Thomson Lamp Formalized

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Abstract.-The discussions on Thomson's lamp can be formalized (at least up to a certain point) by introducing a simple symbolic notation that allows to define the lamp and its functioning in abstract terms. The symbolic definition can then be used to develop formulas that represent the functioning laws of the lamp. Being independent of the number of times the lamp is turned on/off, these laws represent the universal attributes and the universal behaviour of a Thomson's lamp. As we will see, some of those laws are not compatible with the assumption that a Thomson's lamp can be switched infinitely many times during a finite interval of time. This conclusion proves that, as its author defended, Thomson supertask could be inconsistent.

1 Symbols and definitions

The symbols '*' and 'o' will be used to represent the lamp is on and off respectively. The clicks will be represented with the letter 'c'. We will also use standard symbols of logic and mathematics. So we will write (TL denotes Thomson's lamp):

TL is on at instant t: *[t] (1)

TL is off at instant t: o[t] (2)

TL is on along the interval (t_a, t_b): *(t_a, t_b) (3)

TL is off along the interval (t_a, t_b): o(t_a, t_b) (4)

Being on, TL is cliked at instant t: c$\{[t], *\}$ (5)

Being off, TL is cliked at instant t: c$\{[t], o\}$ (6)

Being on, TL is cliked at least one time along (t_a, t_b): c$\{(t_a, t_b), *\}$ (7)

Being off, TL is cliked at least one time along (t_a, t_b): c$\{(t_a, t_b), o\}$ (8)

TL is not clciked since t_b: $\neg c\{[t_b, \infty)\}$ (9)

Note the expressions 'Being on' and 'Being off', and recall that in the spacetime continuum no instant has an immediate preceding (or succeeding) instant: between any two instants, however close they may be, there are another $2^{\aleph_o}$ instants, the same number of instants as in the entire history of the universe ($\approx$ 13800 millions years). We can now formalize the definition of Thomson's lamp by means of the following four axioms:

$$\text{Thomson's lamp} \begin{cases} c\{[t], o\} \Rightarrow *[t] \\ c\{[t], *\} \Rightarrow o[t] \\ *[t] \vee o[t] \\ \neg(*[t] \wedge o[t]) \end{cases} \tag{10}$$

Some basic laws of Thomson's lamp can now be immediately established, for example:

$$c\{(t_a, t_b), o\} \Rightarrow \exists t \in (t_a, t_b) : *[t] \tag{11}$$

$$c\{(t_a, t_b), *\} \Rightarrow \neg * (t_a, t_b) \tag{12}$$

$$o[t_b] \Rightarrow \neg * [t_b, \infty) \tag{13}$$

$$* [t_a, t_b] \Rightarrow \neg c\{(t_a, t_b)\} \tag{14}$$

$$c\{[t], o\} \Rightarrow \neg o\{[t, \infty)\} \tag{15}$$

etc. (16)

2 Discussion

Consider the following two laws: Thomson lamp BT1 and BT2 laws

$$\text{BT1: } c\{(-\infty, t_b), *\} \wedge *[t_b, \infty) \Rightarrow \exists t \leq t_b : \ c\{[t], o\} \wedge \neg c\{(t, \infty), *\} \tag{17}$$

$$\text{BT2: } c\{(-\infty, t_b), o\} \wedge o[t_b, \infty) \Rightarrow \exists t \leq t_b : \ c\{[t], *\} \wedge \neg c\{(t, \infty), o\} \tag{18}$$

The first law (BT1) reads: if the lamp's button has been clicked at least once within the interval $(-\infty, t_b)$, the lamp being previously on, and the lamp stays on from t_b, then there is an instant t equal or prior to t_b such that the button is clicked at t, the lamp being previously off, and the button is no longer clicked from t. The second law (BT2) reads equal except we must replace *on* with *off* and vice versa.

Let us now prove BT1 (BT2 would be proved in a similar way). Assume that:

$$\neg\exists t \leq t_b : \ c\{[t], o\} \tag{19}$$

We can write:

$$\neg c\{(-\infty, t_b], o\} \tag{20}$$

Taking into account the antecedent of BT1 we have:

$$c\{(-\infty, t_b), *\} \Rightarrow \exists t < t_b : \ c\{[t], *\} \tag{21}$$

and then:

$$o[t] \tag{22}$$

From (20) and (22), and taking into account that $t < t_b$ we deduce:

$$o[t_b] \tag{23}$$

and then:

$$\neg * [t_b, \infty) \tag{24}$$

which goes against the second term of the antecedent of BT1. Therefore if that antecedent is true then assumption (19) is false.

Assume now that it holds:

$$\neg\exists t \leq t_b : \ \neg c\{(t, \infty), *\} \tag{25}$$

We will have:

$$c\{[t_b, \infty), *\} \tag{26}$$

which goes against the second term $*[t_b, \infty)$ of BT1 antecedent. Consequently, if this antecedent is true then assumption (25) must be false.

The falsehood of assumptions (19) and (25) proves BT1. It is worth noting that BT1 is not derived from the successively performed clicks but from the laws defining Thomson's lamp. Thus, if we assume the Principle of Invariance (PI), BT1 must always hold: before, during and after the performing of any finite or infinite sequence of clicks.

Consider Thomson's supertask $\langle c_n \rangle$, being each click c_i performed at the precise instant t_i of the strictly increasing sequence of instants $\langle t_n \rangle$ within (t_a, t_b) and whose limit is t_b. Assume the state S_b of the lamp at t_b is *on* (a similar argument could be developed if it were *off* though making use of BT2 in the place of BT1). In these conditions the antecedent of BT1 would be true: the lamp has been clicked at least once along the interval (∞, t_b) and it is on from t_b. Therefore BT1 consequent must also be true. We will now prove, however, it is not. Indeed, on the one hand, if $t < t_b$, and with t_b being the limit of the sequence $\langle t_n \rangle$, there would exist a t_v in the sequence $\langle t_n \rangle$ such that:

$$t_v \leq t < t_{v+1} \tag{27}$$

and therefore only a finite number v of clicks would have been performed. On the other hand, the instant t cannot be the limit t_b either, because at t_b the button of the lamp has not been clicked. Consequently, t cannot be an element of $(t_a, t_b]$. Therefore, to perform Thomson's supertask implies the violation of BT1, which goes against the Principle of Invariance. Thomson supertask seems to be inconsistent.

References

[1] Antonio León and Ana C. León, *Supertasks, Physics and the Axiom of Infinity*, Truth, Objects, Infinity. New Perspectives on the Philosophy of Paul Benacerraf. (Fabrice Pataut, ed.), Springer, Switzerland, 2017, pp. 223–259.