6 Extending Cantor paradox

(Draft chapter of the book *Infinity Put to the Test* by Antonio León[1])

Abstract.-This chapter discusses Cantor's paradox of the set all cardinals, and proves that in Cantor's set theory every set of cardinal C originates at least 2^C inconsistent infinite sets.
Keywords: Cantor Paradox, Buralli-Forti's Paradox, inconsistencies in the foundations of naive set theories.

6.1 Introduction

1. Cantor's paradox is not a paradox but a true inconsistency related to the set of all cardinals. This is why that set is explicitly rejected in modern axiomatic set theories. The following discussion proves, however, that not only the set of all cardinals is inconsistent, it proves that in Cantor's naive set theory a set of cardinal C originates at least 2^C inconsistent infinite sets, where 'infinite' refers to the *actual infinite*.

2. Although Burali-Forti was the first to publish the proof of an inconsistency derived from the existence of an infinite set [1], [6], Cantor was the first to discover one of those infinitist paradoxes (inconsistencies): the maximum cardinal paradox [6], [4]. There is no agreement regarding the date Cantor discovered his paradox [6] (the proposed dates range from 1883 [9] to 1896 [7]). Burali-Forti paradox (inconsistency) on the set of all ordinals and Cantor paradox on the set of all cardinals are both related to the size of the considered totalities, perhaps too big as to be consistent according to Cantor. It seems somewhat ironic that an infinite set may be inconsistent just because of its excessive size. By the way, note we use the euphemism 'paradox' to denote what really is an inconsistency, i.e. a pair of contradictory terms that surely derive from a common precedent hypothesis. From which hypothesis? Perhaps from the hypothesis of the actual infinity according to which infinite sets do exist as complete totalities?

3. Indeed, the simplest explanation for both paradoxes is that they really are inconsistencies derived from the hypothesis of the actual infinity, i.e.

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from assuming the existence of infinite sets as complete totalities. But no one has dared to analyze this alternative. It was finally accepted that some infinite totalities (as the totality of the real numbers) do exist while others (as the totality of cardinals, or the totality of ordinals) do not because they lead to contradictions.

6.2 Cantor paradox

4. The easiest and shortest version of Cantor paradox[2] (for a detailed analysis see [6, pp. 66-74]) goes as follows: Let U be the set of all sets, the so called universal set,[3] and $P(U)$ its power set, the set of all its subsets. Let us denote by $|U|$ and $|P(U)|$ their respective cardinals. Being U the set of *all* sets it must contain all sets and then we can write:

$$|U| \geq |P(U)| \tag{1}$$

On the other hand, and according to Cantor's theorem on the power set [3], it holds:

$$|U| < |P(U)| \tag{2}$$

which contradicts (1). Equations (1)-(2) represent our simplified version of Cantor's paradox.

5. As is well known, Cantor gave no importance to that inconsistency [5] and clinched the argument by assuming the existence of two types of infinite totalities, the consistent and the inconsistent ones [2]. As noted above, in Cantor's opinion the inconsistency of those infinite totalities would be due to their excessive infinitude. In fact, we would be in the face of the mother of all infinities, the absolute infinity which, according to Cantor, leads directly to God, being just the divine nature of this absolute infinitude what makes it inconsistent for our poor human minds [2].

6. As we will immediately see, it is possible to extend Cantor paradox to other sets much more modest than the set of all sets. But neither Cantor nor his successors considered such a possibility. We will do it here. This is just the objective of the discussion that follows. A discussion that will take place within the framework of Cantor naive, and then non axiomatized, set theory.

[2]Note that, usual as it may be, the expression 'Cantor's paradox' is at least confusing since it is not a paradox but a true contradiction.

[3]Cantor's naive set theory admits sets as the universal set U that are forbidden in modern axiomatic theories.

6.3 An extension of Cantor's paradox

7. Since in naive set theory the elements of a set can be sets, sets of sets, sets of sets of sets and so on, we will begin by defining the following binary relation $\mathcal{R}$ between two sets: we will say that a set A is $\mathcal{R}$-related to a set B, written $A \mathcal{R} B$, if B contains at least one element which forms part of the definition of at least one element of A. For instance, if:

$$A = \{\ \{\{a,\ \{b\}\}\},\ \{c\},\ d,\ \{\{\{\{e\}\}\}\}, f\} \tag{3}$$

$$B = \{1, 2, b\} \tag{4}$$

$$C = \{1, 2, 3\} \tag{5}$$

then A is $\mathcal{R}$-related to B because the element b of B forms part of the definition of the element $\{\{a,\ \{b\}\}\}$ of A, while A is not R-related to C because no element of C is involved in the definition of A's elements.

8. In these conditions, let X be any non empty set and Y any of its subsets. From Y we define the set $T_{\bar{Y}}$ according to:

$$T_{\bar{Y}} = \{Z \mid \neg\exists V(V \cap Y \neq \emptyset \wedge Z \mathcal{R} V)\} \tag{6}$$

$T_{\bar{Y}}$ is, therefore, the set of all sets Z that are not $\mathcal{R}$-related to any set V containing one or more elements of the set Y. Notice that if $Y = \emptyset$ then $T_{\bar{Y}}$ is the inconsistent universal set.

9. It is immediate to prove that $T_{\bar{Y}}$ is infinite. In fact, let n be any finite natural number and assume $|T_{\bar{Y}}| = n$. Let x be any element of any set of $T_{\bar{Y}}$. By definition of $T_{\bar{Y}}$, the sets $\{x\}$, $\{\{x\}\}$, $\{\{\{x\}\}\}$,. . . , $\{\{_{\overset{n+1}{\dots}}\{\{x\}\}_{\overset{n+1}{\dots}}\}\}$ belong all of them to $T_{\bar{Y}}$, and then $|T_{\bar{Y}}| > n$. Therefore the cardinal $|T_{\bar{Y}}|$ can only be infinite.

10. Let us now consider the set $P(T_{\bar{Y}})$, the power set of $T_{\bar{Y}}$. The elements of $P(T_{\bar{Y}})$ are all of them subsets of $T_{\bar{Y}}$ and therefore sets of sets that are not $\mathcal{R}$-related to sets that contain elements of the set Y:

$$\forall D \in P(T_{\bar{Y}}):\ \neg\exists V(V \cap Y \neq \emptyset \wedge D \mathcal{R} V) \tag{7}$$

Consequently, it holds:

$$\forall D \in P(T_{\bar{Y}}):\ D \in T_{\bar{Y}} \tag{8}$$

And then:

$$P(T_{\bar{Y}}) \subseteq T_{\bar{Y}} \tag{9}$$

Accordingly, we can write:

$$|P(T_{\bar{Y}})| \leq |T_{\bar{Y}}| \tag{10}$$

11. On the other hand, and in accordance with Cantor's theorem it holds:

$$|P(T_{\bar{Y}})| > |T_{\bar{Y}}| \tag{11}$$

Again a contradiction. But now X is any non empty set and Y any of its subsets. Therefore, and taking into account that every set of cardinal C has 2^C different subsets, we have proved the following:

> **Theorem 11 (of Cantor Paradox).**-In Cantor's set theory, every set of cardinal C gives rise to at least 2^C inconsistent infinite sets.

12. The above argument not only proves the number of inconsistent infinite totalities is much greater than the number of consistent ones, it also suggests the excessive size of the sets could not be the cause of the inconsistency. Consider, for example, the set X of all sets whose elements are exclusively defined by means of the natural number 1:

$$X = \{1, \mathbf{\{}1\mathbf{\}},\ \mathbf{\{}1, \{1\}, \{1, \{1\}\}\mathbf{\}},\ \mathbf{\{}\{\{1\}\}\mathbf{\}},\ \mathbf{\{}\{\ 1, \{1\}\ \}\mathbf{\}} \dots \} \tag{12}$$

An argument similar to **8-11** would immediately prove it is an inconsistent infinite totality, although compared with the universal set it is an insignificant totality.[4]

13. Notice that sets as the set X defined by (12) are inconsistent only when considered from the perspective of the actual infinity, i.e. when considered as *complete* totalities. And recall that from the potential infinite point of view these sets make no sense because from this perspective the only *complete* totalities are the finite totalities, as large as we wish but always finite.

14. Had we known the existence of so many inconsistent infinite sets, and not necessarily so great as the absolute infinity, and perhaps Cantor transfinite set theory would have been received in a different way. Perhaps the very notion of the actual infinity would have been put into question in set theoretical terms; and perhaps we would have found the way to prove it is an inconsistent notion. But, as we know, this was not the case.

15. The history of the reception of set theory and the way of dealing with its inconsistencies (all of them promoted by the actual infinity hypothesis and by self-reference) is well known. From the beginnings of the XX

[4]Recall, for instance, that between any to real numbers an uncountable infinitude ($2^{\aleph_o}$) of other different reals numbers do exist. What, as Wittgenstein would surely say, makes one feel dizzy [11]

century a great deal of effort has been carried out to found set theory on a formal basis free of inconsistencies. Although the objective could only be accomplished with the aid of the appropriate axiomatic patching. At least half a dozen of axiomatic set theories have been developed ever since.[5] Some hundreds of pages are needed to explain in detail all axiomatic restrictions of contemporary axiomatic set theories. Just the contrary one could expect from the axiomatic foundation of a formal science.

16. As noted above, the simplest explanation of Cantor and Burali-Forti inconsistencies is that they are true contradictions derived from the inconsistency of the hypothesis of the actual infinity. The same applies to the set of all sets, and to the set of all sets that are not member of themselves (Russell paradox), although in this case there is an additional cause of inconsistency related to self-reference. All sets involved in the paradoxes of naive set theory were finally removed from the theory by the opportune axiomatic restrictions. No one dared to suggest the possibility that those paradoxes were in fact contradictions derived from the hypothesis of the actual infinity; i.e. from assuming the existence of infinite sets as complete totalities.

17. What is really true is that Cantor set of *all* cardinals, Burali-Forti set of *all* ordinals, the set of *all* sets, and Russell set of *all* sets that are not members of themselves, are all of them inconsistent totalities when considered from the perspective of the actual infinity hypothesis. Even Turing's famous halting problem is related to the hypothesis of the actual infinity because it also assumes the existence of all pairs (programs, inputs) as a complete infinite totality [10]. Under the hypothesis of the potential infinity, on the other hand, none of those totalities makes sense because from this perspective only finite totalities can be considered, indefinitely extensible, but always finite.

18. As noted above, Cantor (or Burali-Forti) paradox is not a paradox but an inconsistency, a pair of contradictory terms:

$$\begin{cases} |U| \geq |P(U)| \\ |U| < |P(U)| \end{cases} \tag{13}$$

Recall that we are discussing within the framework of Cantor's naive set theory, where axiomatic restrictions had not yet been established. In those conditions, the contradictory terms (13) can only derive from some previous inconsistent assumption. And the only assumption to get (13) is the

[5] There are also some contemporary attempts to recover naive set theory [8]

hypothesis of the actual infinity. It is then striking Cantor's conclusion that (13) is a consequence of the *excessive infinitude* of the involved set. Any thing but to put in question his profound infinitist convictions, *as firm as a rock*.

Chapter References

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