Nested sets inconsistency

(Version 2019, adapted from a draft chapter of the book Infinity put to the test[1])

Antonio León Sánchez
Instituto de Enseñanza Secundaria F. Salinas.
Salamanca, Spain (Profesor jubilado)

This chapter examines a denumerable version of the nested-set theorem and derives from it a contradiction involving the formal consistency of the actual infinity hypothesis subsumed into the axiom of infinity.

A denumerable version of the nested-set theorem

1. Let $A_1 = \{a_1, a_2, a_3 \dots\}$ be any ω-ordered set and consider de following recursive definition:

$$A_{i+1} = A_i - \{a_i\}; \; i = 1, 2, 3, \dots \tag{1}$$

which yields the ω-ordered sequence of nested sets:

$$S \; = \; A_1 \supset A_2 \supset A_3 \supset \dots \tag{2}$$

being each set $A_n = \{a_n, a_{n+1}, a_{n+2}, \dots\}$ a denumerable proper subset of all its predecessors, as well as a superset of all its successors.

2. The sequence S of sets $\langle A_n \rangle$ satisfies:

$$\bigcap_S A_i = \emptyset \tag{3}$$

where $\bigcap_S A_i$ stands for the intersection of all sets of the sequence S. This result is a *denumerable version* of the nested-sets theorem[2] that, for the sake of simplicity, will be referred to as NST. The proof of NST is immediate: if an element a_k would belong to the intersection then only a finite number of sets equal or less than k would have been defined since a_k does not belong to A_{k+1}, A_{k+2}, A_{k+3}, ...

3. NST is a trivial result in modern infinitist mathematics. As far as I know, it has never been involved in any discussion on the formal nature of infinity. The theorem simply states that the sets $\langle A_n \rangle$ have no common element. The implications of the fact that *each* A_i is a denumerable proper subset of *all* its predecessors have never been examined. In the discussion that follows, however, we will have the opportunity to examine some of those implications.

4. Before beginning our discussion, let us examine an elementary *physical* version of NST. Let B be a box containing a denumerable collection of balls labeled as b_1, b_2, b_3, ..., and let $\langle t_n \rangle$ be any ω-ordered sequence of instants within the real interval (t_a, t_b) whose limit is just t_b. Now consider the following supertask [1]: at each instant t_i remove the ball b_i from the box. The one to one correspondence f between $\langle t_n \rangle$ and $\langle b_n \rangle$ defined by $f(t_i) = b_i$ proves that at t_b all balls will have been removed from B.

5. According to the *one by one* way of removing the balls, which means that between the removing of two successive balls b_i b_{i+1} an interval of time greater than zero $\Delta t = t_{i+1} - t_i > 0$ always elapses, it could be expected that just before completing the removal, the box will contain ...5, 4, 3, 2, 1, 0 balls. Nothing further from the (infinitist) truth: the box will never contain a finite number n of balls simply because these balls would be the impossible n last balls of an ω-ordered collection of balls, and the successive instants at which they had to be removed the impossible last n instants of an ω-ordered sequence of instants.

6. Let $f(t)$ be the number of balls within the box at any instant t in $[t_a, t_b]$, i.e. the number of balls to be removed at t. As a consequence of ω-ordered, we will have the following inevitable dichotomy:

$$f(t) = \begin{cases} \aleph_0, \; \forall t \in [t_a, t_b) \\ 0 \text{ if t} = t_b \end{cases} \tag{4}$$

Otherwise, if for a t in $[t_a, t_b)$ we would have $f(t) = n$, being n a natural number, then there would exist the impossible last n terms of an ω-ordered sequence.

[1] Next publication.

[2] The original version deals with non-denumerable sets, and the conclusion is exactly the contrary, i.e. that the intersection is nonempty.

7. In accordance with the one to one correspondence $f(t_i) = b_i$, all balls $\langle b_n \rangle$ are removed *one by one* from the box B, one after the other and in in such a way that an interval of time greater than zero $t_{i+1} - t_i$ always elapses between the removal of two successive balls b_i, b_{i+1}. But according to the above $\aleph_0$ or 0 dichotomy (4), this is impossible because the number of balls to be removed from the box has to change *directly*[3] from $\aleph_0$ to 0, and this is only possible by simultaneously removing $\aleph_0$ balls.

8. Evidently, B plays the role of the set A_1 and the successive removing of balls represents the successive steps of the recursive definition $A_{i+1} = A_i - \{a_i\}$. Since the successive elements a_1, a_2, a_3, … of A_1 are successively removed in order to define the successive terms A_1, A_2, A_3, … of the sequence S, we could write:

$$\{\not{a}_1, a_2, a_3, a_4, \dots\} \tag{5}$$
$$\{\not{a}_1, \not{a}_2, a_3, a_4, \dots\} \tag{6}$$
$$\{\not{a}_1, \not{a}_3, \not{a}_3, a_4, \dots\} \tag{7}$$
$$\dots \tag{8}$$

where $\not{a}_1, \not{a}_2, \not{a}_3, \dots$ indicate the removal of the successive elements a_1, a_2, a_3, … in order to define the successive members A_1, A_2, A_3, … of the sequence S.

9. As in the case of the box B, and for the same reasons, if we focus our attention on the number of elements that remain unmarked in (5)-(8) as the recursive definition (1) progresses, then we will immediately come to the conclusion that such a number can only take two values: either $\aleph_0$ or 0.

10. The $\aleph_0$ or 0 dichotomy implies that the number of unmarked elements in (5)-(8) changes directly from $\aleph_0$ to 0, and this is only possible by simultaneously marking $\aleph_0$ elements, i.e. by simultaneously defining $\aleph_0$ sets of the sequence S, which evidently is not compatible with the recursiveness of the definition in question.

Nested-set inconsistency

11. The above discussion on NST suggests that this theorem is not as trivial as it seems to be. It, in fact, motivates the short discussion that follows, whose main objective is to put into question the formal consistency of the actual infinity hypothesis.

12. It seems convenient at this point to recall that Cantor took it for granted the existence of the set of all finite cardinals (natural numbers) as a complete infinite totality (axiom of infinity in modern terms), and that from that initial assumption he successfully derived (theorem K-15) the infinite sequence of transfinite ordinals of the first and of the second kind, being ω the least of them and being of all of them based on it [2]. Thus any result affecting the consistency of ω will affect the whole sequence of transfinite ordinals as well as the consistency of the actual infinity hypothesis subsumed into the axiom of infinity.

13. Let us just begin by assuming the axiom of infinity and then the existence of ω-ordered sets and ω-ordered sequences as complete infinite totalities.

14. Consider again the above sequence of sets $S = A_1, A_2, A_3, \dots$. From S we will define the sequence S^* by successively adding to it the successive sets A_n of S if, and only if, the successive intersections $\cap_{i=1}^{i=n} A_i$ are nonempty. In symbols:

$$n = 2, 3, 4, \dots \tag{9}$$
$$\bigcap_{i=1}^{i=n} A_i \neq \emptyset \Rightarrow S^* = A_1, A_2 \dots A_n \tag{10}$$

[3] Without intermediate finite states at which only a finite number of balls remain to be removed.

15. Assume that while the successive sets A_n of S can be added to S^* they are added. Once all possible sets A_n of S have been added to S^*, this sequence will be formed by a certain number (finite or infinite) of sets that by construction have a nonempty intersection. Let, therefore, a_v be any element of that intersection. Evidently, we have:

$$a_v \notin A_{v+1} \tag{11}$$

And in consequence A_{v+1} is not a member of the sequence S^*

16. But, on the other hand, we also have:

$$\begin{aligned}
&A_1 \cap A_2 = A_2 \neq \emptyset \\
&A_1 \cap A_2 \cap A_3 = A_3 \neq \emptyset \\
&\dots \\
&A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_v \cap A_{v+1} = A_{v+1} \neq \emptyset
\end{aligned}$$

Therefore, and also by construction, A_{v+1} is in S^*

17. We have, therefore, derived a contradiction from our initial assumption: the set A_v is and is not in the sequence S^*.

18. The alternative to the above contradiction is another contradiction: after having performed all possible additions of sets A_i to S^* not all possible additions of sets A_i to S^* have been performed.

References

[1] Jon Pérez Laraudogoitia. Supertasks. In E. N. Zaltax, editor, *The Stanford Encyclopaedia of Philosophy.* Standford University, URL = http://plato.stanford.edu, 2001.

[2] Georg Cantor. *Contributions to the founding of the theory of transfinite numbers.* Dover, New York, 1955.