9 Cantor's 1874 argument revisited

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Abstract.-This chapter explains in detail the first Cantor's proof of the uncountable nature of the real numbers, and discusses the conditions under which it could also be applied to the set of the rational numbers.
Keywords: Cantor's 1874 argument, cardinal of the set of real numbers, cardinal of the set of rational numbers.

Introduction

P157 In 1874, 17 years before the publication of his famous diagonal argument, Cantor proved for the first time the set of the real numbers cannot be denumerable. That early Cantor's proof is one of the objectives of this chapter. The other is the analysis of the conditions under which that proof could also be applied to the set of the rational numbers. It will necessary, therefore, to prove such conditions can never be satisfied in order to ensure the impossibility of a contradiction on the cardinality of the set of the rational numbers, which was proved to be numerable by Cantor himself in the same publication [2]. A conflicting rational variant of Cantor's argument is also discussed at the end of the chapter.

Cantor's 1874-argument

P158 This section explains in detail the first Cantor's proof of the uncountable nature of the set $\mathbb{R}$ of the real numbers, published in the year 1874 in a short paper [2] that also included a proof of the denumerable nature of the set $\mathbb{A}$ of the algebraic numbers and then of the set of the rational numbers $\mathbb{Q}$, a subset of $\mathbb{A}$ (English edition [1], French edition [3], Spanish edition [4]).

P159 Assume the set $\mathbb{R}$ is denumerable. In such a case, there would be at least one bijection between the ω-ordered set $\mathbb{N}$ of the natural numbers and $\mathbb{R}$. Let f be any of such bijections. The elements of $\mathbb{R}$ would be reordered by f in the sequence $\langle r_i \rangle$ (Theorem of the Indexed Sets):

$$\langle r_i \rangle = r_1,\ r_2,\ r_3,\ \dots \tag{1}$$

being $r_i = f(i), \forall i \in \mathbb{N}$. Obviously, the sequence $\langle r_i \rangle$ defined by f *would contain* all real numbers if $\mathbb{R}$ were actually denumerable, and it would be possible to consider all of them successively and one by one. This *one by one* consideration is the basis of Cantor's proof.

P160 Consider now any real interval (a, b). Cantor's 1874 argument consists in proving the existence of a real number s in (a, b) which is not in the sequence $\langle r_i \rangle$. The existence of s would prove that $\langle r_i \rangle$ does not contain all real numbers. Therefore, the one to one correspondence f, whatsoever it be, would be impossible. And the initial assumption on the denumerable nature of $\mathbb{R}$ would be false. The proof goes as follows.

P161 Starting from r_1, find the *first two* elements of $\langle r_i \rangle$ within (a, b). Denote the smaller of them by a_1 and the greater by b_1. Define the real interval (a_1, b_1) (see Figure 9.1). Starting from r_1, find the *first two* elements

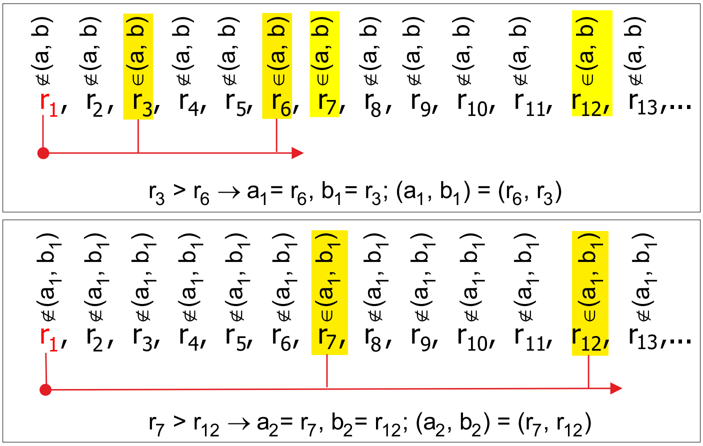

Figura 9.1 – Definition of the first two intervals (a_1, b_1), (a_2, b_2).

of $\langle r_i \rangle$ within (a_1, b_1). Denote the smaller of them by a_2 and the greater by b_2. Define the real interval (a_2, b_2). Evidently it holds:

$$(a_1, b_1) \supset (a_2, b_2) \tag{2}$$

Starting from r_1, find the *first two* elements of $\langle r_i \rangle$ within (a_2, b_2). Denote the smaller of them by a_3 and the greater by b_3. Define the real interval (a_3, b_3). Evidently it holds:

$$(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3) \tag{3}$$

The continuation of the above Procedure P161 defines a sequence of real nested intervals (R-intervals):

$$(a_1,\, b_1) \supset (a_2,\, b_2) \supset (a_3,\, b_3) \supset \ldots \tag{4}$$

whose left endpoints $a_1, a_2, a_3, \ldots$ form a strictly increasing sequence of real numbers, and whose right endpoints $b_1, b_2, b_3, \ldots$ form a strictly decreasing sequence also of real numbers, being every element of the first sequence smaller than every element of the second one.

P162 It is important to highlight the fact that an element r_n of $\langle r_i \rangle$ cannot belong to the successive nested real intervals $(a_n, b_n) \supset (a_{n+1}, b_{n+1}) \supset (a_{n+2}, b_{n+2}) \supset \ldots$ Indeed, the first time the Procedure P161 considers r_n, a maximum of $n/2$ of those intervals will have been defined. Therefore either r_n is used to define an endpoint of a new real interval $(a_{i<n}, b_{i<n})$, or it does not belong to the last defined interval. In consequence, r_n cannot belong to the successive nested real intervals $(a_n, b_n) \supset (a_{n+1}, b_{n+1}) \supset (a_{n+2}, b_{n+2}) \supset \ldots$

P163 The number of R-intervals will be finite or infinite, and both possibilities have to be considered. Assume in the first place the number of R-intervals is a finite natural number n. In this case, there will be a last R-interval (a_n, b_n) in the sequence of R-intervals, because the successive R-intervals have been indexed by the successive finite natural numbers. This last R-interval will contain, at most, one element r_v of $\langle r_i \rangle$, otherwise it would be possible to define at least a new R-interval (a_{n+1}, b_{n+1}). Let, therefore, s be any element within (a_n, b_n), different from r_v, if r_v does exist. Evidently s is a real number within (a, b) which does not belong to the sequence $\langle r_i \rangle$. Consequently, the sequence $\langle r_i \rangle$ does not contain all real numbers, and the one to one correspondence f is impossible.

P164 Consider now the number of R-intervals is infinite (note this case implies the completion of a procedure of infinitely many successive steps). The sequence $\langle a_i \rangle$ is strictly increasing and upper bounded by any b_i, therefore the limit L_a of $\langle a_i \rangle$ exists. On its part, the sequence $\langle b_i \rangle$ is strictly decreasing and lower bounded by any a_i, in consequence the limit L_b of this sequence also exists. Taking into account that every a_i is less than every b_i it must hold: $L_a \leq L_b$ (Figure 9.2).

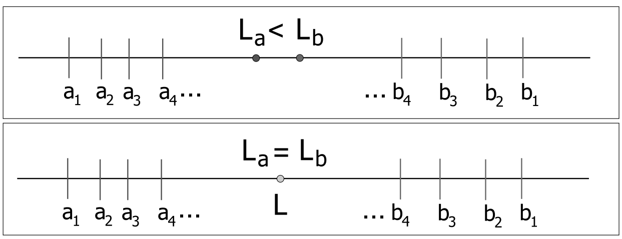

Figura 9.2 – Convergence of $\langle a_i \rangle$ and $\langle b_i \rangle$.

P165 Assume that $L_a < L_b$. In this case, any of the infinitely many elements within the real interval $(L_a,\ L_b)$ is a real number s within (a, b) which does not belong to the sequence $\langle r_i \rangle$ because, according to P162, if it were an element r_v of $\langle r_i \rangle$ it could not belong to the successive $(a_v, b_v) \supset (a_{v+1}, b_{v+1}) \supset (a_{v+2}, b_{v+2}) \supset \dots$, while s belongs to all of them. Therefore, the one to one correspondence f is impossible.

P166 Finally, assume that $L_a = L_b = L$. It is immediate to prove that L is a real number within (a, b) which is not in $\langle r_i \rangle$. Indeed, assume that L is an element r_v of $\langle r_i \rangle$. According to P162, r_v does not belong to the successive R-intervals $(a_v, b_v) \supset (a_{v+1}, b_{v+1}) \supset (a_{v+2}, b_{v+2}) \supset \dots$, while L belongs to all of them. Therefore, L cannot be r_v. The limit L is a real number in (a, b) which is not in $\langle r_i \rangle$. So, the bijection f is impossible.

P167 According to P159-P166, and being f any supposed bijection between $\mathbb{N}$ and $\mathbb{R}$, it must be concluded that a bijection (one to one correspondence) between the set $\mathbb{N}$ of the natural numbers and the set $\mathbb{R}$ of real numbers is impossible. Therefore, $\mathbb{R}$ is not denumerable.

Rational version of Cantor's 1874-argument

P168 The argument that follows is identical to the previous one, except in that it applies to the set $\mathbb{Q}$ of the rational numbers.

P169 Assume the set $\mathbb{Q}$ of the rational numbers is denumerable. In such a case, there would be at least one bijection between the ω-ordered set $\mathbb{N}$ of the natural numbers and $\mathbb{Q}$. Let f be any of such bijections. The elements of $\mathbb{Q}$ would be reordered by f in the sequence $\langle q_i \rangle$:

$$\langle q_i \rangle = q_1,\ q_2,\ q_3,\ \dots \tag{5}$$

being $q_i = f(i), \forall i \in \mathbb{N}$ (Theorem of the Indexed Sets). Obviously, the sequence $\langle q_i \rangle$ defined by f *would contain* all rational numbers if $\mathbb{Q}$ were actually denumerable, and it would be possible to consider all of them successively and one by one

P170 Consider any real interval (a, b). Starting from q_1, find the *first two* elements of $\langle q_i \rangle$ within (a, b). Denote the smaller of them by a_1 and the greater by b_1. Define the real interval (a_1, b_1). Starting from q_1, find the *first two* elements of $\langle q_i \rangle$ within (a_1, b_1). Denote the smaller of them by a_2 and the greater by b_2. Define the real interval (a_2, b_2). Evidently it holds:

$$(a_1, b_1) \supset (a_2, b_2) \tag{6}$$

Starting from q_1, find the *first two* elements of $\langle q_i \rangle$ within (a_2, b_2). Denote the smaller of them by a_3 and the greater by b_3. Define the real interval (a_3, b_3). Evidently it holds:

$$(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3). \tag{7}$$

P171 The continuation of the above Procedure P170 defines a sequence of real nested intervals (R'-intervals):

$$(a_1,\, b_1) \supset (a_2,\, b_2) \supset (a_3,\, b_3) \supset \dots \tag{8}$$

whose left endpoints a_1, a_2, a_3,... form a strictly increasing sequence of rational numbers, and whose right endpoints b_1, b_2, b_3,... form a strictly decreasing sequence of rational numbers, being every element of the first sequence smaller than every element of the second one.

P172 It is important to highlight the fact that an element q_n of $\langle q_i \rangle$ cannot belong to the successive nested real intervals $(a_n, b_n) \supset (a_{n+1}, b_{n+1}) \supset (a_{n+2}, b_{n+2}) \supset \dots$ Indeed, the first time the Procedure P170 considers q_n, a maximum of $n/2$ of those intervals will have been defined. Therefore either q_n is used to define an endpoint of a new real interval $(a_{i<n}, b_{i<n})$, or it does not belong to the last defined interval. In consequence, q_n cannot belong to the successive nested real intervals $(a_n, b_n) \supset (a_{n+1}, b_{n+1}) \supset (a_{n+2}, b_{n+2}) \supset \dots$

P173 The number of R'-intervals will be finite or infinite, and both possibilities have to be considered. Assume in the first place that the number of R'-intervals is a finite natural number n. In this case, there will be a last R'-interval (a_n, b_n) in the sequence of R'-intervals, because the successive R'-intervals have been indexed by the successive finite natural numbers. This last R'-interval will contain, at best, one element q_v of $\langle q_i \rangle$, otherwise it would be possible to define at least one new R-interval $(a_{n+1},\, b_{n+1})$. Let, therefore, s be any rational number within (a_n, b_n), different from q_v, if q_v does exist. Evidently s is a rational number within (a, b) which does not belong to the sequence $\langle q_i \rangle$. Consequently, the sequence $\langle q_i \rangle$ does not contain all rational numbers, and the one to one correspondence f is impossible.

P174 Consider now the number of R'-intervals is infinite (note this case implies the completion of a procedure of infinitely many successive steps). The sequence $\langle a_i \rangle$ is strictly increasing and upper bounded by any b_i, therefore the *real* limit L_a of $\langle a_i \rangle$ does exist. On its part, the sequence $\langle b_i \rangle$ is strictly decreasing and lower bounded by any a_i, in consequence the *real* limit L_b of this sequence also exists. Taking into account that every a_i is less than every b_i it must hold: $L_a \leq L_b$, being L_a and L_b two real (rational or irrational) numbers.

P175 Assume that $L_a < L_b$. In this case, any of the infinitely many rationals within the real interval (L_a, L_b) is a rational number s within (a, b) which does not belong to the sequence $\langle q_i \rangle$, because according to P172, if it were an element q_v of $\langle q_i \rangle$ it could not belong to the successive R'-intervals $(a_v, b_v) \supset (a_{v+1}, b_{v+1}) \supset (a_{v+2}, b_{v+2}) \supset \dots$, while s belongs to all of them. Therefore $\langle q_i \rangle$ does not contain all rational numbers, and the one to one correspondence f is impossible.

P176 Finally, assume that $L_a = L_b = L$. It is immediate that L is a *real* number within the *real interval* (a, b) which is not in $\langle q_i \rangle$. In fact, if L is irrational then it is clear that it is not in $\langle q_i \rangle$; assume then L is rational, and assume also it is an element q_v of $\langle q_i \rangle$. According to P172, q_v does not belong to the successive R'-intervals $(a_v, b_v) \supset (a_{v+1}, b_{v+1}) \supset (a_{v+2}, b_{v+2}) \supset \dots$, while L belongs to all of them. Therefore, L cannot be q_v. The limit L is a real number (rational or irrational) in the real interval (a, b) which is not in $\langle q_i \rangle$. Thus, if L were rational then $\langle q_i \rangle$ would not contain all rational numbers, and the one to one correspondence f would be impossible.

P177 We have just proved that, as in Cantor's 1874 argument, the bijection f, which is any assumed bijection between the sets $\mathbb{N}$ and $\mathbb{Q}$, is impossible in all cases, except that the sequences $\langle a_i \rangle$ and $\langle b_i \rangle$ have a common irrational limit. Thus, except in that case, and for the same reasons as in Cantor's 1874 argument, we would have proved the set $\mathbb{Q}$ of the rational numbers is non-denumerable.

P178 Evidently, If Cantor's 1874-argument could be extended to the rational numbers we would have a contradiction: the set $\mathbb{Q}$ would and would not be denumerable. In consequence, and in order to ensure the impossibility of that contradiction, it must be proved that whatsoever be the rational interval (a, b) and the reordering of $\langle q_i \rangle$, the number of R'-intervals can never be finite and the sequences of endpoints $\langle a_i \rangle$ and $\langle b_i \rangle$ have always a common *irrational* limit. Until then, the consistency of transfinite set theory will be at stake. However, 146 years after the publication of Cantor's article, the problem has not even been raised. The following chapter deals with that problem.

A variant of Cantor's 1874 argument

P179 The argument that follows is a variant of the above Cantor's first proof of the uncountable nature of the set of the real numbers, though applied to the set of the rational numbers $\mathbb{Q}$.

P180 Since, according to Cantor, the set $\mathbb{Q}$ of the rational numbers is denumerable we can consider a one to one correspondence f between the ω-ordered set $\mathbb{N}$ of the natural numbers and $\mathbb{Q}$. Let $\langle q_i \rangle$ be the reordered sequence (Theorem of the Indexed Sets) of rational numbers defined by:

$$f(i) = q_i, \ \forall i \in \mathbb{N} \tag{9}$$

Obviously $\langle q_i \rangle$ contains all rational numbers, so that it is possible to consider all of them successively and one by one

P181 Let x be a rational variable whose domain is any rational interval (a, b), and let x_o be any element within (a, b). Now consider the following sequence of successive recursive definitions $\langle D_i(x) \rangle$ of x:

$$\begin{cases} D_1(x) = x_o \\ D_i(x) = \text{mín}\left(\{D_{i-1}(x), q_i\} \cap (a,b)\right), \ i = 2, 3, 4, \dots \end{cases} \tag{10}$$

where mín stands for the smallest (in the natural order of precedence of $\mathbb{Q}$) of the two numbers in brackets, or the only number in bracket if $q_i \notin (a, b)$. $\langle D_i(x) \rangle$ compares x with the successive elements of $\langle q_i \rangle$ that belong to (a, b), and defines x as the compared element each time the compared element is smaller than the current value of x.

P182 Unnecessary as it may seem, we will impose the following restriction to the successive definitions $\langle D_i(x) \rangle$:

Restriction P182.-*Each successive definition $D_i(x)$ will be carried out if, and only if, x results defined as a rational number within its domain (a, b).*

We will prove now that for any natural number v, the first v successive definitions (10) can be carried out according to Restriction P182.

P183 The first definition $D_1(x)$ can be carried out according to Restriction P182 because $D_1(x) = x_o$, and $x_o \in (a, b)$. Assume that, being n any natural number, the first n definitions $\langle D_i(x) \rangle_{i=1,2,\dots n}$ can be carried according to Restriction P182, so that $D_n(x) \in (a, b)$. Since q_{n+1} is a well defined rational number, we will know if, being in (a, b), it is less than

$D_n(x)$. If this is the case $D_{n+1}(x) = q_{n+1}$; otherwise $D_{n+1}(x) = D_n(x)$. In both cases x results defined within its domain (a, b). This proves $D_{n+1}(x)$ can also be performed according to Restriction P182. Consequently, for any natural number v, the first v definitions $\langle D_i(x)\rangle_{i=1,2,\ldots v}$ can be carried out according to Restriction P182.

P184 Assume that all definitions $\langle D_i(x)\rangle$ that observe Restriction 182 are carried out (Principle of Execution). The value of x once performed all of them, whatsoever be the finite or infinite number of times it has been defined with a different value, will be a rational number within its domain (a, b) just because *it was always defined within its domain* (a, b). Thus, we can affirm:

> Undeterminable as the current value of x may be once performed all definitions $\langle D_i(x)\rangle$ according to Restriction 182, it will be a certain rational number r within its domain (a, b) (Principle of Invariance).

P185 Obviously a variable can be properly defined within its domain even if we cannot know its current value. Some infinitists argue, however, that although Restriction 182 applies to each of the infinitely many successive definitions of x, once completed the infinite sequence of those definitions we cannot ensure x continue to be a rational variable defined within its domain (a, b), despite the fact that each of those definitions defined x as a rational number within its domain (a, b). As if the completion of an infinite sequence of definitions had arbitrary additional effects on the defined object, as losing the condition of being a rational variable defined within its domain. Obviously this goes against the Principle of Invariance.

P186 The same unknown additional effects on the defined objects could, then, be expected in any other definition, procedure or proof consisting of infinitely many successive steps, in which case infinitist mathematics would have no sense. For instance, in Cantor's 1874 argument if the number of R-intervals is infinite, and due to those unknown additional effects of the completion on the defined object, we could not ensure these intervals continue to be the real intervals within (a, b) they were defined to be.

P187 Thus, if to complete the infinite sequence of definitions (10) means to perform each and every definition of the sequence, and only them, each of which defines x within its domain (a, b), and if the completion of the sequence, as such a completion, has not unknown arbitrary effects on x, then, once performed all possible definitions (Principle of Execution), x can only be defined as a certain rational number r (whatsoever it be) within its domain (a, b) (Principle of Invariance).

P188 Consider the rational interval (a, r) and any element s within (a, r). It is quite clear that $s \in (a, b)$ and $s < r$. We will prove s cannot belong to $\langle q_i \rangle$. In fact, assume s belongs to $\langle q_i \rangle$. In such a case there will be an element q_v in $\langle q_i \rangle$ such that $q_v = s$, and being s in (a, r), we will have $q_v \in (a, r)$, and therefore $q_v < r$. But this is impossible because:

a) The index v of q_v is a natural number.

b) According to 183, for each natural number v, it is possible to carry out the first v definitions $\langle D_i(x) \rangle_{i=1,2,\ldots v}$ satisfying Restriction P182.

c) All definitions $\langle D_i(x) \rangle$ satisfying Restriction P182 have been carried out.

d) At least the first v definitions $\langle D_i(x) \rangle_{i=1,2,\ldots v}$ satisfying Restriction P182 have been carried out (Principle of Execution).

e) $D_v(x) = \text{mín} \left(\{D_{v-1}(x), q_v\} \cap (a, b) \right)$ and then $D_v(x) \leq q_v$. Therefore $r \leq q_v$

f) It is then impossible that $q_v < r$.

In consequence s cannot be an element of $\langle q_i \rangle$.

P189 The rational number s proves, therefore, the existence of rational numbers within (a, b) that are not in $\langle q_i \rangle$, which in turn proves the falseness of the initial assumption on the denumerable nature of $\mathbb{Q}$. Now then, taking into account that Cantor proved $\mathbb{Q}$ is denumerable, the final conclusion can only be that $\mathbb{Q}$ is and is not denumerable.

P190 The sequence of definitions $\langle D_i(x) \rangle$ leads to some other contradictory results the reader can easily find. Evidently, contradictory results do not invalidate one another, they simply prove the existence of contradictions (this obviousness is often ignored in the discussions on the actual infinity). If, starting from the same hypothesis, two independent arguments lead to contradictory results they prove the inconsistency of the initial hypothesis. It is quite clear, then, that an argument cannot be refuted by another argument even if this last argument comes to conclusions that contradict the conclusions of the first one. An argument can only be refuted by indicating where and why *that* argument fails. These obviousness are not necessary to be recalled in other areas of discussion, but they do if the area is that of the hypothesis of the actual infinite. Or that of any other hypothesis or axiom used to support a hegemonic stream of scientific thought, as if hegemony were synonymous with truth. Hegemony, almost always hostile to disagreement, takes for granted that its foundational assumptions are indisputable.

Chapter References

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