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The Shortest Proof of the Inconsistent Infinity

Antonio Leon
Retired Professor. Independent researcher in the foundations of science.
Links to other author's works.

Abstract.-This article contains the shortest proof I have been able to develop of the inconsistency of the actual Infinity. The proof is based on the dual numerable and densely ordered nature of the rational interval $(0, 1)$, and is a consequence of assuming that there exist all rational numbers greater than zero and less than 1, without there being a first rational number greater than zero (a similar proof could be based on the assumed existence of all rational number less than 1 without there being a last rational number less than 1).

1 On the inconsistency of the actual infinity

I invite the reader to spend five minutes in reading the proof of the following theorem, the importance of which need not be emphasized. Forty other proofs are examined in [1]

Theorem 1 (of Actual Infinity) *The actual infinity subsumed in the set of all natural numbers and in the rational interval* $(0, 1)$ *is inconsistent.*

Proof.- The open interval of rational numbers $(0, 1)$ is densely ordered in the natural order of precedence (in symbol $<$) defined by the natural values of the rational numbers. It is also a denumerable set, so it can be put in one-to-one correspondence f with the set $\mathbb{N}$ of natural numbers in their natural order of precedence. Consequently, $0, 1$ can be rewritten as the set $\mathbb{Q}_{01} = \{q_1, q_2, q_3, \dots\}$, where $q_i = f(i), \forall i \in \mathbb{N}$, and the successive elements $q_1, q_2, q_3, \dots$ of $\mathbb{Q}_{01}$ are ordered by their respective subscripts, and not by their natural values as rational numbers. Obviously, these subscripts are the successive natural numbers of the domain $\mathbb{N}$ of the one-to-one correspondence f between $\mathbb{N}$ and $(0, 1)$. Let now x be a rational variable initially defined as q_1; and let (the current value of) x be $<$-compared (i.e. compared according to the natural values of rational numbers) with all the successive elements of the set $\{q_1, q_2, q_3, \dots\}$, so that x is redefined as q_i if, and only if (iff), $q_i < x$, i.e. iff q_i is LESS than the current value of x. Since, according to the Axiom of Infinity, all elements $q_1, q_2, q_3, \dots$ of $\mathbb{Q}_{01}$ are rational numbers that exist as a complete totality, x can be successively $<$-compared with ALL of them:

$$\begin{aligned} &\forall n \in \mathbb{N}: \ x \text{ is } <\text{-compared with } q_n, \text{ and} \\ &\text{redefined as } q_n \text{ iff } q_n < x \end{aligned} \tag{1}$$

Let us denote by $<$-comparison* this $<$-comparison and redefinition of x iff the element compared is less than the current value of x. It is immediate to prove that for any natural number v it is possible to perform the first v $<$-comparisons* of x with the first successive v elements of $\mathbb{Q}_{01}$. Indeed, if it were not possible to do so, there would exist at least one natural number $n \leq v$ such that x could not be $<$-compared* with q_n, which is impossible because q_n is a rational number in $\mathbb{Q}_{01}$ that can be $<$-compared* with the current value of x, which is also a rational number. Once all possible $<$-comparisons* of x with the successive elements $q_1, q_2, q_3, \dots$ of $\mathbb{Q}_{01}$ have been carried out, the current value of x, whatever it is, will be the smallest rational number in that set. Indeed, if once all possible $<$-comparisons* of x with the successive elements of $\mathbb{Q}_{01}$ have been performed, the current value of x were not the smallest rational number in $\mathbb{Q}_{01}$, there would exist at least one element q_n in $\mathbb{Q}_{01}$ such that $q_n < x$. But this is impossible because n is a natural number, the first n $<$-comparisons* have been performed, and then x was $<$-compared* with q_n and redefined as q_n, and in all subsequent $<$-comparisons*, x could only be redefined with values less than q_n. So, it is impossible that $q_n < x$. But, on the other hand, it is also immediate to prove that once all possible $<$-comparisons* of x with the successive elements of $\mathbb{Q}_{01}$ have been performed, the current value of x is not the smallest rational in that set. In effect, each element of the infinite set $\{x/2, x/3, x/4 \dots\}$ is an element of $\mathbb{Q}_{01}$ less than x. This contradiction proves the assumed actual infinity of the denumerable sets $\mathbb{N}$ and $\mathbb{Q}_{01}$ is inconsistent. □

Bibliographical Reference

[1] A. León. *Infinity put to the test.* Self edition in KDP. Printed at amazon.com. Free pdf, 2023 (2021).