12 An irrational source of rational numbers

(Draft chapter of the book *Infinity Put to the Test* by Antonio León[1])

Abstract.-This chapter introduces the expofactorial and n-expofactorial numbers: finite numbers of colossal size which are not anodyne sequences of zeroes but precise sequences of different numerals whose standard writing would be strings of numerals of a length millions of times greater than the diameter of the visible universe. It also introduces the method of the successive expansions from which it is possible to define a different rational number from each different irrational number within the real interval $(0, 1)$.

12.1 n-Expofactorial numbers

1. This chapter introduces the expofactorial and the n-expofactorial numbers, as well as the method of the successive decimal expansions by means of which it is possible to define a different rational number from the infinite decimal expansion of each irrational number within the real interval $(0, 1)$. Evidently, this conclusion goes against other well known results on the cardinality of the set Q of the rational numbers.

2. Although the method of the successive decimal expansions we will make use of in the next section works with natural numbers of any size, we will use natural numbers unimaginably large: the n-expofactorials numbers defined in **5**.

3. The expofactorial[2] of a natural number n, written $n^!$ (note the factorial symbol '!' appears as exponent), is the factorial $n!$ raised to a power tower of order $n!$ of the same exponent $n!$:

$$n! \tag{1}$$
$$(\overset{n!}{\cdot\cdot\cdot}) \tag{2}$$
$$n! \tag{3}$$
$$n! \tag{4}$$
$$n^! = n! \tag{5}$$

[1]Next publication

[2]The first time I considered this type of numbers, I didn't know they have already been defined by C. A. Pickover ([1] cited in [3]) with the name of *superfactorials* and the symbols n$, the same name and symbols used by Sloane and Plouffe to define $n\$ = \prod_{k=1}^{n} k!$ [3]. That said, I will retain my original notation and name.

Or in Knuth's notation:

$$n^{!} = n! \uparrow\uparrow n! \tag{6}$$

4. These numbers growth so rapidly that while the expofactorial of 2 (in symbols $2^{!}$) is 16, the expofactorial of 3 (in symbols $3^{!}$) is practically incalculable even with the aid of the most powerful computers:

$$3^{!} = 6^{6^{6^{6^{6^{6^{6}}}}}} = 6^{6^{6^{6^{6^{46656}}}}} = 6^{6^{6^{6^{26591197721532267796824894043879185949053422002699 24 \ldots}}}}$$

where the incomplete exponent of the last term on the right has nothing less than 36306 digits, a string of figures over seven meters long, if each figure is 5mm. And there still remains four steps to go. Indeed, the expofactorial of any natural number greater than 2 is so large that it is practically incalculable (it is not an anodyne power of ten but a precise sequence of different figures).

5. Expofactorials are insignificant compared with n-expofactorials, recursively defined from expofactorials as follows: the 2-expofactorial of a natural number n, denoted by $n^{!2}$, is the expofactorial $n^{!}$ raised to a power tower of order $n^{!}$ of the same exponent $n^{!}$; the 3-expofactorial of n, denoted by $n^{!3}$, is the 2-expofactorial of n ($n^{!2}$) raised to a power tower of order $n^{!2}$ of the same exponent $n^{!2}$; the 4-expofactorial of n, denoted by $n^{!4}$, is the 3-expofactorial of n ($n^{!3}$) raised to a power tower of order $n^{!3}$ of the same exponent $n^{!3}$; and so on:

$$n^{!2} = n^{!^{n^{!^{\left(\overset{n^{!}}{\cdots}\right)^{n^{!}}}}}} \qquad n^{!3} = n^{!2^{n^{!2^{\left(\overset{n^{!2}}{\cdots}\right)^{n^{!2}}}}}} \qquad n^{!4} = n^{!3^{n^{!3^{\left(\overset{n^{!3}}{\cdots}\right)^{n^{!3}}}}}} \qquad \ldots$$

Or in Knuth's notation:

$$n^{!2} = n^{!} \uparrow\uparrow n^{!} \tag{7}$$
$$n^{!3} = n^{!2} \uparrow\uparrow n^{!2} \tag{8}$$
$$n^{!4} = n^{!3} \uparrow\uparrow n^{!3} \tag{9}$$
$$n^{!5} = n^{!4} \uparrow\uparrow n^{!4} \tag{10}$$
$$\ldots$$

The *grandeur* of, for example, $9^{!9}$ (9-expofactorial of 9) is far beyond human imagination. Three standard arithmetic symbols, just $9^{!9}$, is all we need to define a *finite* number so large that the standard writing of its precise sequence of figures would surely be a string of numerals of a length billions of times greater than the diameter of the visible universe. If we use the hexadecimal numeral system, $F^{!F}$ would be inconceivable greater.

6. The discussion that follows makes use of the 9-expofactorial of 9. For simplicity, it will be denoted by the letter 'h' (for huge). So, in what follows h will stand for $9^{!9}$.

12.2 An irrational source of rational numbers

7. The real numbers within the interval $(0, 1)$ with an infinite decimal expansion are arithmetically defined as

$$r = 0.d_1 d_2 d_3 \dots \tag{11}$$
$$= d_1 \times 10^{-1} + d_2 \times 10^{-2} + d_3 \times 10^{-3} + \dots \tag{12}$$

where the sequence of decimals digits $d_1 d_2 d_3 \dots$ is ω-ordered, as the set $\mathbb{N}$ of the natural numbers in their natural order of precedence 1, 2, 3, ...

8. In accordance with the hypothesis of the actual infinity, subsumed into the Axiom of Infinity, the infinite decimal expansion $0.d_1 d_2 d_3 d_4 \dots$ of any real number (with an infinite decimal expansion) within the real interval (0, 1) does exist as a complete ω-ordered totality: it has a first decimal digit (decimal hereafter), d_1, and each decimal d_n (except d_1) has an *immediate predecessor* d_{n-1} and an *immediate successor* d_{n+1}, so that no last decimal exists, and where immediate predecessor (successor) means that no other decimal exists between any two successive decimals d_n, d_{n+1} (ω-discreteness). Since the argument that follows deals exclusively with ω-ordered infinities, from now on, and for simplicity, they will be referred to simply as infinities.

9. A point of note is that ω, the ordinal of the ω-ordered sequences, is *the smallest infinite ordinal.* Therefore, if r and s are two real numbers within the real interval $(0, 1)$ and they coincide in their first successive ω decimals, then both numbers are identical. On the contrary, and taking into account that every ordinal less than ω is finite, if r and s are different then they can only coincide in a finite number of their first successive decimals.

10. Let $\mathbb{N}$ be the ω-ordered set of the natural numbers, h the 9-expofactorial of 9 (in symbols $9^{!9}$), and m_α any element of the set M_I of the irrational numbers within the real interval (0, 1). The exclusive decimal expansion of m_α:

$$m_\alpha = 0.d_1 d_2 d_3 \dots \tag{13}$$

defines the following ω-ordered sequence $\langle q_\alpha, nh\rangle$ of rational numbers:

$$q_{\alpha,h} = 0.d_1 d_2 \dots d_h \tag{14}$$
$$q_{\alpha,2h} = 0.d_1 d_2 \dots d_h d_{h+1} \dots d_{2h} \tag{15}$$
$$q_{\alpha,3h} = 0.d_1 d_2 \dots d_h d_{h+1} \dots d_{2h} d_{2h+1} \dots d_{3h} \tag{16}$$
$$\dots$$
$$q_{\alpha,nh} = 0.d_1 d_2 \dots d_h d_{h+1} \dots d_{2h} d_{2h+1} \dots d_{3h} d_{3h+1} \dots d_{nh} \tag{17}$$
$$\dots$$

being $q_{\alpha,nh}$ (for every n in $\mathbb{N}$) the rational number within (0, 1) whose finite decimal expansion $0.d_1 d_2 \dots d_{nh}$ coincides with the first nh decimals of m_α. For this reason, m_α will be said the *source* of the sequence $\langle q_\alpha, nh\rangle$, and α will appear as a part of the subindex of each $q_{\alpha,nh}$. The rational $q_{\alpha,(n+1)h}$ will be said the h-expansion of the rational $q_{\alpha,nh}$ because $q_{\alpha,nh}$ is expanded with the next h successive decimals (starting from d_{nh+1}) of the source m_α in order to define $q_{\alpha,(n+1)h}$. Don't forget the unimaginable grandeur of $h = 9^{!9}$.

11. From the perspective of the actual infinity hypothesis, the result of defining the infinitely many natural numbers by adding infinitely many successive times one unit to the first natural number (1+1, 2+1, 3+1,...), is a set of infinitely many increasing finite numbers, without ever reaching an infinite number.[3] Or in other words, infinitists defend that by adding to a first unit an infinite number of successive units we never reach an number of infinite size but infinitely many finite numbers, each one unit greater than its immediate predecessor. The same would happen if instead of a unit we add any finite number of units. Even h units.

12. Consequently, and being h a natural number, the result of defining the infinitely many elements of $\langle q_\alpha, nh\rangle$ by adding infinitely many successive times h new decimals to the decimal expansion of $q_{\alpha,h}$, yield infinitely many decimal expansions, explosively increasing but always finite: $nh \in \mathbb{N}$ for each $n \in \mathbb{N}$ because the semiring $(\mathbb{N}, +, *)$ is closed with respect to addition and multiplication. Therefore, all of them will be rational numbers.

13. This infinitist consequence will be essential for the next argument since it legitimates the actual existence of the *infinitely many* rational numbers

[3] The recursive definition of the natural numbers in set theoretical terms leads to the same conclusion.

in $\langle q_\alpha, nh\rangle$, all of them with *finitely many decimals*, nh for each n in $\mathbb{N}$. In the same way $\mathbb{N}$ contains infinitely many finite natural numbers, each of them one unit greater than its immediate predecessor, $\langle q_\alpha, nh\rangle$ contains infinitely many rational numbers with a finite decimal expansion, each with h decimals more than its immediate predecessor. This is, in fact, infinitist orthodoxy.

14. Let P be the set of *all* pairs $(m_\alpha,\, q_{\alpha,h})$ whose first component is a different element m_α of the set M_I of the irrational numbers in (0, 1), and whose second component is the rational number $q_{\alpha,h}$ within (0, 1) defined by the first h successive decimals $d_1, d_2, \dots d_h$ of m_α:

$$(m_\alpha,\, q_{\alpha,h}) \in P \Leftrightarrow \begin{cases} m_\alpha = 0.d_1d_2\dots d_hd_{h+1}\dots \in M_I \\ q_{\alpha,h} = 0.d_1d_2\dots d_h \end{cases} \tag{18}$$

Although the first element m_α of each pair is a different irrational number, the second one $q_{\alpha,h}$ will be repeated a certain number of times in the different pairs of P.

15. Notice that if there is no irrational number in $(0, 1)$ with the same first h decimals, then the second element of each pair of P would be a different rational number. In these conditions the discussion that follows would be unnecessary: there would be as many rationals as irrationals within $(0, 1)$.

16. Let now $q_{\alpha,h}$ be any of the repeated rationals in P, and let P_α be the subset of P of all pairs $(m_\varphi,\, q_{\varphi,h})$ whose second rational component $q_{\varphi,h}$ coincides with $q_{\alpha,h}$:

$$P_\alpha = \{(m_\varphi,\, q_{\varphi,h}) \,|(m_\varphi,\, q_{\varphi,h}) \in P \;\wedge\; q_{\varphi,h} = q_{\alpha,h}\,\} \tag{19}$$

For simplicity, the repeated rational numbers in P_α will be called p-repetitions.

17. By definition, the irrational numbers of all pairs of P_α are irrationals numbers within $(0, 1)$ with the same first h decimals. Obviously, some of these numbers will also have the first $2h$ decimals and some will not (change, for instance, any decimal $d_{(h+i)0<i\leq h}$ in any irrational in $(0, 1)$ and you will get an irrational with the same first h decimals but not with the same $2h$ decimals). Of the first ones, some will have the first $3h$ decimals and some will not. And so on.

18. In accord with **17**, if we replace each repeated rational in P_α with its h-expansion the number of p-repetitions would decrease. And if we replace the remaining repeated rationals with their corresponding h-expansions, the number of p-repetitions would decrease again. And so on. The problem is that after each h-replacement we would have a new set P'_α, P''_α ... and we could not demonstrate if the repeated rationals disappear or not (see Chapter 13). To avoid this problem we will have to redefine the set P_α after each h-replacement.

19. Each pair $(m_\varphi, q_{\varphi,h})$ of P_α defines a sequence$\langle q_\varphi, nh\rangle$ of rational numbers similar to the sequence $\langle q_\alpha, nh\rangle$ defined in **10**, except in that the source is now the irrational number m_φ in the place of m_α. The assumed actual existence, all at once, of the infinitely many decimals of the ω-ordered decimal expansion of any irrational number in $(0, 1)$ as a complete totality, legitimates the definitions of the sets P, P_α, as well as the sequences $\langle q_\varphi, nh\rangle$, all of them as complete totalities.

20. Let A be any set of pairs of numbers (a, b) whose first component a is an irrational number within the real interval $(0, 1)$ and whose second component b is a rational number within the same real interval $(0, 1)$. Let us define the following two set operators:

1) $D(A)$ = set of all pairs of A whose rational components are different, not repeated.

2) $R(A)$ = set of all pairs of A whose rational components are repeated.

Evidently:

$$A = D(A) \cup R(A) \tag{1}$$
$$D(A) \cap R(A) = \emptyset \tag{2}$$

21. Consider now the following sequence of (re)definitions of the set P_α:

$$n = 1, 2, 3, \dots \begin{cases} \text{If } R(P_\alpha) = \emptyset \text{ Then End. Else:} \\ \quad P^d_\alpha = D(P_\alpha) \\ \quad P^r_\alpha = \{(m_\varphi, q_{\varphi,(n+1)h}) \mid (m_\varphi, q_{\varphi,nh}) \in R(P_\alpha)\} \\ \quad P_\alpha = P^d_\alpha \cup P^r_\alpha \end{cases} \tag{3}$$

In each definition (3) of the set P_α its repeated rationals are replaced with their corresponding h-expansions. For this reason definitions (3) will be called h-replacement. In agreement with **17**, in each h-replacement the number of repeated rationals in P_α decreases.

22. We will now try to prove that, by successive h-replacements, it is possible to replace each repeated rational in P_α with a different rational within the interval $(0, 1)$.

23. Let us assume that while $R(P_\alpha) \neq \emptyset$ and P_α can be h-replaced, it is h-replaced in accordance with (3). Once all possible h-replacements have been carried out, there will be two mutually exclusive alternatives regarding $R(P_\alpha)$ (the subset of P_α of all pairs with repeated rationals):

1.- $R(P_\alpha)$ is not empty.

2.- $R(P_\alpha)$ is empty.

Consider the first alternative: $R(P_\alpha)$ is not empty. We know that for each element $(m_\lambda, q_{\lambda,vh})$ in $R(P_\alpha)$ there is an ω-ordered sequence $\langle q_\lambda, nh\rangle$ of rationals with a finite decimal expansion. So that each $(m_\lambda, q_{\lambda,vh})$ in $R(P_\alpha)$ can be replaced with $(m_\lambda, q_{\lambda,(v+1)h})$. Consequently a new h-replacement of P_α is possible, which contradict the fact that, being $R(P_\alpha) \neq \emptyset$, all possible h-replacements of P_α have been carried out. Therefore, and by Modus Tollens, The first alternative is false and then, once performed all possible h-replacements of P_α the set $R(P_\alpha)$ is empty.

24. Note that argument **23** has nothing to do with constructive reasonings based on the successively performed h-replacements. It is a single Modus Tollens: once performed all possible h-replacements, the hypothesis that $R(P_\alpha)$ is not empty leads to the contradictory conclusion that not all possible h-replacements have been carried out. That hypothesis must be, therefore, false.

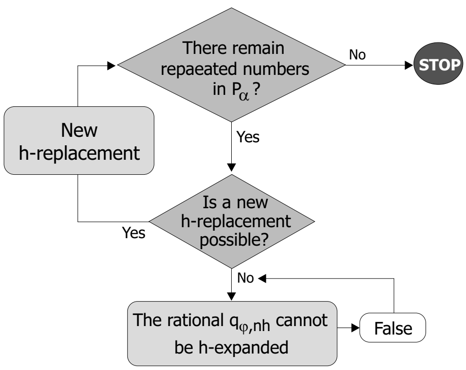

Figura 13.1 – The consequences of being a complete sequence without a last element completing the sequence.

25. Argument **23** takes advantage of the fact that, in accord with the hypothesis of the actual infinity, ω-ordered sequences do exist as complete totalities in which *each element* has infinitely many successors (Figure 13.1). This assumption, makes it possible to ensure that while P_α contains p-repetitions, i.e. while $R(P_\alpha)$ is not empty, the repeated numbers can be replaced with their corresponding successive h-expansions by means of successive h-replacements of P_α. And that this sequence of h-replacements can *actually be completed* because of the *actual completeness* of each infinite sequence $\langle q_\varphi, nh \rangle$. Consequently, only when P_α no longer contains p-repetitions, i.e. when $R(P_\alpha)$ is empty, it will be possible to ensure that all possible h-replacements have been carried out (under penalty of contradiction).

26. By contrast, from the potential infinity perspective the existence of completed infinite totalities without a last element that completes them, makes no sense. Thus, from this perspective we are not legitimated to consider the completion of the sequence of h-replacements if this sequence is potentially infinite.

27. Once removed all p-repetitions, the resulting numbers can only be rational numbers with a finite decimal expansion since all elements of all sequences $\langle q_\varphi, nh \rangle$ are rational numbers with a finite decimal expansion, for the same reason that each of the infinitely many natural numbers is a finite number.

28. In accordance with the definition **16** of P_α, the rational numbers resulting from the removal of all p-repetitions cannot be repeated in the set $P - P_\alpha$ because all rational numbers in this last set differ from the rationals of P_α in at least one of their first h decimals.

29. The above argument **16-28** can be applied to any other repeated rational in the set P of pairs $(m_\alpha, q_{\alpha,nh})$. In consequence, all repeated rationals can be replaced with a different rational number derived from the decimal expansion of the first irrational component of the pair. In these conditions each pair of P will be formed by a different irrational number m_α and a different rational number q_α. The one to one correspondence f defined by:

$$f(m_\alpha) = q_\alpha \tag{4}$$

would be proving the set of rationals and the set of irrationals in $(0, 1)$ have the same cardinality.

13.1 Discussion

30. The hypothesis of the actual infinity subsumed into the Axiom of Infinity legitimizes the following line of reasoning on which argument **14-29** is grounded:

30-1. The infinitely many decimals of the decimal expansion of any irrational number within $(0, 1)$ do exist as an actual complete totality.

30-2. The infinite decimal expansions of the irrational numbers in $(0, 1)$ are ω-ordered, being ω the smallest infinite ordinal.

30-3. Two different irrational numbers in $(0, 1)$ can only coincide in a finite number of their first successive decimals.

30-4. The infinitely many h-expansions $\langle q_\varphi, nh \rangle$ defined from the decimal expansion of each irrational m_φ in the real interval $(0, 1)$ do exist as an actual complete totality.

30-5. Each of the infinitely many h-expansions of $\langle q_\varphi, nh \rangle$ is a rational number with finitely many decimals: nh for each n in $\mathbb{N}$.

30-6. In accordance with 30-4 and 30-5, the repeated rationals of P_α can be successively replaced with their corresponding successive rational h-expansions any finite or infinite number of times.

30-7. In these conditions, and by Modus Tollens **23**, all p-repetitions can be removed from P_α, and then from P, so that each pair will finally be formed by a different irrational and a different rational derived from its irrational partner.

30-8. Consequently each irrational number within (0, 1) defines a different rational number within the same interval.

31. Conclusion **30-8** contradicts other well known results on the cardinality of the set of the rational numbers.

32. To define rational numbers, and ω-ordered sequences of rational numbers, from the decimal expansion of the irrational numbers leads to some other contradictory results we have not dealt with here.

13.2 Epilog

33. As it has been repeatedly said, from the perspective of the actual infinity hypothesis, the infinitely many decimals of a real number with an infinite decimal expansion do exist as a complete ω-ordered totality. In consequence, to consider that a real number *does exist* as the complete totality of its infinitely many decimals, means to consider that number is either a mind-independent entity, or an unverifiable assumption, because human mind cannot embrace the actual infinity (we can not even imagine numbers as $9^{!9}$, which are minuscule compared to the actual infinitude of for instance $\aleph_o$). Thus, from the infinitist perspective, all real numbers would be (platonic) mind-independent entities.

34. From the hypothesis of the potential infinity, however, an irrational number is not a mind-independent entity formed by a complete ω-ordered sequence of decimals that exist all at once and by themselves. From this hypothesis, irrational numbers result from endless calculations that cannot be replaced with a division between two integers, although at each stage of the calculation the number coincides with a rational number of finitely many decimals. In this sense the irrational numbers are also definable as (potentially infinite) sequences of rational numbers.

35. In the case of the rational numbers the calculations can be replaced with a division between two integers, which is not necessarily endless. In its turn, integer numbers would result from the endless process of counting. Naturally, the existence of endless processes of counting and calculations does not necessarily mean the existence of their corresponding finished results as complete totalities, as is assumed from the infinitist point of view.

36. We must decide which of the two alternatives is the most appropriate to found a theory of numbers. And the election is not irrelevant: we need mathematics to explain the world. Think, for example, of the problems posed by the actual infinity in certain areas of physics, as quantum electrodynamics (*renormalization*) or quantum gravity [2]. Or the assumed dense ordering of the *continuum* spacetime[1] versus the discontinuous nature of ordinary matter, electric charge or energy.

[1]Founded on the assumed uncountable cardinality $2^{\aleph_o}$ of the real numbers.

Chapter References

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[2] Lee Smolin, *The trouble with physics*, Allen Lane. Penguin Books, London, 2007.

[3] Eric W. Weistein, *Superfactorial*, Eric Weisstein World of Mathematics, Wolfram Research Inc., http://mathworld.wolfram.com, 2009.