27 Supertasks sets and boxes

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Abstract.-By making use of sequences of definitions and supertasks, this chapter discusses on sets and boxes containing infinitely many elements. The discussion leads to several contradictory results pointing to the inconsistency of the hypothesis of the actual infinity.

Keywords: supertask, inconsistency of the actual infinity.

Introduction: sets and boxes

P499 From the platonic point of view (the dominant perspective in contemporary mathematics), all attempts to define the concept of set have been circular, so that it is now considered a primitive notion, a concept that cannot be defined in terms of other more basic concepts.

P500 From a non-platonic point of view, however, it is possible to define the notion of set as a mental construct. For instance, Charles Dogson (better known as Lewis Carroll) proposed the following concept [2, p. 31]:

> Classification, or the formation of Classes, is a Mental Process, in which we imagine that we have put together, in a group, certain Things. Such a group is called a Class.

Carroll's notion of class leads immediately to the following definition:

> *A set is a theoretical object that results from a mental process of grouping arbitrary objects previously defined.*

It could be proved this definition is not compatible with self-reference, one of the main sources of inconsistency in naive (Cantorian) set theory. But this type of non-platonic definitions are ignored in contemporary mathematics. Some of them will be introduced in Appendix 39.

P501 We could imagine a set as a sort of box that contains objects. And while the number of objects is finite the comparison will always be consistent. But when the number of objects is infinite some significant differences appear between sets and boxes. As we will see in this chapter the consideration of an infinite set as a box that contains infinitely many objects leads to contradictions.

Emptying sets and boxes

P502 Consider a box BX containing an ω-ordered collection $\langle b_i \rangle$ of identical balls indexed as b_1, b_2, $b_3, \dots$ And consider also an ω-ordered set $B = \{b_1, b_2, b_3 \dots\}$ whose elements are also a denumerable collection of identical balls indexed as b_1, b_2, $b_3, \dots$.

P503 From the set B let us define the following ω-ordered sequence of sets $\langle B_n \rangle$:

$$\begin{cases} B_1 = B - \{b_1\} \\ B_i = B_{i-1} - \{b_i\}, \quad i = 2, 3, 4, \dots \end{cases} \tag{1}$$

$\langle B_n \rangle$ is, therefore, the sequence of nested sets:

$$B_1 \supset B_2 \supset B_3 \supset \dots \tag{2}$$

each of whose members $B_n = \{b_{n+1}, b_{n+2}, b_{n+3}, \dots\}$ is a denumerable set.

P504 Let now (t_a, t_b) be a finite interval of time and $\langle t_n \rangle$ an ω-ordered and strictly increasing sequence of instants within (t_a, t_b) whose limit is t_b. Assume that at each instant t_i of $\langle t_n \rangle$ the ball b_i is removed from the box BX. Let $BX(t_i)$ be the state of the box (the remaining collection of balls within the box) at the instant t_i, just the instant at which the ball b_i has been removed from the box. The successive states $\langle BX(t_i) \rangle$ of the box BX can be symbolically expressed in a form similar to (1):

$$\begin{cases} BX(t_1) = BX(t_a) - b_1 \\ BX(t_i) = BX(t_{i-1}) - b_i, \; i = 2, 3, 4, \dots \end{cases} \tag{3}$$

P505 The one to one correspondence $f(t_i) = b_i$ proves that at t_b all balls will have been removed from the box, and BX will be empty. By comparing (1) with (3) we will have:

$$BX(t_i) = B_i, \forall i \in \mathbb{N} \tag{4}$$

P506 There is, however, a fundamental difference between the sequence of sets $\langle B_n \rangle$ and the sequence of states $\langle BX(t_i) \rangle$ of the box BX: in each of the successive states $BX(t_i)$ defined by (3), the box BX is always the same box BX, while the successive sets B_i defined by the successive definitions (1) are different from one another. As a consequence we will have a final empty box BX but not a final empty set. How is this possible? Why and when the symmetry between both sequences of definitions (sets and boxes) get broken?

P507 On the other hand, and regarding the sequence of states $\langle BX(t_i)\rangle$ of the box BX defined by (3), it is worth noting that at *each* instant t in (t_a, t_b) the box BX contains $\aleph_o$ balls, whereas at t_b it is empty. In fact, since t_b is the limit of the sequence $\langle t_n\rangle$, we will have:

$$\forall t \in (t_a, t_b) : \ \exists v : \ t_v \leq t < t_{v+1} \tag{5}$$

Therefore, at t only the first v balls $b_1, b_2, \ldots\ b_v$ have been removed from BX, and BX still contains infinitely many balls $b_{v+1}, b_{v+2}, b_{v+3}, \ldots$ So then, at each instant t within (t_a, t_b) the box BX contains $\aleph_o$ balls. Or in other words, if T is the set of all instants of the interval of time (t_a, t_b) at which the box BX contains $\aleph_o$ balls, the complement $\overline{T}$ of T in $(t_a, t_b]$ can only be the singleton $\{t_b\}$.

P508 In these conditions, the only way for the box BX to become empty at t_b would be by removing simultaneously infinitely many balls just at t_b. How is this possible if at t_b no ball is removed from the box? How is this possible if all balls have been removed *one by one*, and with an interval of time greater than zero between any two successive removals? How is it possible that, in those conditions, and for any natural number n, the box *never* contains $n \ldots$, 3, 2, 1 balls? And recall we are not subtracting cardinals (Chapter 21) but removing one by one the balls from a box that contains balls (see P321).

P509 Let us go a step further in this discussion. Consider the following sequence of definitions of the sets X and Y by means of the above ω-ordered sequence of sets $\langle B_n\rangle$:

$$i = 1, 2, 3 \ldots \begin{cases} B_i \neq \emptyset \Rightarrow X = B_i \\ Y = B_2 \end{cases} \tag{6}$$

While the sequence of definitions (6) of the set Y poses no problem and we will finally have $Y = B_2$, the successive definitions (6) of the set X poses the following problem: Definitions (6) can only leave X defined as the empty set, otherwise only a finite number of definitions would have been performed, because any element b_n in X would be proving the nth redefinition (that defines X as $\{b_{n+1}, b_{n+2}, b_{n+3}, \ldots\}$) would not have been carried out. The problem is that no definition (6) defines X as the empty set, simple because all sets B_i of $\langle B_n\rangle$ are denumerable. All.

P510 An interesting variant of the above argument is the following one. Let $A_1 = \{a_1, a_2, a_3, \ldots\}$ be a denumerable set and consider the following sequence of definitions of the set B:

$$i = 1, 2, 3 \cdots : \text{iff } |A_i| > 1 \text{ then } \begin{cases} A_{i+1} = A_i - \{a_i\} \\ A_i = A_{i+1} \\ B = A_i \end{cases} \tag{7}$$

According to (7), B is defined as A_i if, and only if, the cardinal of A_i is equal or greater than 1. Therefore, (7) can only define B as a singleton $\{a_\nu\}$. But this is impossible because, having being successively defined according to the ω-order of the successive indexes $1, 2, 3, \dots$, the index ν of a_ν could only be an impossible last natural number.

The last ball supertask

P511 The above set theoretical argument P510 can be reanalyzed by means of a conditional supertask S_{bx}. Indeed, consider again the same above box BX with the same collection of indexed balls $\langle b_n \rangle$, and the same sequence of instants $\langle t_n \rangle$ within (t_a, t_b). Let the conditional supertask S_{bx} be defined according to:

> *At each precise instant t_i of $\langle t_n \rangle$, remove from BX the ball b_i if, and only if, the box BX contains at least two balls.*

Note, the successive balls are removed from BX *one by one*, one after the other, and in such a way that a time greater than zero always elapses between two successive removals: $\Delta t = t_{i+1} - t_i > 0, \forall i \in \mathbb{N}$. And note also the balls are successively removed from BX according to the ω-order of their respective indexes 1, 2, 3,. . .

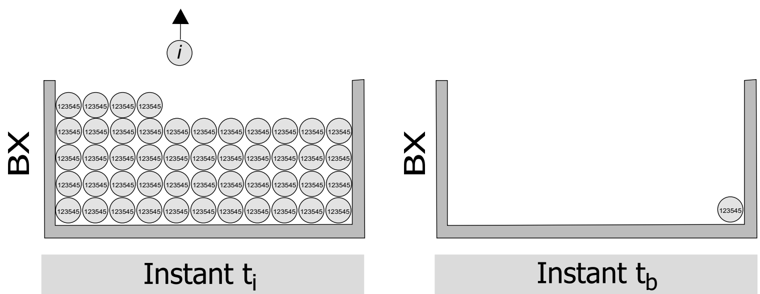

Figura 27.1 – The last ball supertask S_{bx}: remove from BX the ball b_i at t_i if, and only if, BX contains at least two balls.

P512 The one to one correspondence f between $\langle t_n \rangle$ and $\langle b_n \rangle$ defined by $f(t_i) = b_i$ proves that, being t_b the limit of $\langle t_n \rangle$, at t_b the supertask S_{bx}

has been completed. Indeed, for all $i \in \mathbb{N}$, it is always possible to remove from BX the ball b_i at the instant t_i iff BX contains at least two balls. But, on the other hand, the completion of S_{bx} is impossible because it can only left one ball within BX, and that ball could only be a ball indexed by an impossible last natural number. S_{bx} leads, then, to a contradiction: it can, and cannot, be completed.

P513 Consider now the following variant S'_{bx} of S_{bx}:

At each successive instant t_i of $\langle t_n \rangle$ remove from BX any ball b_k.

In this case, it is immediate to prove that at t_b the supertask S'_{bx} has been completed, leaving one ball b_p within BX, where the index p is any natural number (in the place of the impossible last natural number of S_{bx}). That BX contains the unique ball b_p is, in fact, a possible result for S'_{bx}. Therefore, in this sense S'_{bx} is not contradictory.

P514 In consequence, we must conclude that it is possible, and it is not possible, to remove from BX one by one all balls but one of $\langle b_n \rangle$, depending on the order they are removed: if they are removed at random, the removal is possible; if they are removed following the ω-ordered sequence of their respective indexes, the removal is impossible. It is hard to accept that, being it possible a random removal, the removal is impossible if the balls are removed by following the ω-order of their respective indexes.

P515 The supertask S'_{bx} poses an additional problem related to the instant at which it takes place the removal of the last ball that leaves BX with only one ball. Indeed, let t be any instant within the finite interval of time (t_a, t_b). Being t_b the limit of the sequence $\langle t_n \rangle$, it holds:

$$\exists t_v \in \langle t_n \rangle : t_v < t < t_{v+1} \tag{8}$$

So that, at t only a finite number v of balls have been removed from BX. Consequently, if T is the set of all instants of (t_a, t_b) at which BX contains infinitely many balls, then the complement $\overline{T}$ of T in (t_a, t_b) can only be the singleton $\{t_b\}$. Hence, the last removal that left BX with only one ball inside it, could only take place at t_b, just the first instant at which no removal takes place; and that removal had to remove from BX infinitely many balls at once, which obviously goes against the own definition of the supertask S'_{bx}.

P516 As in previous chapters of this book, the above contradictory results deduced from the supertasks S_{bx} and S'_{bx} point to the same suspicious hypothesis: the hypothesis of the actual infinity; the belief that an infinite list exist as a complete totality without a last element completing the list; the believing that it is possible to complete the incompletable, as Aristotle would surely say [1, p. 291]. A hypothesis, on the other hand, subsumed into the Axiom of Infinity founding infinitist mathematics, the main, and *almost* unique, stream in contemporary mathematics.

Catching a fallacy

P517 Consider again the collection of indexed balls $\langle b_n \rangle$. We can consider a denumerable set B whose elements are the collection of balls $\langle b_n \rangle$. We can also consider a box BX that contains all of them. But could we consider a hollow cylinder AB, with the same diameter as the balls, that contains the same collection of balls $\langle b_n \rangle$? Obviously, in this case the balls could only be aligned in straight line, one after the other, just as the sequence of the natural numbers 1, 2, 3,... Naturally, both the box and the cylinder would have to have infinite sizes, but the existence of such objects can be assumed without that assumption affecting the arguments that such containers illustrate (Principle of Autonomy).

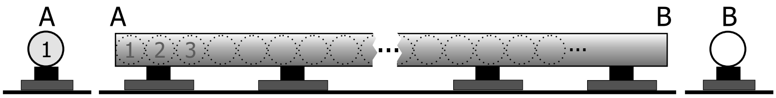

Figura 27.2 – The hollow cylinder AB containing the collection of balls $\langle b_n \rangle$ as a complete totality. The cylinder appears occupied when observed from its end A with the ball b_1 at sight. But it appears empty when observed from its end B, otherwise we would be observing the impossible last ball of an ω-ordered collection of balls.

P518 From the point of view of the hypothesis of the actual infinity, the answer to the question posed in P517 can only be negative: the cylinder AB would appear occupied when observed from its end A with the ball b_1 at sight, and empty when observed from its end B, otherwise the impossible last ball of an ω-ordered sequence of balls (the collection $\langle b_n \rangle$) would be at sight.

P519 In consequence, while we can consider the set B of the ω-ordered collection of balls $\langle b_n \rangle$, and the box BX with the ω-ordered collection of balls $\langle b_n \rangle$ inside it, we cannot consider the hollow cylinder AB with the same ω-ordered collection of balls $\langle b_n \rangle$ inside it. Or in other more general words, the possibility to consider an ω-ordered collection of objects inside a container depends on the shape of the container. Some shapes, as the hollow cylinder AB, cannot be permitted under penalty of inconsistency.

P520 Ridiculous as it may seem, axiomatic set theories should face the above inconsistency [P519]. They would have to include a new axiom restricting the shapes of the containers capable of containing ω-ordered sequences of objects. For example, the hollow cylinder above would have to be declared inconsistent as a container of the balls $\langle b_i \rangle$.

P521 Or, alternatively, the hollow cylinder AB could be considered a trap to catch a fallacy: the fallacy of completing the incompletable; the fallacy of the existence of ω-ordered lists of elements as complete totalities without a last element completing the lists.

Chapter References

[1] Aristóteles, *Metafísica*, Espasa Calpe, Madrid, 1995.

[2] Lewis Carroll, *El juego de la Lógica*, 6 ed., Alianza, Madrid, 1982.