5. The paradoxes of reflexivity revisited

Chapter of the book *Infinity Put to the Test* by A. León (Free pdf)

5.1 Introduction

If after pairing each element of a set A with a different element of another set B all elements of B result paired, it is said both sets have the same number of elements (the same cardinality). But if one or more elements of B result unpaired and B is infinite, it is not always allowed to say both sets have a different number of elements, a different cardinality. In this chapter we discuss why it is not.

An injection is a correspondence between the elements of two sets A and B such that each element of A is paired off with a different element of B. If all elements of B are also paired, the injection is said exhaustive or surjective (it is also said a bijection or one to one correspondence); otherwise it is said non-exhaustive, or non-surjective. As we will see, the existence of both exhaustive and non-exhaustive injections between two infinite sets could be indicating they have and have not the same cardinality. Thus, the arbitrary distinction of the exhaustive injections to the detriment of the non-exhaustive ones *could be* concealing a fundamental contradiction in set theory.

Most of the paradoxes related to the actual infinity result from the violation of the Axiom of the Whole and the Part (the assumption that the whole is greater than the part), one of the Common Notions assumed in the First Book of Euclid's *Elements* [87, p 19]. Among the paradoxes resulting from that violation are the so called paradoxes of reflexivity in which the elements of a whole are paired off with the elements of one of its proper parts [219, 73]. A well-known example of this kind of paradox is Galileo Paradox: the elements of the set of the natural numbers can be paired with the elements of one of its proper subsets, the subset of their squares [99]):

$$f(n) = n^2,\ \forall n \in \mathbb{N}:\ 1 \leftrightarrow 1^2, 2 \leftrightarrow 2^2, 3 \leftrightarrow 3^2 \dots \qquad (1)$$

Authors as Proclus, J. Filopón, Thabit ibn Qurra al-Harani, R. Grosseteste, G. of Rimini, W. of Ockham etc. found many other examples [219].

The strategy of pairing off the elements of two sets is not just a modern invention. In a certain way, Aristotle used it when trying to solve Zeno's Dichotomy in its two variants [10, 11]. And since then, it has been frequently used by different authors with different level of formalism and different purposes, although, before Dedekind and Cantor, they were never used (including the case of Bolzano [29]) as an instrument to consummate the violation of the old Euclidean axiom. Of course, the existence of a one to one correspondence between two infinite sets does not prove both sets are actually infinite because they could also be potentially infinite.

Things began to change with Dedekind, who stated the definition of infinite set (Definition (p. 25)) just on the basis of that violation. Dedekind and Cantor inaugurated the so called paradise of the actual infinity, where exhaustive injections (bijections or one to one correspondences) play a major role.

5.2 Paradoxes or contradictions?

As indicated above, an exhaustive injection of a set A into another set B is a correspondence between the elements of both sets in which each element of A is paired off with a different element of B, and all elements of A and B result paired. When at least one element of the set B results unpaired the injection is said non-exhaustive. Exhaustive and non-exhaustive injections can be used to compare the cardinality of the finite sets. But if the compared sets are infinite, then only exhaustive injections are permitted. An inevitable consequence of assuming that the infinite sets violate, by definition, the Axiom of the Whole and the Part.

But definitions can also be inconsistent. Specially when the definition is based on the violation of a basic axiom, as is the case of Dedekind's Definition (p. 25) of infinite set. The infinite sets could have been defined inconsistently on the basis of one of the terms of a contradiction: there is an exhaustive injection between a set A and one of its supersets B. The other part of the contradiction would be: there is a non-exhaustive injection between the set and the same superset. No one has ever explained why to have an exhaustive injection with a superset ($|A| = |B|$) and at the same time to have a non-exhaustive injection with the same superset ($|A| < |B|$) is not contradictory. The problem has simply been ignored (justifying it with Dedekind's Definition (p. 25)), and set theory has been raised on the basis of that ignorance.

If the notion of set is primitive (undefinable), as it seems to be, then only operational definitions of set could be given. And if sets may have different cardinalities, then an appropriate basic method for comparing cardinalities should be established *before* defining the types of sets that could be defined according to their cardinals, especially if the comparing method has to form part of the definition, as is the case of the Definition (p. 25) of infinite set.

To pair off the elements of two sets is a basic and legitimate method for comparing their respective cardinalities, being unnecessary any other arithmetical or set theoretical operation. It is at this foundational level of set theory where it would have to be discussed if exhaustive and non exhaustive injections are appropriate operations to get conclusions on the cardinality of any two sets. So, this question should be elucidate before trying any definition involving cardinalities, as the definition of infinite set.

It seems reasonable to assume that if after pairing every element of a set A with a different element of a set B, all elements of B result paired, then A and B have the same number of elements. But it seems also reasonable, and for the same elementary reasons, to assume that if after pairing every element of a set A with a different element of a set B one or more elements of the set B remain unpaired, then A and B do not have the same number of elements. It is worth noting that both exhaustive and non-exhaustive injections make use of *the same basic method of pairing elements*, without carrying out any finite or transfinite arithmetic operation. We are not counting but pairing, we are discussing at the most basic foundational level of set theory.

It should be recalled at this point that the arithmetic peculiarities of transfinite cardinals, as $\aleph_0 = \aleph_0 + \aleph_0$ and the like (some of them are discussed in Chapter 20), are of all them derived from the hypothetical existence of the infinite sets (Axiom of Infinity), i.e. of sets whose elements can, by definition, be paired with the elements of some of their proper subsets. So, under penalty of circular reasoning, we cannot infer from the deduced existence of those arithmetical peculiarities the existence of just the sets from which those arithmetic peculiarities of infinite cardinals have been deduced (peculiarities that could be used to justify the existence of exhaustive and non exhaustive injections between an infinite set and some of its supersets). This is an unacceptable circular argument. Here, we are simply discussing if the method of pairing the elements of two sets is appropriate to compare their respective cardinalities; and if it is, why non-exhaustive injections are rejected, because that rejection could be concealing a fundamental contradiction.

P9 For example, consider the set $\mathbb{N}$ of the natural numbers, the sets $\mathbb{E}$ and $\mathbb{O}$ of even and odd numbers respectively, and the injection f from $\mathbb{E}$ to $\mathbb{N}$ defined by:

$$f(e) = e; \ \forall e \in \mathbb{E} \tag{2}$$

The injection f is non-exhaustive since all odd numbers in $\mathbb{O} \subset \mathbb{N}$ remains unpaired. Assume that, consequently, we write:

$$|\mathbb{E}| < |\mathbb{N}| \tag{3}$$

On the other hand, the injection g of $\mathbb{E}$ in $\mathbb{N}$ defined by:

$$g(e) = e/2; \ \forall e \in \mathbb{E} \tag{4}$$

is exhaustive. Therefore, and according to Dedekind's Definition (p. 25), $\mathbb{N}$ is infinite, and $\mathbb{E}$ has the same cardinality as $\mathbb{N}$. In consequence:

$$|\mathbb{E}| = |\mathbb{N}| \tag{5}$$

that contradicts (3). Consequently, to say that (5) invalidates (3) because (5) is Dedekind's Definition (p. 25), can be legitimately interpreted as if one term of a contradiction ($|\mathbb{E}| = |\mathbb{N}|$) is used to define a class of objects (the infinite sets), then the other term of the contradiction ($|\mathbb{E}| < |\mathbb{N}|$) is invalidated. We would have finally found the ultimate way to end all contradictions.

Exhaustive and non-exhaustive injections should have the same validity as instruments to compare the cardinalities of the infinite sets just because they use exactly the same comparison method: to pair elements. However, only exhaustive injections can be used with that purpose. But why? Why some pairings are valid while some others are not, if all of them have the same basic legitimacy? The problem here is that the existence of both exhaustive and non-exhaustive injections between two infinite sets could be indicating the existence of an elementary contradiction (that both infinite sets have and have not the same cardinality). In this case the distinction of the exhaustive injections would be the distinction of a term of a contradiction ($|\mathbb{E}| = |\mathbb{N}|$) to the detriment of the other ($|\mathbb{E}| < |\mathbb{N}|$). Or in other words, one term of a contradiction ($|\mathbb{E}| = |\mathbb{N}|$) would be being used to define an object (the infinite sets), while ignoring the other term of the contradiction ($|\mathbb{E}| < |\mathbb{N}|$).

At the very least, the alternative to consider a set as inconsistent because of the existence of both exhaustive and non-exhaustive injections with the elements of the same superset is as legitimate as the alternative to consider it as consistent. Thus, at the very least, the arbitrary election of the second alternative should be explicitly declared at the foundational level of the theory, which is not the case in current set theories. Current set theories systematically ignore the first alternative. It could be argued that Dedekind's Definition (p. 25) implies to assume the existence of sets for which there exist both exhaustive and non-exhaustive injections with at least one of its supersets. But, for the reason given in P9, a simple definition does not guarantee the defined object is consistent, and then the alternative of the inconsistency has also to be considered. To propose such an alternative is the main objective of this chapter. An alternative that, for all I know, has never been proposed.

Assume, only for a moment, that exhaustive and non exhaustive

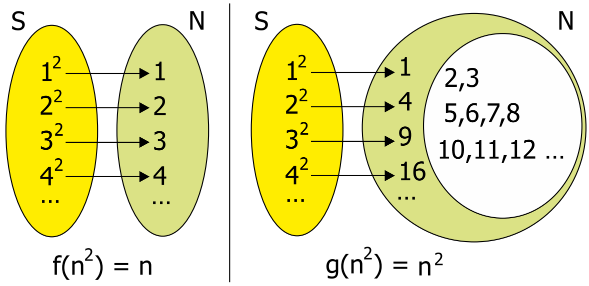

Figure 5.1 – The suspicious power of the ellipsis: the sets S and N have (left) and not have (right) the same number of elements.

injections were valid instruments to compare the cardinality of any two sets. In these conditions, let N be an infinite set (Figure 5.1). By definition, there exists a proper subset S of N and an exhaustive injection f from S to N proving both sets have the same number of elements. Consider now the injection g from S to N defined by:

$$g(x) = x, \ \forall x \in S \tag{6}$$

which evidently is non-exhaustive (the elements of the nonempty set N-S remain unpaired). The injections f and g would be proving that S and N have (f) and not have (g) the same number of elements, i.e. that the infinite sets are inconsistent.

We must therefore decide if exhaustive and non-exhaustive injections do have the same validity as instruments to compare the number of elements of any two sets. If they do, then the actually infinite sets are inconsistent. If they do not, at least one non-circular reason (i.e. unrelated to Dedekind's definition) should be given to explain why they do not. If no reason can be given, then the arbitrary distinction in favor of the exhaustive injections should be declared in an appropriate ad hoc axiom.

Although less satisfactory, it would also be a valid alternative that the Axiom of Infinity states explicitly that the infinite set whose existence is being proposing meets Dedekind's definition, because the set the Axiom of Infinity proposes:

$$\exists N((\emptyset \in N) \wedge (\forall x \in N, \ x \cup \{x\} \in N)) \tag{7}$$

could, or could not, be considered as a set satisfying that definition. Until this problem is solved, the foundation of set theory rests on the basis of one of the terms of a contradiction. Unbelievable as it may seem, the axiomatic foundation of set theory has always ignored this problem.

As could be expected from a theory with such initial foundations, inconsistencies appeared immediately: the set of all ordinals and the set of all cardinals were proved to be inconsistent by Burali-Forti [33] and Cantor respectively. According to Cantor, those sets are inconsistent because of their excessive infinitude (letter to Dedekind quoted in [68, pag. 245], [100, 90]). A set can be infinite but not too infinite. By the appropriate axiomatic restrictions, it was finally stated that some infinite totalities, as the totality of cardinals or the totality of ordinals, do not exist because they lead to contradictions. It can easily be proved, as we will see in the next chapter, that in a set theory without axiomatic restrictions, as Cantor's set theory, each (finite or infinite) set of cardinal C originates nothing less than 2^C inconsistent infinite totalities. Even Riemann's Series Theorem can be reinterpreted as the proof of the existence of another infinitude of inconsistent infinite totalities (Chapter 34)