7 A rational inconsistency

(Draft chapter of the book *Infinity Put to the Test* by Antonio León[1])

Abstract.-This chapter proves an inconsistency in the set of the rational numbers as a consequence of the fact that, being densely ordered, the rational numbers can also be ω-ordered by means of any bijection with the set of the natural numbers.

Keywords: rational numbers, dense order, ω-order, recursive definitions.

7.1 Introduction

1. The set $\mathbb{Q}$ of the rational numbers, in its natural ordering, is densely ordered: between any two rational numbers infinitely many different rational numbers do exist. But, being denumerable [1], $\mathbb{Q}$ can also be ω-ordered, so that between any two successive rational numbers no other rational number does exist. The argument that follows makes use of this attribute of the rational numbers.

7.2 Discussion

2. For the sake of simplicity, we will deal with the set $\mathbb{Q}^+$ of the positive rational numbers greater than zero, which is also denumerable and densely ordered. Let then f be a one to one correspondence between the set $\mathbb{N}$ of the natural numbers and $\mathbb{Q}^+$. It is evident that f makes it possible to ω-order the elements of $\mathbb{Q}^+$ so that they can be written as $\{q_1, q_2, q_3, \dots\}$, being $q_i = f(i),\ \forall i \in \mathbb{N}$.

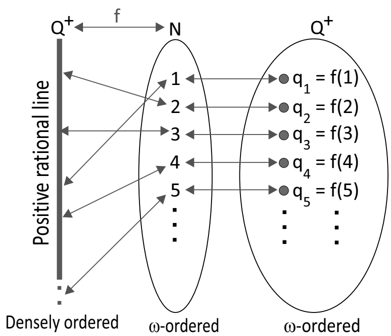

Figura 7.1 – ω-Ordering the positive rational line.

[1]Next publication

3. Let x be a rational variable whose domain is the rational interval $(0, 1)$ and whose initial value x_o is any rational number within $(0, 1)$. Consider the following sequence $\langle D_i(x)\rangle$ of recursive definitions of x:

$$\begin{cases} D_1(x) = x_o \\ D_i(x) = \text{mín}(D_{i-1}(x), |q_i - q_1|), \;\; i = 2, 3, 4, \ldots \end{cases} \tag{1}$$

where $D_i(x)$ is the ith definition of x; $\text{mín}(D_{i-1}(x), |q_i - q_1)|)$ is the smallest (in the usual dense ordering of $\mathbb{Q}$) of the two values in brackets; and $|q_i - q_1|$ is the absolute value of $q_i - q_1$. So, the successive recursive definitions $\langle D_i(x)\rangle$ define x as $|q_i - q_1|$ if $|q_i - q_1|$ is less than $D_{i-1}(x)$, or as $D_{i-1}(x)$ if it is not.

4. Definitions, procedures and proofs consisting of infinitely many successive steps, as definition (1), are usual in infinitist mathematics (see, for instance, Cantor 1874 argument, or Cantor ternary set, later in this book). Unnecessary as it may seem, we will impose to the successive definitions $\langle D_i(x)\rangle$ the following:

> **Restriction 4**-Each successive definition $D_i(x)$ will be carried out if, and only if, x results defined as a positive rational number within its domain $(0, 1)$.

In what follows we will say a definition $D_i(x)$ is possible if, and only if, it satisfies the above restriction.

5. By induction, it is immediate to prove that for each natural number v, the first v successive definitions $\langle D_i(x)\rangle_{i=1,2,\ldots v}$, can be carried out. Evidently $D_1(x)$ can be carried out since $D_1(x) = x_o$, and $x_o \in (0, 1)$. Assume that, being n any natural number, the first n successive definitions $\langle D_i(x)\rangle_{i=1,2,\ldots n}$ can be carried out, which means x is defined with a certain value $D_n(x)$ within its domain $(0, 1)$. Since $|q_{n+1} - q_1|$ is a well defined positive rational number it will be, or not, less than $D_n(x)$. Consequently $D_{n+1}(x)$ defines x as $|q_{n+1} - q_1|$ if this number is less than $D_n(x)$ or as $D_n(x)$ if it is not. In any case $D_{n+1}(x)$ defines x within its domain $(0, 1)$. Therefore, the first $(n + 1)$ successive definitions $\langle D_i(x)\rangle_{i=1,2,\ldots n+1}$ can be carried out. Hence, for any natural number v, the first v successive definitions $\langle D_i(x)\rangle_{i=1,2,\ldots v}$ can be carried out

6. We will begin by proving that once performed all possible[2] successive definitions $\langle D_i(x)\rangle$, the rational number $q_1 + x$ is not the smallest rational

[2]Note that if it were impossible to perform all possible successive definitions $\langle D_i(x)\rangle$ we would be in the face of the elementary contradiction of an impossible possibility

greater than q_1. In fact, whatsoever be the value of x once performed all possible successive definitions $\langle D_i(x)\rangle$, the rational number $q_1 + 0{,}1 \times x$, for instance, is greater than q_1 and less then $q_1 + x$. Notice this argument is a consequence of the natural dense ordering of $\mathbb{Q}^+$.

7. We will prove now, however, that once performed all possible successive definitions $\langle D_i(x)\rangle$, the rational number q_1+x is the smallest rational greater than q_1. In effect, assume that once performed all possible successive definitions $\langle D_i(x)\rangle$ the rational number $q_1 + x$ is not the smallest rational greater than q_1. In such a case there would be a positive rational q_v greater than q_1 and less than $q_1 + x$:

$$q_1 < q_v < q_1 + x \tag{2}$$

and then, by subtracting q_1 to the three members (all of them proper rational numbers) of the above two inequalities, we will have:

$$0 < q_v - q_1 < x \tag{3}$$

which is impossible because:

a) The index v of q_v is a natural number.

b) In accordance with P5, it is possible to perform the first v successive definitions $\langle D_i(x)\rangle_{i=1,2,\ldots v}$.

c) All possible successive definitions $\langle D_i(x)\rangle$ have been carried out.

d) So, at least the first v successive definitions $\langle D_i(x)\rangle_{i=1,2,\ldots v}$ have been carried out.

e) As a consequence of $D_v(x)$, we can assert that $x \leq q_v - q_1$.

f) It is then impossible that $x > q_v - q_1$.

In consequence our initial hypothesis must be false and $q_1 + x$ is the smallest rational number greater than q_1. Notice this amazing conclusion is a legitimate consequence of the ω-order of $\mathbb{Q}^+$ induced by the one to one correspondence f defined in P2. Indeed, it is that correspondence what makes it possible to consider *in a successive way, and one by one*, all rational numbers q_i in Q^+ and then to calculate, one by one, all $|q_i - q_1|$.

8. Once completed the sequence of all possible definitions $\langle D_i(x)\rangle$, the defined variable x could have been defined an infinite number of times without a last definition. For this reason it will be impossible to know the current value of x once completed the sequence of definitions $\langle D_i(x)\rangle$.

But, in any case, x will continue to be a rational variable properly defined within its domain $(0,1)$ (Principle of Invariance, see Chapter **??**). Thus, indeterminable as its current value may be, x will continue to be a rational variable properly defined within its domain $(0,1)$. And this is all we need in order to make the above argument conclusive.

9. We can, therefore, conclude that once performed all possible definitions $\langle D_i(x)\rangle$, the rational variable x is and is not a rational variable defined within its rational domain $(0,1)$.

10. Otherwise, if after completing the sequence of definitions $\langle D_i(x)\rangle$, the rational variable x had lost its condition of being a rational variable properly defined in its domain $(0,1)$, we would have to admit that the completion of an infinite sequence of successive definitions, as such a completion, has additional and arbitrary effects on the defined object (which goes against the Principle of Invariance). But if that were the case, the same *additional arbitrary effects* could be expected from any other definition, procedure or proof consisting of an infinite sequence of successive steps, and then anything could be expected from infinitist mathematics.

11. We could even timetable the sequence of definitions $\langle D_i(x)\rangle$ by performing each definition $D_i(x)$ at the precise instant t_i of the ω-ordered and strictly increasing sequence of instants $\langle t_n\rangle = t_1,\ t_2,\ t_3 \ldots$ within the finite interval (t_a, t_b), whose limit is t_b. In these conditions, x could only lose its condition of rational variable properly defined within its domain $(0,1)$ at the precise instant t_b, the first instant *after* having completed the sequence of definitions $\langle D_i(x)\rangle$. In fact, being t_b the limit of $\langle t_n\rangle$ we will have:

$$\forall t \in (t_a, t_b) : \ \exists v : t_v \leq t < t_{v+1} \tag{4}$$

$$\therefore \ \text{at } t,\ x \text{ is well defined by } D_v(x) \tag{5}$$

and then, at every instant t within (t_a, t_b), x is a well defined rational variable within its rational domain $(0,1)$.

12. Therefore, if S is the set of all instants within the closed interval $[t_a, t_b]$ at which x is a rational variable defined within its domain $(0,1)$, the complement S' of S in $[t_a, t_b]$ is just t_b. In consequence only at the precise instant t_b, the first instant *after* having completed the sequence of definitions $\langle D_i(x)\rangle$, could x lose its condition of rational variable properly defined within its domain $(0,1)$.

13. Thus, we would have to admit not only that completing a sequence of infinitely many successive definitions, all of them possible, has additional and arbitrary effects on the defined object, but also that those effects unexpectedly appear after completing the sequence of definitions. And the same would apply to any other definition, procedure or proof composed of infinitely many successive steps.

Chapter References

[1] Georg Cantor, *Über eine eigenschaft aller reallen algebraishen zahlen*, Journal für die reine und angewandte Mathematik **77** (1874), 258–262.