set theory, based on this very hypothesis, that provided the author with the tools to prove its formal inconsistency. This paper denounces the scarce echo of these proofs (some of which were published 16 years ago) and invites the reader to examine one of them, the shortest I have been able to develop. It also points out the extraordinary importance of this inconsistency in a large part of physics, especially in cosmological theories and those that make use of the spacetime continuum. The question, then, is inescapable: what can be done to get physics to consider the possibility of this inconsistency and its consequences?

1 The actual infinity

As is well known, and according to R. Dedekind's classical definition, a set is infinite if, and only if, it can be put into one-to-one correspondence with one of its proper subsets [9, p. 115]. The Axiom of Infinity establishes the existence of one such set. It is therefore not a self-evident axiom. It can be proved immediately that only the actual infinity can be the infinity of the Axiom of Infinity [13, p. 2]. The other infinity, the potential infinity, or the improper infinity as Cantor called it [5, p. 70], has been ignored by mathematics and physics for more than a century. Thus, I will follow here the universal rule of using the word infinity only to refer to the actual infinity; the potential infinity will always be referred to with these two words, "potential" and "infinity."

The difference between the two infinities is very clear and significant: every infinite set exists as a complete totality: a totality of a certain type of elements to which it is not possible to add new elements of that type because it already contains them all. It is not possible, for example, to add new natural numbers to the set of natural numbers because it already contains them all. On the contrary, a potential infinite set would always exist as an incomplete totality of a certain type of elements: a totality to which it is always possible to add new elements of that type because it never contains all the elements of that type. Thus, according to (the actual) infinity, the ordered list of natural numbers in their natural order of precedence exists as a complete ordered list containing all natural numbers, even if there is no last natural number that completes that list. According to the potential infinity, that list can only exist as an incomplete and finite list to which it is always possible to add new natural numbers greater than all the elements in that finite list. But, as indicated above, the potential infinity is completely outside the realm of contemporary science, despite Aristotle's view [1, p. 291], and common sense, that the *incompletable* cannot be *completed*.

However, the ordinary language of physics reveals how far infinity is from physical descriptions: expressions such as contiguous points, adjacent points, point to point, at the next instant, etc., are very common in the physical literature, but all of them are impossible because in densely ordered infinite sets, such as the spacetime continuum, there are neither adjacent points nor adjacent instants: between any two such points (instants) there is always the same number of different points (instants): $2^{\aleph_0}$ points (instants). The Dimension Problem demonstrated by G. Cantor [2, 10, 14, 15, 11, 8, 6, 7] is also never discussed in physical terms: any region of the universe, for example a line one billionth of a millimeter long has as many points as the entire three-dimensional universe. Or: one billionth of a second has the same number of instants as the entire history of the universe. It is then legitimate to affirm that light crosses in one billionth of a second the same number of (space) points as in 13.8 billion years (exactly $2^{\aleph_0}$ points). Or that light takes the same number of (time) instants to travel one billionth of a millimeter as it does to travel 90 billion light-years (exactly $2^{\aleph_0}$ instants). Physicists prefer to accept this kind of things rather than question the formal consistency of the hypothesis that produces them: the hypothesis of actual infinity, according to which the only infinity is the actual infinity; a hypothesis that, like any other, can also be put to the test.

The original publication [13, Link] of the Theorem of Actual Infinity, which is stated and proved in the following section, includes certain results prior to the theorem, but they are not essential to the proof of the theorem itself. It also includes some later results deduced from the proved theorem, of which only their respective statements are shown here, not their proofs. These statements are included so that the reader can appreciate the enormous importance of the inconsistency of infinity proved in this theorem. Obviously, the reader can immediately access the original publication referred to, but they can also access [12, Link], where they will find more than 40 other different proofs of the inconsistent nature of infinity. Why so many demonstrations? Because the infinitist mathematics built on the Axiom of Infinity is, practically, the only mathematics for more than a century, and it is not tolerated that someone questions it, as I have been doing for more than 30 years now. Although I must recognize that things are beginning to change. Anyway, it seems very difficult to respond to more than 120 years of absolutely hegemonic and dominant mathematical infinitism. And to do so from my insignificant academic position. But I have to try because it is not a trivial matter: the inconsistency of infinity changes everything, not only in mathematics but also in a good part of physical theories, especially those that, being committed to the infinitist spacetime continuum, cannot replaced this continuum with a discrete space and time.

2 On the inconsistency of the actual infinity

In the proof of the following theorem, I will use some well-established results from classical set theory: the denumerable nature of the sets $\mathbb{N}$ of the natural numbers, and $\mathbb{Q}$ of the rational numbers [3], the latter being, moreover, densely ordered (between any two rationals other different rationals always exist). The same applies to the open rational interval $(0, 1)$ [4, §9]. The expression "complete totality" (used in the demonstration) refers to a set of a type of elements that contains all elements of that type.

Theorem 1 (of Actual Infinity) *The actual infinity subsumed in the set of all natural numbers and in the rational interval* $(0, 1)$ *is inconsistent.*

Proof.-The open interval of rational numbers $(0, 1)$ is densely ordered in the natural order of precedence (in symbol $<$) defined by the natural values of the rational numbers. It is also a denumerable set, so it can be put in one-to-one correspondence f with the set $\mathbb{N}$ of natural numbers in their natural order of precedence. Consequently, $(0, 1)$ can be rewritten as the set $\mathbb{Q}_{01} = \{q_1, q_2, q_3, \dots\}$, where $q_i = f(i), \forall i \in \mathbb{N}$, and the successive elements $q_1, q_2, q_3, \dots$ of $\mathbb{Q}_{01}$ are ordered by their respective subscripts, and not by their natural values as rational numbers. Obviously, these subscripts are the successive natural numbers of the domain $\mathbb{N}$ of the one-to-one correspondence f between $\mathbb{N}$ and $(0, 1)$. Let now x be a rational variable initially defined as q_1; and let (the current value of) x be $<$-compared (i.e. compared according to the natural values of rational numbers) with all the successive elements of the set $\{q_1, q_2, q_3, \dots\}$, so that x is redefined as q_n if, and only if (iff), $q_n < x$, i.e. iff q_n is LESS than the current value of x. Let us denote by $<$-comparison* this $<$-comparison and redefinition of x iff the element $<$-compared is less than the current value of x. Since, according to the Axiom of Infinity, all elements $q_1, q_2, q_3, \dots$ of $\mathbb{Q}_{01}$ are rational numbers that exist as a complete totality, x can be successively $<$-compared* with ALL OF THEM:

$$\forall n \in \mathbb{N} : \ x \text{ is } <\text{-compared* with } q_n \tag{1}$$

It is immediate to prove that for any natural number v it is possible to perform the first v $<$-comparisons* of x with the first successive v elements of $\mathbb{Q}_{01}$. Indeed, if it were not possible to do so, there would exist at least one natural number $n \leq v$ such that x could not be $<$-compared* with q_n, which is impossible because q_n is a rational number in $\mathbb{Q}_{01}$ that can be $<$-compared* with the current value of x, which is also a rational number. Once all possible $<$-comparisons* of x with the successive elements $q_1, q_2, q_3, \ldots$ of $\mathbb{Q}_{01}$ have been carried out, the current value of x, whatever it is, will be the smallest rational number in that set. Indeed, if once all possible $<$-comparisons* of x with the successive elements of $\mathbb{Q}_{01}$ have been performed, the current value of x were not the smallest rational number in $\mathbb{Q}_{01}$, there would exist at least one element q_n in $\mathbb{Q}_{01}$ such that $q_n < x$. But this is impossible because n is a natural number, the first n $<$-comparisons* have been performed, and then x was $<$-compared* with q_n and redefined as q_n, and in all subsequent $<$-comparisons*, x could only be redefined with values less than q_n. So, it is impossible that $q_n < x$. But, on the other hand, it is also immediate to prove that once all possible $<$-comparisons* of x with the successive elements of $\mathbb{Q}_{01}$ have been performed, the current value of x is not the smallest rational in that set. In effect, each element of the infinite set $\{x/2, x/3, x/4 \ldots\}$ is an element of $\mathbb{Q}_{01}$ less than x. This contradiction proves the assumed actual infinity of the denumerable sets $\mathbb{N}$ and $\mathbb{Q}_{01}$, is inconsistent. □

Comment.- It is important to remember here that the actual infinity of $\mathbb{N}$ implies the existence of the totality of natural numbers, without there being a last natural number completing that totality. And that the set $(0, 1)$ contains all rational numbers greater than zero and less than 1, without there being a first rational greater than 0, nor a last rational less than 1. It seems then reasonable to suppose that the reason for the above contradiction could be the hypothesis that a totality can be complete without a last (first) element completing it, as is assumed by the actual infinity subsumed in the Axiom of Infinity.

From the above theorem we can immediately deduce, among many others, the following results (see [13, link]):

1. **Theorem of the Infinite Sets**: All infinite sets are inconsistent.
2. **Corollary of the Axiom of Infinity**: The Axiom of Infinity is inconsistent.
3. **Corollary of Infinite Divisibility**: The infinite divisibility of any formal or physical object is inconsistent.
4. **Theorem of the Inconsistent Continuum**: The spacetime continuum is inconsistent.
5. **Theorem of Space Units**: No space, physical or abstract, of extent greater than zero can be constituted by points of a null extent, but by units of extent greater than zero.
6. **Theorem of Consistent Reality**: A consistent reality cannot consist of an infinite number of universes; nor can a consistent universe be infinite in extension, duration, number of components, or cycles of creation and destruction.

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