8 Cantor's 1874 argument revisited

(Draft chapter of the book *Infinity Put to the Test* by Antonio León[1])

Abstract.-This chapter explains in detail the first Cantor's proof of the uncountable nature of the real numbers, and discusses the conditions under which it could also be applied to the set of the rational numbers.

Keywords: Cantor's 1874 argument, cardinal of the set of real numbers, cardinal of the set of rational numbers.

8.1 Introduction

1. In 1874, 17 years before the publication of his famous diagonal argument, Cantor proved for the first time the set of the real numbers cannot be denumerable. This early Cantor's proof is one of the objectives of this chapter. The other is the analysis of the conditions under which that proof could also be applied to the set of the rational numbers. It will necessary, therefore, to prove those conditions can never be satisfied in order to ensure the impossibility of a contradiction on the cardinality of the set of the rational numbers, which was proved to be numerable by Cantor himself in the same publication [1]. A short rational variant of Cantor's argument is also discussed at the end of the chapter.

8.2 Cantor's 1874-argument

2. This section explains in detail the first Cantor's proof of the uncountable nature of the set $\mathbb{R}$ of the real numbers, published in the year 1874 in a short paper that also included a proof of the denumerable nature of the set $\mathbb{A}$ (also denoted by $\overline{\mathbb{Q}}$) of the algebraic numbers and then of the set of the rational numbers $\mathbb{Q}$, a subset of $\mathbb{A}$ [1], (French edition [2], Spanish edition [3]).

3. Assume the set $\mathbb{R}$ is denumerable. In those conditions there would be at least one bijection between the set $\mathbb{N}$ of the natural numbers and $\mathbb{R}$. Let f be any of such bijections. The elements of $\mathbb{R}$ could be ω-ordered by f as:

$$r_1,\ r_2,\ r_3,\ \ldots \tag{1}$$

[1]Next publication

being $r_i = f(i), \forall i \in \mathbb{N}$. Obviously, the sequence $\langle r_i \rangle$ defined by f *would contain* all real numbers if $\mathbb{R}$ were actually denumerable.

4. Consider now any real interval (a, b). Cantor's 1874-argument consists in proving the existence of a real number s in (a, b) which is not in the ω-ordered sequence $\langle r_i \rangle$. The existence of s would prove that $\langle r_i \rangle$ does not contain all real numbers. Therefore, the one to one correspondence f, whatsoever it be, would be impossible. And the initial assumption on the denumerable nature of $\mathbb{R}$ would be false. The proof goes as follows.

5. Starting from r_1, find the *first two* elements of $\langle r_i \rangle$ within (a, b). Denote the smaller of them by a_1 and the greater by b_1. Define the real interval (a_1, b_1) (see Figure 8.1).

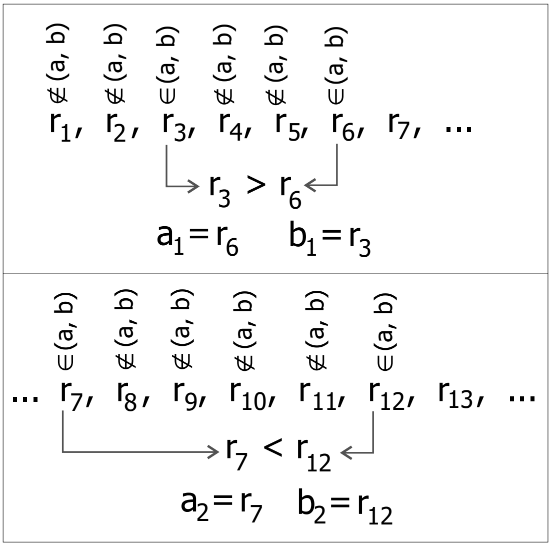

Figura 8.1 – Definition of the first two intervals (a_1, b_1), (a_2, b_2).

6. Starting from r_1, find the *first two* elements of $\langle r_i \rangle$ within (a_1, b_1). Denote the smaller of them by a_2 and the greater by b_2. Define the real interval (a_2, b_2). Evidently it holds:

$$(a_1, b_1) \supset (a_2, b_2) \tag{2}$$

7. Starting from r_1, find the *first two* elements of $\langle r_i \rangle$ within (a_2, b_2). Denote the smaller of them by a_3 and the greater by b_3. Define the real interval (a_3, b_3). Evidently it holds:

$$(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3). \tag{3}$$

8. The continuation of the above procedure (R-procedure from now on) defines a sequence of real nested intervals (R-intervals):

$$(a_1,\, b_1) \supset (a_2,\, b_2) \supset (a_3,\, b_3) \supset \dots \tag{4}$$

whose left endpoints $a_1, a_2, a_3, \dots$ form a strictly increasing sequence of real numbers, and whose right endpoints $b_1, b_2, b_3, \dots$ form a strictly decreasing sequence also of real numbers, being every element of the first sequence smaller than every element of the second one.

9. From the ω-order of $\langle r_i \rangle$ and the ordered way by which R-procedure defines the successive R-intervals (starting from r_1 find the first two elements...), it immediately follows that if r_n defines an endpoint a_i or b_i, then it must hold $i \leq n$. In consequence, if r_n is any element of $\langle r_i \rangle$, it will not belong to the successive R-intervals:

$$(a_n, b_n) \supset (a_{n+1}, b_{n+1}) \supset (a_{n+2}, b_{n+2}) \supset \dots \tag{5}$$

10. The number of R-intervals will be finite or infinite, and both possibilities have to be considered. Assume in the first place the number of R-intervals is a finite natural number n.[2] In this case there would be a last R-interval[3] (a_n, b_n) in the sequence of R-intervals, because the successive R-intervals have been indexed by the successive finite natural numbers in their natural order of precedence. This last R-interval would contain, at best, one element r_v of $\langle r_i \rangle$, otherwise it would be possible to define at least one new R-interval $(a_{n+1},\, b_{n+1})$. Let, therefore, s be any element within (a_n, b_n), different from r_v if r_v does exist. Evidently s is a real number within (a, b) which does not belong to the sequence $\langle r_i \rangle$. Consequently, the sequence $\langle r_i \rangle$ does not contain all real numbers, and the one to one correspondence f is impossible.

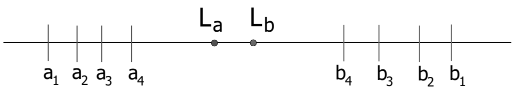

Figura 8.2 – Convergence of $\langle a_i \rangle$ and $\langle b_i \rangle$.

11. Consider now the number of R-intervals is infinite.[4] Since the sequence $\langle a_i \rangle$ is strictly increasing and upper bounded by every element of $\langle b_i \rangle$,

[2]Including the case that R-procedure defines no R-interval.
[3]Or the whole interval (a, b) in the case that R-procedure defines no R-interval.
[4]Note this case implies the completion of a procedure of infinitely many successive steps.

the limit L_a of $\langle a_i \rangle$ does exist. On its part, the sequence $\langle b_i \rangle$ is strictly decreasing and lower bounded by every element of $\langle a_i \rangle$, in consequence the limit L_b of this sequence also exists. Taking into account that every a_i is less than every b_i it must hold: $L_a \leq L_b$.

12. Assume that $L_a < L_b$. In this case any of the infinitely many elements within the real interval (L_a, L_b) is a real number within (a, b) which does not belong to the sequence $\langle r_i \rangle$, and then the one to one correspondence f is impossible.

13. Finally, assume that $L_a = L_b = L$. It is immediate to prove that L is a real number within (a, b) which is not in $\langle r_i \rangle$. In fact, assume that L is an element r_v of $\langle r_i \rangle$. According to 9, r_v does not belong to the successive R-intervals:

$$(a_v, b_v) \supset (a_{v+1}, b_{v+1}) \supset (a_{v+2}, b_{v+2}) \supset \dots, \tag{6}$$

while L belongs to all of them. Therefore, L cannot be r_v. The limit L is a real number in (a, b) which is not in $\langle r_i \rangle$. The one to one correspondence f is impossible. Consequently, and according to 3-13, a one to one correspondence between the set $\mathbb{N}$ of the natural numbers and $\mathbb{R}$ is impossible, and $\mathbb{R}$ is not denumerable.

8.3 Q-VERSION OF CANTOR'S 1874-ARGUMENT

14. The argument that follows is identical to the previous one, except in that it applies to the set $\mathbb{Q}$ of the rational numbers.

15. Assume the set $\mathbb{Q}$ of the rational numbers is denumerable. In these conditions there would be at least one bijection between the set $\mathbb{N}$ of the natural numbers and $\mathbb{Q}$. Let f be any of such bijections. The elements of $\mathbb{Q}$ could be ω-ordered by f as:

$$q_1, \; q_2, \; q_3, \; \dots \tag{7}$$

being $q_i = f(i), \forall i \in \mathbb{N}$. Obviously, the sequence $\langle q_i \rangle$ defined by f *would contain* all rational numbers if $\mathbb{Q}$ were actually denumerable.

16. Consider any rational interval (a, b). Starting from q_1, find the *first two* elements of $\langle q_i \rangle$ within (a, b). Denote the smaller of them by a_1 and the greater by b_1. Define the rational interval (a_1, b_1).

17. Starting from q_1, find the *first two* elements of $\langle q_i \rangle$ within (a_1, b_1). Denote the smaller of them by a_2 and the greater by b_2. Define the rational interval (a_2, b_2). Evidently it holds:

$$(a_1, b_1) \supset (a_2, b_2) \tag{8}$$

18. Starting from q_1, find the *first two* elements of $\langle q_i \rangle$ within (a_2, b_2). Denote the smaller of them by a_3 and the greater by b_3. Define the rational interval (a_3, b_3). Evidently it holds:

$$(a_1, b_1) \supset (a_2, b_2) \supset (a_3, b_3). \tag{9}$$

19. The continuation of the above procedure (Q-procedure from now on) defines a sequence of rational nested intervals (Q-intervals):

$$(a_1,\, b_1) \supset (a_2,\, b_2) \supset (a_3,\, b_3) \supset \dots \tag{10}$$

whose left endpoints a_1, a_2, a_3,... form a strictly increasing sequence of rational numbers, and whose right endpoints b_1, b_2, b_3,... form a strictly decreasing sequence of rational numbers, being every element of the first sequence smaller than every element of the second one.

20. From the ω-order of $\langle q_i \rangle$ and the ordered way by which Q-procedure defines the successive Q-intervals (starting from q_1 find the first two elements...), it immediately follows that if q_n defines an endpoint a_i or b_i, then it must hold $i \leq n$. In consequence, if q_n is any element in $\langle q_i \rangle$, it will not belong to the successive Q-intervals:

$$(a_n, b_n) \supset (a_{n+1}, b_{n+1}) \supset (a_{n+2}, b_{n+2}) \supset \dots \tag{11}$$

21. The number of Q-intervals will be be finite or infinite, and both possibilities have to be considered. Assume in the first place that the number of Q-intervals is a finite natural number n.[5] In this case there would be a last Q-interval[6] (a_n, b_n) in the sequence of Q-intervals, because the successive Q-intervals have been indexed by the successive finite natural numbers in their natural order of precedence. This last Q-interval would contain, at best, one element q_v of $\langle q_i \rangle$, otherwise it would be possible to define at least one new R-interval $(a_{n+1},\, b_{n+1})$. Let, therefore, s be any rational number within (a_n, b_n), different from q_v if q_v does exist. Evidently s is a rational number within (a, b) which does not belong to the sequence $\langle q_i \rangle$. Consequently, the sequence $\langle q_i \rangle$ does not contain all rational numbers, and the one to one correspondence f is impossible.

22. Consider now the number of Q-intervals is infinite.[7] Since the sequence $\langle a_i \rangle$ is strictly increasing and upper bounded by every element of $\langle b_i \rangle$, the

[5] Including the case that Q-procedure defines no Q-interval.
[6] Or the whole interval (a, b) in the case that Q-procedure defines no Q-interval.
[7] Note this case implies the completion of a procedure of infinitely many successive steps.

real limit L_a of $\langle a_i \rangle$ does exist. On its part, the sequence $\langle b_i \rangle$ is strictly decreasing and lower bounded by every element of $\langle a_i \rangle$, in consequence the *real* limit L_b of this sequence also exists. Taking into account that every a_i is less than every b_i it must hold: $L_a \leq L_b$, being L_a and L_b two real (rational or irrational) numbers.

23 . Assume that $L_a < L_b$. In this case, any of the infinitely many rationals within the real interval (L_a, L_b) is a rational number within (a, b) which does not belong to the sequence $\langle q_i \rangle$, the one to one correspondence f is then impossible.

24 . Finally, assume that $L_a = L_b = L$. It is immediate that L is a *real* number within the *real interval* (a, b) which is not in $\langle q_i \rangle$. In fact, if L is irrational then it is clear that it is not in $\langle q_i \rangle$; assume then L is rational, and assume also it is an element q_v of $\langle q_i \rangle$. According to 20, q_v does not belong to the successive intervals:

$$(a_v, b_v) \supset (a_{v+1}, b_{v+1}) \supset (a_{v+2}, b_{v+2}) \supset \dots \tag{12}$$

while L belongs to all of them. Therefore, L cannot be q_v. The limit L is a real number (rational or irrational) in the real interval (a, b) which is not in $\langle q_i \rangle$. Thus, if L were rational then the one to one correspondence f would be impossible.

25 . We have just proved that, as in Cantor's 1874 argument, the bijection f is impossible in all cases, except that the sequences $\langle a_i \rangle$ and $\langle b_i \rangle$ have a common irrational limit. Thus, except in that case, and for the same reasons as in Cantor's 1874 argument, we would have proved the set $\mathbb{Q}$ of rational numbers is non-denumerable.

26 . Evidently, If Cantor's 1874-argument could be extended to the rational numbers we would have a contradiction: the set $\mathbb{Q}$ would and would not be denumerable. In consequence, and in order to ensure the impossibility of that contradiction, we must prove that whatsoever be the rational interval (a, b) and the reordering of $\langle q_i \rangle$, the number of Q-intervals can never be finite and the sequences of endpoints $\langle a_i \rangle$ and $\langle b_i \rangle$ have always a common *irrational* limit. Until then, the consistency of transfinite set theory will be at stake.

8.4 A variant of Cantor's 1874 argument

27 . The argument that follows is a variant of the above Cantor's first proof of the uncountable nature of the set of the real numbers.

28. Since, according to Cantor, the set $\mathbb{Q}$ of the rational numbers is denumerable we can consider a one to one correspondence f between this set and the set $\mathbb{N}$ of the natural numbers. Let $\langle q_i \rangle$ be the ω-ordered sequence of rational numbers defined by:

$$q_i = f(i), \ \forall i \in \mathbb{N} \tag{13}$$

Obviously $\langle q_i \rangle$ contains all rational numbers.

29. Let x be a rational variable whose domain is any rational interval (a, b) and whose initial value is any element x_o within (a, b). Let $\langle q_i \rangle$ be the sequence of rational numbers defined by (13). Now consider the following sequence of successive recursive definitions $\langle D_i(x) \rangle$ of x:

$$\forall i \in \mathbb{N} \begin{cases} D_1(x) = x_o \\ D_i(x) = \text{mín}\Big(\{D_{i-1}(x), q_i\} \cap (a, b) \Big) \end{cases} \tag{14}$$

where mín stands for the smallest (in the natural dense ordering of $\mathbb{Q}$) of the two numbers in brackets, or the only number in bracket if $q_i \notin (a, b)$. $\langle D_i(x) \rangle$ compares x with the successive elements of $\langle q_i \rangle$ that belong to (a, b), and defines x as the compared element each time the compared element is smaller than the current value of x.

30. Unnecessary as it may seem, we will impose the following restriction to the successive definitions (14):

> **Restriction 30**.-Each successive definition $D_i(x)$ will be carried out if, and only if, x results defined as a rational number within its domain (a, b).

We will prove now that for any natural number v the first v successive definitions (14) can be carried out.

31. The first definition $D_1(x)$ can be carried out because $D_1(x) = x_o$, and $x_o \in (a, b)$. Assume that, being n any natural number, the first n definitions $\langle D_i(x) \rangle_{i=1,2,\ldots n}$ can be carried, so that $D_n(x) \in (a, b)$. Since q_{n+1} is a well defined rational number, we will know if, being in (a, b), it is less than $D_n(x)$. If this is the case $D_{n+1}(x) = q_{n+1}$; otherwise $D_{n+1}(x) = D_n(x)$. In both cases x results defined within its domain (a, b). This proves $D_{n+1}(x)$ can also be performed. Consequently, for any natural number v, the first v definitions $\langle D_i(x) \rangle_{i=1,2,\ldots v}$ can be carried out.

32. Assume that while the successive definitions(14) that observe Restriction 30 can be carried out, they are carried out. The value of x once performed all possible[8] definitions (14), whatsoever be the finite or infinite number of times it has been defined, will be a rational number within its domain (a, b) just because *it was always defined within its domain* (a, b). Thus, we can affirm:

> Undeterminable as the current value of x once performed all possible definitions (14) may be, it will be a certain rational number r within its domain (a, b) (Principle of Invariance).

33. Obviously a variable can be properly defined within its domain even if we cannot know its current value. Some infinitists argue, however, that although Restriction 30 applies to each of the infinitely many successive definitions of x, once completed the infinite sequence of those definitions we cannot ensure x continue to be a rational variable properly defined within its domain (a, b), despite the fact that each of those definitions defined x as a rational number within its domain (a, b). As if the completion of an infinite sequence of definitions had arbitrary additional effects on the defined object, as losing the condition of being a rational variable properly defined within its domain. Obviously this goes against the Principle of Invariance.

34. The same unknown additional effects on the defined objects could, then, be expected in any other definition, procedure or proof composed of infinitely many successive steps, in which case infinitist mathematics would have no sense. For instance, in Cantor's 1874 argument if the number of R-intervals is infinite, and due to those unknown additional effects of the completion on the defined object, we could not ensure these intervals continue to be the real intervals within (a, b) they were defined to be.

35. Thus, if to complete the infinite sequence of definitions (14) means to perform each and every definition of the sequence (and only them) each of which defines x within its domain (a, b), and if the completion of the sequence has not unknown arbitrary effects on x, then, once performed all possible definitions, x can only be defined as a certain rational number r (whatsoever it be) within its domain (a, b) (Principle of Invariance).

36. Consider the rational interval (a, r) and any element s within (a, r). It is quite clear that $s \in (a, b)$ and $s < r$. We will prove s cannot belong to $\langle q_i \rangle$. In fact, assume s belongs to $\langle q_i \rangle$. In such a case there will be an element q_v in $\langle q_i \rangle$ such that $q_v = s$, and being s in (a, r), we will have $q_v \in (a, r)$, and therefore $q_v < r$. But this is impossible because:

[8] If it were impossible to perform all possible definitions(14) we would be in the face of the elementary contradiction of an impossible possibility.

1.- The index v of q_v is a natural number.

2.- According to 31, for each natural number v, it is possible to carry out the first v definitions (14).

3.- All possible definitions (14) have been carried out.

4.- At least the first v definitions (14) have been carried out.

5.- $D_v(x) = \text{mín}\Big(\{D_{v-1}(x), q_v\} \cap (a,b)\Big)$ and then $D_v(x) \leq q_v$. Therefore $r \leq q_v$

6.- It is then impossible that $q_v < r$.

In consequence s cannot be an element of $\langle q_i \rangle$.

37. The rational number s proves, therefore, the existence of rational numbers within (a, b) which are not in $\langle q_i \rangle$, which in turn proves the falseness of the initial assumption on the denumerable nature of $\mathbb{Q}$. Now then, taking into account that Cantor proved $\mathbb{Q}$ is denumerable, the final conclusion can only be that $\mathbb{Q}$ is and is not denumerable.

38. The sequence of definitions (14) leads to some other contradictory results the reader could easily find. Evidently, contradictory results do not invalidate one another, they simply prove the existence of contradictions.[9] If, starting from the same hypothesis, two independent arguments lead to contradictory results they prove the inconsistency of the initial hypothesis. It is quite clear, then, that an argument cannot be refuted by another argument even if this last argument comes to conclusions that contradict the conclusions of the first one. An argument can only be refuted by indicating where and why *that* argument fails.

[9] This obviousness is often ignored in the discussions on the actual infinity.

Chapter References

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