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Infinity, Language, and non-Euclidean Geometries

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Abstract. This article includes a very simple and short proof of the inconsistency of the Hypothesis of the Actual Infinity subsumed in the Axiom of Infinity. And it reminds us that contemporary mathematics, geometry and mathematical physics are infinitist disciplines that assume that axiom. The second part of the article recalls the infinite (in this case potential) regress of arguments, definitions and causes that, among other things, makes inevitable the use of primitive concepts in all sciences. As we will see here, from this necessity derive certain abuses of language that, in the case of non-Euclidean geometries, have ended up giving a very distorted and unreal image of their relationship with Euclidean geometry. It is also recalled that since 2021 there is a new foundational basis of this Euclidean geometry in which the Parallel Postulate can be proved.

Keywords: actual infinity, potential infinity, infinite regress, straight lines, parallelism, equidistance, non-Euclidean geometries, Euclidean geometries.

1 Introduction

Geometries, Euclidean and non-Euclidean, are infinitist: they assume the Hypothesis of the Actual Infinite and its metric consequences. And not only geometries, contemporary physics is also unequivocally infinitist: its mathematical language includes among its foundations the Hypothesis of the Actual Infinity subsumed in the Axiom of Infinity. An axiom that could be inconsistent, although this possibility has not even been contemplated for more than a century. In this article, and after a brief explanation of the differences between the actual infinity and the potential infinity, a short basic demonstration of such inconsistency is provided, and the reader is invited to analyze other different more advanced proofs. Obviously, the inconsistency of the actual infinity would change all things in mathematics, geometry, and mathematical physics.

Related to infinity is the infinite (in this case potential) regress of arguments, definitions and causes, which imposes on human knowledge certain insurmountable limitations that, at least partially, explain the abuses of language common in all scientific disciplines, including formal sciences such as geometry. In the case of geometry, abuses are committed especially in non-Euclidean geometries and in relation to the concepts of straight line and parallelism. Abuses that have never been considered in its two centuries of history and that give a very distorted, even erroneous, image of its relations with Euclidean geometries. The last section of this article considers these abuses and distortions.

2 Actual infinity and potential infinity

Before starting the announced argument about the inconsistency of the actual infinity, which will be developed in the next section of this article, it is worth recalling the Aristotelian distinction between the actual infinity and the potential infinity. As is well known to mathematicians (and somewhat less to physicists), the infinity of contemporary mathematics, defined in set theoretical terms by R. Dedekind [8, p. 115] and G. Cantor [6], and assumed by the Axiom of Infinity, is the actual infinity. It can be proved in formal terms that this is the case (Theorem of Actual Infinity [21, 23]).

According to the Hypothesis of the Actual Infinity, the list L_n of the natural numbers in their natural order of precedence[1] 1, 2, 3,... exists as a complete totality, a totality containing all the natural numbers. The ellipsis (...) in 1, 2, 3,... represents all the natural numbers. All. The word "actual" in actual infinity means, therefore, that all the elements of an infinite collection, such as L_n, exist all of them in the act, forming a *complete totality*. To assume the Hypothesis of Actual Infinity means, therefore, to assume that it is possible to complete the incompletable, as Aristotle would surely say [3, p. 291]. Or that the incomplete can exist as completed. And it is this completion that distinguishes the actual infinite from the potential infinite.

The alternative to the Hypothesis of the Actual Infinity is the Hypothesis of the Potential Infinity, which rejects the existence of complete infinite totalities. In this view, the natural numbers result from the endless process of counting: it is always possible to count a number greater than any previously counted number (Peano's Axiom of the Successor [26, p. 1]). But it is impossible to complete the process of counting them all, simply because there is no last natural number to complete the process. Therefore, the complete list of all natural numbers is meaningless. The word "potential" in potential infinity means, then, that the elements of an infinite collection do not exist all at once, but potentially, as possible. For the potential infinity there is no last natural number (it is always possible to consider a number greater than any number previously considered), but neither is there the complete collection of all natural numbers. We can, therefore, consider the following:

Definition 1 *A strictly increasing collection of integer numbers is infinite if there is no last element that completes it, or a first element that initiates it. The collection is actually infinite if it is considered as a complete totality, and potentially infinite if it is not considered as a complete totality.*

[1] ω-order, being ω the least of the infinite ordinals.

Unless otherwise stated, in the remainder of this article, the word "infinity" will always be used with the meaning of actual infinity.

3 On the inconsistency of the actual infinity

Recall that a supertask consists of performing an infinite number of actions or tasks in a finite interval of time [39, 5, 7, 34, 4, 40, 34]. It is a usual and legitimate theoretical infinitist instrument whose physical possibilities have also been discussed [27, 30, 34, 37, 13, 15, 14, 30, 31, 32, 12, 33, 25, 1, 2, 35, 40, 17, 10, 11, 25, 9, 38], even making use of relativity theories. The following argument is based on one such supertask.

Consider again the list L_n of natural numbers in their natural order of precedence (ω-order). In L_n, each natural number n occupies exclusively its n-th row. Let us then define the following supertask E: let $\langle t_i \rangle$ be a strictly increasing sequence of instants within the finite time interval $[t_a, t_b]$, the mathematical limit of $\langle t_i \rangle$ being the extreme t_b of the interval $[t_a, t_b]$; at each instant t_i of $\langle t_i \rangle$, and only at each instant t_i of $\langle t_i \rangle$, the exchange e_i that exchanges the number 1 of L_n with the number placed in the row r_{i+1} of L_n is carried out, so that at the instant t_i the number 1 is in the $i+1$-th row of L_n, and the number $i+1$ is in the i-th row of L_n. Although unnecessary, we will impose the following restriction R on the supertask E: each of the successive exchanges e_i will be performed if, and only if, the number 1 remains in some row of L_n, otherwise the exchange will not be carried our and the supertask E stops.

Since t_b is the mathematical limit of the sequence of instants $\langle t_i \rangle$, the instant t_b is the first instant after ALL instants of $\langle t_i \rangle$, and therefore the first instant after performing all successive exchanges $\langle e_i \rangle$ of the supertask E, whose number could be finite or infinite according to the restriction R. Assuming the Hypothesis of the Actual Infinity subsumed in the Axiom of Infinity implies assuming the existence as complete totalities of all elements of L_n, $\langle t_i \rangle$ and $\langle e_i \rangle$. Consequently, and since t_b is the mathematical limit of the strictly increasing sequence $\langle t_i \rangle$ whose limit is just t_b, the instant t_b is the first instant after all instants of $\langle t_i \rangle$. Or, what is the same, t_b is the first instant after performing all the successive possible exchanges of the number 1 with each and every one of the successive elements of the original list L_n.

We will now have the occasion to examine the formal consequences of accepting that the list of the natural numbers ordered in their natural order of precedence EXISTS AS A COMPLETE TOTALITY WITHOUT A LAST NUMBER COMPLETING THE LIST. For this purpose let us consider the ordered list $L_n^{1\downarrow}$ of the natural numbers at the instant t_b, after all possible exchanges $\langle e_i \rangle$ have been performed. It will be $L_n^{1\downarrow} = 2, 3, 4, \ldots$. Since each of the exchanges was performed if, and only if, the number 1 was placed in some row of the table L_n, once all of them have been performed the number 1 must be in some row of the resulting table $L_n^{1\downarrow}$. Let r_n be any row of $L_n^{1\downarrow}$ and suppose that the number 1 is in r_n. That the number 1 is in the r_n row of $L_n^{1\downarrow}$ means that the exchange e_n that exchanges the number 1 in the r_n row of L_n with the number $n+1$ located in the $(n+1)$-th row of L_n could not be performed, which is only possible if there is no number in the $(n+1)$-th row of L_n. But according to Peano's Axiom of the Successor [26, p. 1], for every natural number n exists the number $n+1$, which by definition must be placed in the $(n+1)$-th row of L_n. Therefore, n-th exchange e_n was carried out, and the number 1 cannot be in the r_n-row of $L_n^{1\downarrow}$. And being r_n ANY row of $L_n^{1\downarrow}$, we must conclude that the set of rows of $L_n^{1\downarrow}$ in which the number 1 could be found at the instant t_b is the empty set. Thus the number 1, which must be in $L_n^{1\downarrow}$, cannot be in $L_n^{1\downarrow}$. And this is a contradiction, whose only cause is to accept the existence of the complete list of the natural numbers in their natural order of precedence without a last element completing the list. The reader may find another very simple and brief proof in [21] and another 40 somewhat more advanced ones in [24].

4 The inevitable incompleteness of human knowledge

Everything seems to indicate that our observable universe evolves in the direction marked by the continuous increase of its entropy (isotropy). It is, on the other hand, significant that from this inductive principle the following result can be deduced almost immediately [22, 23]:

Theorem 1 (of the Formal Dependence) . *No concept defines itself; no statement proves itself; no physical object is the cause of itself; and no cause is the cause of itself.*

in which one can recognize the Aristotelian infinite regress of arguments, extended also to definitions and causes. Naturally, such restrictions inevitably limit human knowledge, and in a much more general and severe way than Gödel's incompleteness theorems, to which, however, much more attention is paid [20, 18].

Here we are interested in the potentially infinite regress of definitions, which makes the use of primitive (undefinable) concepts inevitable in all sciences, including formal sciences such as geometry. Obviously, since concepts do not define themselves (Theorem of the Formal Dependence), if one were to succeed in defining a primitive concept, the definition would have to include at least another concept that would become the new primitive concept replacing the newly defined one. There is no way to get rid of primitive concepts.

But instead of explicitly admitting the need for indefinable basic concepts, attempts are often made to define them more or less ambiguously, or circularly, or invalid for other reasons. This fact has given rise to a certain chaos in the formal use of the most basic concepts in geometry (and in the rest of the sciences), and of some not so basic ones such as the concept of straight line, for which it is possible to give a formal definition, although based on two primitive concepts: the concept of point and that of line. In the last section of this article we will have the opportunity to prove that the definition of straight line poses a serious conflict in the commonly accepted relations of non-Euclidean geometries with Euclidean geometries, particularly the one developed by Playfair in 1813 [28, 29] and by the author in 2021 [19].

5 Straight lines and parallelism

In Euclid's Elements the following definitions appear [16, p. 153-155]:

1. A point is that of which there is no part.
2. A line is a length without breadth.
3. The extremities of a line are points.
4. A straight-line is a line which lies evenly with the points on itself.

All of them are unsatisfactory: Euclid himself did not use them explicitly in his demonstrations. However, admitting that the straight line is a primitive concept is equivalent to admitting a primitive concept (straight line) that includes another primitive concept (line) that includes another primitive concept (point). Perhaps too many primitive concepts involved in the same concept. As will be seen below, straight lines can indeed be explicitly and uniquely defined, although such a definition is unusual in classical and modern geometries, and could be the reason for the formal conflicts between Euclidean geometry and non-Euclidean geometries.

In J. Playfair's Elements of Geometry [28, 29] we can read [28, p. 8]:

1. A Point is that which has position, but not magnitude.
2. A line is length without breadth.

   Corollary. The extremities of a line are points; and the intersections of one line with another are also points.
3. If two lines are such that they cannot coincide in any two points, without coinciding altogether, each of them is called a straight line.

   Corollary. Hence two straight lines cannot enclose a space. Neither can two straight lines have a common segment; for they cannot coincide in part, without coinciding altogether.

In Playfair's definition of a straight line, an essential characteristic of straight lines, of straightness, already appears: two straight lines cannot have a common segment if they are not part of the same straight line, nor can they enclose a space. And in the foundational basis (29 definitions, 10 axioms and 45 corollaries) of the author's New Elements of Euclidean Geometry[2] we find [19, p. 28-66]:

1. Points and segments that do not belong to the same line are said non-collinear. Non-collinear lines with at least one common segment are said locally collinear.
2. Lines whose segments have all of them the same definition as the whole line are said uniform. Two or more uniform lines are said mutually uniform iff any segment of any of them has the same definition as any segment of any of the others.
3. To extend a given line by a given length is to define a line, said extension of the given line, that is adjacent to the given line, has the given length, and the extension and the extended line are lines of the same class as the given line. Lines that can be extended from each endpoint and by any given length are called extensible lines.
4. **Definition 2** *Straight lines: Extensible and mutually uniform lines that can neither be locally collinear nor have non-common points between common points.*

Therefore, it is possible to give an exclusive definition of straight line not based on metric concepts alien to the nature of lines[3] but on concepts proper to the nature of lines. It is also a functional, productive, definition. That is, a definition that is explicitly used in the demonstrations. Note how "being the line of least length joining two points" does not appear in any of the above definitions of straight line. That is a metrical result that can be deduced, at least in some particular cases [19]. For a general demonstration the reader may consider the demonstrable fact that the straight line AB joining two points A and B has a length less than the sum of the lengths of the other two sides of any triangle ABC; and that the same conclusion applies to the sides AC and CB of any of the triangles constructed on AC and CB; etc.

6 Language abuses in non-Euclidean geometries

In spherical (Riemannian) geometry, the straight lines (or lines of maximum straightness [36, p. 8]) are the great circles (also called geodesics). And parallel straight lines are those that do not intersect each other. Since all the great circles (straight lines in spherical geometry) intersect at two points, it is concluded that in this geometry there are no parallel "straights" lines, and therefore Euclid's Postulate of Parallels is not necessary. The reason why the maximal circles are called straight lines is because they are the lines of least length that join any two points of the sphere that defines the geometry under consideration.

The problem is that the great circles of spherical geometry, their supposedly "straight" lines, do not fulfill the essential and natural properties of Euclidean straight lines, and therefore should not be considered straight lines in any way:

1. Two great circles intersect at two points, and all points between those two common points are non-common points, which is impossible in Euclidean straight lines (see Definition 2 of straight lines given above).
2. Two great circles enclose non-zero surfaces, which is impossible with Euclidean lines.
3. Great circles are not extensible, they all have the

[2] In which the Parallels Postulate is proved.

[3] As is the case of the non-Euclidean definition of straight line as the line that minimizes the distance between any two given points.

same length. Euclidean lines are extensible and can exist with any length and extend (produced) from each endpoint to any given length.

4. There are points outside a Euclidean straight line that are in a straight line with that straight line, and can be joined to it by a straight line forming a new straight line of greater length. This is also impossible with the great circles of spherical geometry.
5. If P is a point between the endpoints A and B of a Euclidean straight line AB, in the direction from P to B the point A is never reached, which is the case if AB is a great circle.
6. Euclidean straight lines remain invariant in shape and position while self-rotating[4], which is not the case with great circles.

   Comment: this condition could be used to define in geometrical terms lines of zero curvature, and include in Definition 2 that straight lines have zero curvature, since it seems reasonable to require that straight lines are not curved lines. I will do so in the next issue of [19]. In such a case the non-Euclidean lines could never be straight since they all have non-zero curvature.

On the other hand, parallel lines (not necessarily straight lines) should not be defined in terms of whether they intersect (non-parallel) or do not intersect (parallel), since there are non-parallel lines that do not intersect each other: any curve with asymptotes and one of its asymptotes. And since parallelism is a metrical relation between different lines rather than a topological property of a line, it makes sense to define parallelism in terms of equidistance. If this is done, and there is no reason not to do so, in spherical geometry there would be a potentially infinite number of parallel lines: all minor circles equidistant from each other.

In the case of hyperbolic geometries the problem is the opposite (apart from calling here also straight lines to curved lines): for a given point there exists an infinity of parallels to a given "straight" line. But here again the problem is solved with the same solution: defining parallelism in terms of equidistance. In this case, from that infinity of parallels to a given line (so called because they do not cut the given line) all those that are not equidistant would have to be eliminated, and only one would remain.

Non-Euclidean geometries have always been presented as alternatives to Euclidean geometry in which the Postulate of Parallels does not exist. But in reality what do not exist are the Euclidean straight lines, the lines of zero curvature. And there are no problems with parallel lines either if parallelism is defined in terms of equidistance. Furthermore, and taking into account that in the New Elements of Euclidean Geometry the Postulate of Parallels is proved as a theorem, to continue insisting on the non-existence of parallels or on the existence of more than one parallel through a given point to a given line will imply rejecting some of the 10 axioms of those New Elements, axioms which are the following:

Axiom 1.-Point, line and surface are primitive concepts of which any number, and in any arrangement, can be considered and drawn.

Axiom 2.-A line has at least two points, at least one point between any two of its points, and at most two endpoints, whether or not in the line.

Axiom 3.-Two adjacent lines make a line, and a point of a line can be common to any number of any other different lines, either collinear, or non-collinear, or locally collinear.

Axiom 4.-Being not a figure, each point of a line, except endpoints, has just two sides in that line, whose lengths are greater than zero and sum the length of the whole line.

Axiom 5.-Any two points can be the endpoints of a straight line, and only both points are necessary to draw the straight line.

Axiom 6.-Any three points lie in a plane, in which any straight line has two, and only two, sides. Any other line is in one of such sides iff its endpoints are in that side.

Axiom 7.-The distances from the points of a line to a fixed point or to another line vary in a continuous way. The distances from a point to itself and to a line to which it belongs are zero.

Axiom 8.-Any point in a plane can be the center of a circle of any radius, and its complementary arcs are each on a different side of its chord.

Axiom 9.-It is possible for two adjacent straight lines to make any angle at their common endpoint. The angle is zero iff both straight lines are superposed.

Axiom 10.-The area of a polygon is greater than zero, and is the sum of the areas of the two adjacent polygons defined by any of its divisors. Equal polygons have equal areas.

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