5 The paradoxes of reflexivity revisited

(Draft chapter of the book *Infinity Put to the Test* by Antonio León[1])

Abstract.-This chapter discusses the paradoxes of reflexivity from a new perspective according to which such paradoxes could be inconsistencies derived from the hypothesis of the actual infinity that legitimizes Dedekind's definition of the infinite sets.
Keywords: Dedekind's definition of the infinite sets, paradoxes of reflexivity, Galileo's paradox, exhaustive and non-exhaustive injections, inconsistencies in the foundation of set theories, actual infinity, potential infinity.

5.1 Introduction

1. If after pairing each and every element of a set A with a different element of another set B all elements of B result paired, we say both sets have the same number of elements (the same cardinality). But if one or more elements of B result unpaired and B is infinite, it is not allowed to say both sets have a different number of elements, a different cardinality. In this chapter we discuss why it is not allowed to do it.

2. As we will see, the existence of both exhaustive and non-exhaustive injections[2] between two infinite sets could be indicating they have and not have the same cardinality. Thus, the arbitrary distinction of the exhaustive injections to the detriment of the non-exhaustive ones could be concealing a fundamental contradiction in set theory.

3. Most of the paradoxes related to the actual infinity result from the violation of the Euclidean Axiom of the Whole and the Part,[3] among them the so called paradoxes of reflexivity in which the elements of a whole are paired off with the elements of one of its proper parts [12], [7]. Galileo's paradox[4] is a well known example of reflexive paradox. Authors as Proclus,

[1] Next publication

[2] An injection is a correspondence between the elements of two sets A and B such that each and every element of A is paired off with a different element of B.

[3] The assumption that the whole is greater than the part is one of the Common Notions (general axioms) that is assumed in the First Book of Euclid's *Elements* [8, p 19].

[4] The elements of the set of the natural numbers can be paired with the elements of one of its proper subsets: the subset of their squares: $1 \leftrightarrow 1^2$, $2 \leftrightarrow 2^2$, $3 \leftrightarrow 3^2$, $4 \leftrightarrow 4^2$,

J. Filopón, Thabit ibn Qurra al-Harani, R. Grosseteste, G. of Rimini, W. of Ockham etc. found many other examples [12].

4. The strategy of pairing off the elements of two sets is not just a modern invention. Aristotle used it when trying to solve Zeno's Dichotomy (in its two variants).[5] And since then, it has been extensively used by different authors with different purposes, although, before Dedekind and Cantor, they were never used (including the case of Bolzano [3]) as an instrument to consummate the violation of the old Euclidean axiom. Of course, the existence of a one to one correspondence between two infinite sets does not prove both set are actually infinite because they could also be potentially infinite.

5. Things began to change with Dedekind, who stated the definition of the infinite sets just on the basis of that violation: a set is infinite if its elements can be put into a one to one correspondence with the elements of one of its proper subsets [6]. Dedekind and Cantor inaugurated the so called paradise of the actual infinity, where exhaustive injections (bijections or one to one correspondences) play a capital role.

5.2 Paradoxes or contradictions?

6. An exhaustive injection between two sets A and B is a correspondence between the elements of both sets in which each element of A is paired off with a different element of B, and all elements of A and B result paired. When at least one element of the set B results unpaired the injection is said non-exhaustive. Exhaustive and non-exhaustive injections can be used to compare the cardinality of the finite sets. But if the compared sets are infinite, then only exhaustive injections are permitted. An inevitable consequence of assuming that the infinite sets violate, by definition, the Axiom of the Whole and the Part.

7. But, since definitions can also be inconsistent,[6] the infinite sets could have been defined inconsistently on the basis of one of the terms of a contradiction: there is an exhaustive injection between the set and one of its proper subsets. The other part of the contradiction would be: there is a non-exhaustive injection between the set and the same proper subset. No one has ever explained why to have an exhaustive injection with a proper

$5 \leftrightarrow 5^2 \ldots$ [10].

[5] Aristotle finally rejected his pairing method and proposed the distinction between the actual and the potential infinity [2], [1].

[6] Specially when the definition is based on the violation of a basic axiom, as is the case of the Dedekind's definition of the infinite sets.

subset and at the same time to have a non-exhaustive injection with the same proper subset is not contradictory. The problem has simply been ignored and set theory has been raised on the basis of that ignoring.

8. If the notion of set is primitive (undefinable), as it seems to be, then we could only provide operational definitions of set. And if sets may have different cardinalities, we should establish an appropriate basic method for comparing cardinalities *before* defining the types of sets that could be defined according to their cardinals, especially if the comparing method forms part of the definition, as is the case of the definition of the infinite sets.

9. To pair off the elements of two sets is a basic and legitimate method for comparing their respective cardinalities, being unnecessary any other arithmetical or set theoretical operation. It is at this foundational level of set theory where we will discuss if exhaustive and non exhaustive injections are appropriate operations to get conclusions on the cardinality of any two sets. So, this question should be elucidate before trying any definition involving cardinalities, as the definition of the infinite sets.

10. It seems reasonable to assume that if after pairing every element of a set A with a different element of a set B, all elements of B result paired, then A and B have the same number of elements. But it seems also reasonable, and for the same elementary reasons, to assume that if after pairing every element of a set A with a different element of a set B one or more elements of the set B remain unpaired, then A and B do not have the same number of elements. It is worth noting that both exhaustive and non-exhaustive injections make use of *the same basic method of pairing elements*, without carrying out any finite or transfinite arithmetic operation. We are not counting but pairing, we are discussing at the most basic foundational level of set theory.

11. It should be recalled at this point that the arithmetic peculiarities of transfinite cardinals, as $\aleph_o = \aleph_o + \aleph_o$ and the like, are of all them derived from the hypothetical existence (Axiom of Infinity) of the infinite sets, i.e. of sets whose elements can, by definition, be paired with the elements of some of their proper subsets. So, under penalty of circular reasoning, we cannot infer from the deduced existence of those arithmetical 'peculiarities' (that could be used to justify the existence of exhaustive and non exhaustive injections between an infinite set and some of its proper subsets), the existence of just the sets from which those arithmetic peculiarities of infinite cardinals have been deduced. This is an unacceptable circular argument.

Here, we are simply discussing if the method of pairing the elements of two sets is appropriate to compare their respective cardinalities; and if it is, why non-exhaustive injections are rejected, because that rejection could be concealing a fundamental contradiction.

12. For example, consider the set $\mathbb{N}$ of the natural numbers, the sets $\mathbb{E}$ and $\mathbb{O}$ of even and odd numbers respectively, and the injection f from $\mathbb{E}$ to $\mathbb{N}$ defined by:

$$f(e) = e; \ \forall\, e \in \mathbb{E} \tag{1}$$

The injection f is non-exhaustive since all odd numbers in $\mathbb{O} \subset \mathbb{N}$ remains unpaired. Assume that, consequently, we write:

$$|\mathbb{E}| < |\mathbb{N}| \tag{2}$$

On the other hand, Dedekind's definition leads immediately to

$$\aleph_o = \aleph_o + \aleph_o \tag{3}$$

and then:

$$|\mathbb{E}| = |\mathbb{E}| + |\mathbb{O}| = |\mathbb{N}| \tag{4}$$

that invalidates (2). So, to say that (3)-(4) invalidate (2), is the same as to say that Dedekind's definition invalidates (2), which could be legitimately interpreted as if a term of a contradiction (to have an exhaustive injection, $|\mathbb{E}| = |\mathbb{N}|$) invalidates the other (to have a non-exhaustive injection $|\mathbb{E}| < |\mathbb{N}|$).

13. Exhaustive and non-exhaustive injections should have the same validity as instruments to compare the cardinality of the infinite sets just because they use exactly the same comparison method: to pair elements. However, only exhaustive injections can be used with that purpose. But why? Why some pairings are valid while some others are not, if all of them have the same basic legitimacy? The problem here is that the existence of both exhaustive and non-exhaustive injections between two infinite sets could be indicating the existence of an elementary contradiction (that both infinite sets have and have not the same cardinality), in which case the distinction of exhaustive injections would be the distinction of a term of a contradiction to the detriment of the other.

14. At the very least, the alternative of considering a set as inconsistent because of the existence of both exhaustive and non-exhaustive injections with the elements of the same proper subset is as legitimate as the alternative of considering it as consistent. Thus, at the very least, the arbitrary election of the second alternative should be explicitly declared at the foundational level of the theory, which is not the case in current set theories. Current set theories systematically ignore the first alternative. It could be argued that Dedekind's definition **5** implies to assume the existence of sets for which there exist both exhaustive and non-exhaustive injections with at least one of its proper subsets, but a simple definition does not guarantee the defined object is consistent, and then the alternative of the inconsistency has also to be considered. The proposal of this consideration is just the main objective of this discussion. A consideration that, for all I know, has never been seriously proposed.

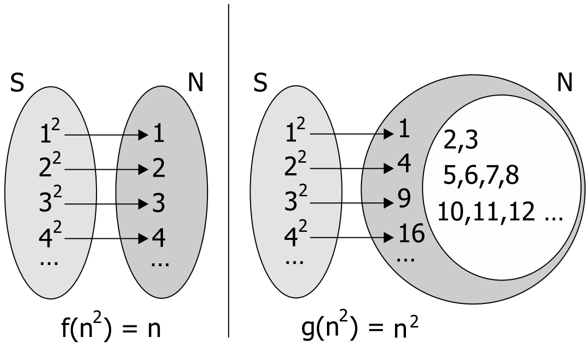

Figura 5.1 – The suspicious power of the ellipsis: the sets S and N have (left) and not have (right) the same number of elements.

15. Assume, only for a moment, that exhaustive and non exhaustive injections were valid instruments to compare the cardinality of any two sets. In these conditions, let N be an infinite set. By definition, there exists a proper subset S of N and an exhaustive injection f from S to N proving both sets have the same number of elements. Consider now the injection g from S to N defined by:

$$g(x) = x, \ \forall x \in S \tag{5}$$

which evidently is non-exhaustive (the elements of the nonempty set N-S remain unpaired). The injections f and g would be proving that S and N have (f) and not have (g) the same number of elements, i.e. that the infinite sets are inconsistent (Figure 5.1).

16. We must therefore decide if exhaustive and non-exhaustive injections do have the same validity as instruments to compare the number of elements of any two sets. If they do, then the actually infinite sets are inconsistent. If they don't, at least one (non-circular, not related to the definition of infinite set) reason should be given to explain why they don't. And, if no reason can be given, then the arbitrary distinction in favor of the exhaustive injections should be declared in an appropriate ad hoc axiom. Until then, the foundation of set theory rests on the basis of one of the terms of a possible contradiction.[7]

17. As could be expected from a theory with such foundations, inconsistencies appeared from its very beginning: the set of all ordinals and the set of all cardinals were proved to be inconsistent by Burali-Forti [4] and Cantor respectively. According to Cantor those sets are inconsistent because of their excessive infinitude.[8] One can be infinite but only within certain *limits*. By the appropriate axiomatic restrictions, it was finally stated that some infinite totalities, as the totality of cardinals or the totality of ordinals, do not exist because they lead to contradictions. It can easily be proved, as we will see in the next chapter, that in a naive (not limited by axiomatic restrictions) infinitist set theory, as Cantor's set theory, each set of cardinal C originates nothing less than 2^C inconsistent infinite totalities.

18. In Chapter 29 we will see that Riemann's series theorem can also be reinterpreted as a proof of the inconsistency of the actual infinity hypothesis. In the remainder of the book, more than twenty arguments will be developed all of them proving the same conclusion.

[7]Unbelievable as it may seem, the axiomatic foundation of set theory has always ignored this problem.

[8]Letter to Dedekind quoted in [5, pag. 245], [11], [9].

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