20 Thomson's lamp revisited

(Adapted from a chapter of the book Infinity put to the test by Antonio León[1])

Abstract.-The argument of Thomson lamp and Benacerraf's critique are reexamined from the perspective of the w-order legitimated by the hypothesis of the actual infinity subsumed into the Axiom of Infinite. The conclusions point to the inconsistency of that hypothesis.

Keywords: Thomson lamp, actual infinity, supertask, P. Benacerraf, counting machine.

20.1 Introduction

1. Although Benacerraf's criticism of Thomson's lamp argument is well founded (see below), it is far from being complete. As we will see here, it is possible to consider a new line of argument, which Benacerraf only incidentally considered, based on the formal definition of the lamp. That line of argument leads to a contradictory result that compromises the formal consistency of the ω-order involved in all ω-supertasks.

2. To perform an ω-supertask (supertask hereafter) means to perform an ω-ordered sequence of actions (tasks) in a finite interval of time. Supertasks are useful theoretical devices for the philosophy of mathematics, particularly for the discussions on certain problems related to infinity.[2] Although their physical possibilities and implications have also been discussed.[3] Notwithstanding, here we will only deal with conceptual supertasks.

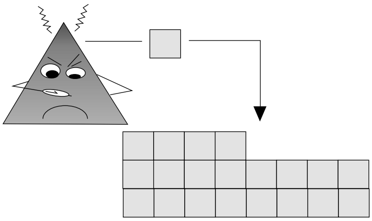

Figura 20.1 – God performing Gregory's supertask.

3. Probably Gregory of Rimini was the first to propose how a supertask

[1] Next publication

[2] [39], [8], [11], [33], [6], [41], [33]

[3] [28], [29], [33], [35], [17], [19], [18], [29], [30], [31], [15], [32], [27], [3], [4], [34] [41], [21], [13], [14], [27], [12], [36]

could be accomplished ([26], p. 53):

> If God can endlessly add a cubic foot to a stone -which He can- then He can create an infinitely big stone. For He need only add one cubic foot at some time, another half an hour later, another a quarter of an hour later than that, and so on *ad infinitum*. He would then have before Him an infinite stone at the end of the hour.

But the term "*supertask*"was introduced by J. F. Thomson in his seminal paper of 1954 [39]. Thomson's paper was motivated by Black's argument [7] on the impossibility to perform infinitely many successive actions and by the discussions of Black's argument by R. Taylor [38] and J. Watling [40]. In his paper Thomson tried to prove the impossibility of supertasks. Thomson argument was, in turn, criticized in another seminal paper, in this case by P. Benacerraf [5]. Benacerraf's successful criticism finally motivated the foundation of a new infinitist theory independent of set theory: supertask theory.

4 . The possibilities to perform an uncountable infinitude of actions were examined, and ruled out, by P. Clark and S. Read [11]. Supertasks have also been considered from the perspective of nonstandard analysis,[4] although the possibilities to perform an *hypertask* along an hyperreal interval of time have not been discussed, despite the fact that finite hyperreal intervals can be divided into hypercountably many successive infinitesimal intervals (hyperfinite partitions).[5] But most of the supertasks are ω-supertasks, i.e. ω-ordered sequences of actions performed in a finite (or perceived as finite) interval of time. As could not be otherwise, all supertask arguments assume both the Principle of Invariance and the Principle of Autonomy (see end note).

5 . The basic idea of Benacerraf's criticism of Thomson's argument is the impossibility to derive formal conclusions on the final state of the supermachine that performs the supertask from the sequence of states the machine traverses as a consequence of performing the supertask. But, as we will see, Benacerraf's analysis of Thomson's lamp argument is incomplete.

6 . In fact, if the world continues to be the same world it was before the execution of a supertask, and one is still allowed to think in rational terms in the same framework of the laws of logic, then Thomson's argument can be reoriented towards the formal definition of the machine that performs the supertask. A definition that is assumed to be independent of the num-

[4] [25], [24], [2], [23]
[5] [37], [16], [22], [20], etc.

ber of performed tasks with that machine (Principle of Autonomy), and then a definition that holds before, during and after performing the supertask, whenever the execution of a supertask does not arbitrarily change a legitimate definition previously established (Principle of Invariance). The discussion that follows on Thomson's lamp presumes that is in fact the case.

20.2 Thomson's lamp

7. As Thomson did in 1954 ([39], p. 5), in the following discussion we will make use of one of those:

> ... reading-lamps that have a button in the base. If the lamp is *off* and you press the button the lamp goes *on*, and if the lamp is *on* and you press the button the lamp goes *off*.

Let us complete Thomson's definition by explicitly declaring the following conditions regarding the (theoretical) functioning of the lamp::

1.- Thomson's lamp has only two states: on and off.

2.- The state of the lamp (on/off) changes if, and only if, its button is pressed down.

3.- Each change of state takes place at a precise and definite instant.

4.- The pressing down (clicking) of the button and the corresponding lamp change of state (on/off) are both instantaneous and simultaneous events.

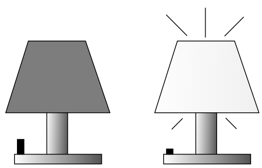

Figura 20.2 – Thomson's lamp has two, and only two, states: off (left) and on (right). The state of Thomson's lamp changes if, and only if, its button is pressed.

8. Assume now the button of Thomson's lamp is clicked at each of the infinitely many successive instants t_i, and only at them, of a strictly increasing ω-ordered sequence of instants $\langle t_n \rangle$ defined within a finite interval of time (t_a, t_b), being t_b the limit of the sequence $\langle t_n \rangle$. In these conditions, at instant t_b the button of the lamp will have undergone an ω-ordered sequence $\langle c_n \rangle$ of clicks (each click c_i performed at the precise instant t_i)

and, consequently, the state of the lamp will have changed an ω-ordered infinitude of times. Or in other words, at t_b Thomson's supertask will have been completed. Don't forget this is a purely conceptual argument, so that we are not concerned here with the physical details.

9. Thomson tried to derive a contradiction from his supertask by speculating on the final state of the lamp at instant t_b in terms of the sequence of switchings completed along the supertask ([39], p. 5):

> [The lamp] cannot be *on*, because I did not ever turn it *on* without at once turning it *off*. It cannot be *off*, because I did in the first place turn it *on*, and thereafter I never turned *off* without at once turning it *on*. But the lamp must be either *on* or *off*. This is a contradiction.

10. It is worth noting, as we have just seen, that Thomson based his argument on the sequence of actions carried out on the lamp: it was never turned on without turning it off after, and viceversa. What Thomson tried to do is to derive the final state of the lamp, the state of the lamp at t_b, from the successive changes of state the lamp underwent during the supertask: The reason why the lamp cannot be *on* is because it was always turned *off* after turning it *on*. And for the same reason it cannot be *off* either. This way of arguing was severely criticized by Benacerraf

11. Benacerraf argued against Thomson's argument as follows ([5], p. 768):

> The only reasons Thomson gives for supposing that his lamp will not be *off* at t_b are ones which hold only for times *before* t_b. The explanation is quite simply that Thomson's instructions do not cover the state of the lamp at t_b, although they do tell us what will be its state at every instant *between* t_a and t_b (including t_a). Certainly, the lamp must be *on* or *off* (provided that it hasn't gone up in a metaphysical puff of smoke in the interval), but nothing we are told implies which it is to be. The arguments to the effect that it can't be either just have no bearing on the case. To suppose that they *do* is to suppose that a description of the physical state of the lamp at t_b (with respect to the property of being *on* or *off*) is a *logical* consequence of a description of its state (with respect to the same property) at times prior to t_b. [t_a and t_b appears respectively as t_0 and t_1 in Benacerraf's paper].

12. In short, according to Benacerraf, the problem posed by Thomson is not sufficiently described since no constraint have been placed on what happens at t_b [1]. But the only constraint on what happens at t_b is that Thomson's lamp continue to be Thomson's lamp. Or in other words, that the execution of a supertask does not change the formal definitions of the involved theoretical artifacts (Principle of Invariance). As we will see, the state of Thomson's lamp at t_b is not 'a *logical consequence of a description of its state (with respect to the same property) at times prior to* t_b' it is a logical consequence of being a Thomson lamp (Principle of Invariance). And this is pertinent to the case. It will be the key of the next argumentation.

13. Consider the instant t_b, the limit of the sequence $\langle t_n \rangle$ of instants at which the successive clicks $\langle c_n \rangle$ have been performed. That instant is, therefore, the first instant *after completing* the sequence of switchings. The first instant at which the button of the lamp is no longer clicked. Let now S_b be the state of the lamp at instant t_b. Being the state of a Thomson lamp, it can only be either *on* or *off*. And this conclusion has nothing to do with the number of previously performed switchings. The lamp will be either *on* or *off* because, being a Thomson's lamp, it has only two states: *on* and *off* (Principle of Invariance).

14. Some infinitist claim, however, that at t_b, after performing Thomson's supertask, the lamp could be in any unknown state, even in an exotic one. But a lamp that can be in an unknown state is not a Thomson's lamp: the only possible states of a Thomson's lamp are *on* and *off*. No other alternative is possible *without arbitrarily violating* the formal legitimate definition of Thomson's lamp. And we presume no formal theory is authorized to violate arbitrarily a formal definition, nor, obviously to change, in the same arbitrary terms, the nature of the world (Principle of invariance). It goes without saying that if that were the case any thing could be expected from that theory.

15. Others claim the state S_b is the consequence of completing the ω-ordered sequence of clicks $\langle c_n \rangle$, since that sequence, and only that sequence, has been carried out. But if to complete the sequence of clicks $\langle c_n \rangle$ means to perform each and every of the infinitely many clicks c_i, and only them, then we have a problem. The problem that no click c_i of $\langle c_n \rangle$ originates S_b. None. Indeed, if c_v is any element of $\langle c_n \rangle$ it cannot originates S_b because in such a case the button would have been clicked only a finite number v of times. That is to say, if we remove from $\langle c_n \rangle$ all clicks that do not originate S_b then all of them would be removed. Or in other, set theoretical, words, if from the set of performed clicking we remove each and every click that does not originate S_b, we would get the empty set.

16. In those conditions, how can it be claimed that the completion of the sequence of clicks $\langle c_n \rangle$, *none* of whose elements originates S_b, originates just S_b? Is the completion of the sequence an additional click different from all elements of $\langle c_n \rangle$? If that were the case the sequence of performed clicks would be $(\omega+1)$-ordered in the place of ω-ordered, but ω-supertasks are ω-ordered not $(\omega+1)$-ordered.

17. At this point some infinitists claim the lamp could be at S_b by reasons unknown. But, once again, that claim violates the definition of the lamp: the state of a Thomson's lamp changes exclusively by pressing down its button, by clicking its button. So a lamp that changes its state by reasons unknown is not, by definition, a Thomson's lamp (Principle of Invariance).

18. In any case, the relevant question on the state S_b is: at which instant Thomson's lamp becomes S_b? It is immediate to prove that instant can only be the precise instant t_b. We know the state of the lamp is S_b at instant t_b, but assume there exist an instant t within (t_a, t_b) at which the lamp becomes S_b. Since t_b is the limit of the sequence $\langle t_n \rangle$, we will have:

$$\exists v : \; t_v \leq t < t_{v+1} \tag{1}$$

which means that at t only a finite number v of clicks have been carried out, and then that infinitely many clickings still remain to be carried out. Therefore, no instant t exists in (t_a, t_b) at which the lamp becomes S_b. None. The precise instant at which the lamp becomes S_b is not within the open interval (t_a, t_b). Consequently, and being S_b the state of the lamp at the precise and definite instant t_b, Thomson's lamp can only becomes S_b at the precise instant t_b.

19. But t_b is not the instant at which the sequence of switchings $\langle c_n \rangle$ is completed; t_b is the first instant *after completing* the sequence. There is not, in fact, an instant at which that sequence is completed[6] because that sequence is ω-ordered and ω-ordered sequences have not last element. At t_b the sequence $\langle c_n \rangle$ of clicks, and then sequence $\langle S_n \rangle$ of changes of state of the lamp, *have already been completed.* At t_b the button of the lamp is not clicked. At t_b nothing happens that can produce a lamp change of state.

20. It makes no sense to argue about the last term of an ω-ordered sequence simply because such a last term does not exist. By contrast, we could always argue about the limit of an ω-ordered sequence, whenever that limit exists, because it is a well defined object, though it is not an element of the sequence. Similarly, whilst it makes no sense to argue about the last instant at which the button of Thomson's lamp is clicked, the instant t_b

[6]This poses an additional problem: how it is possible to complete a sequence of actions within the interval (t_a, t_b) if there is no instant within (t_a, t_b) at which the sequence is completed?

is plenty of meaning: it is limit of the sequence of instants at which the successive switchings are carried out; it is the first instant after completing the sequence of switchings. It is the first instant at which the button of the lamp is no longer clicked.

21. In accordance with 18-20, it cannot be claimed that S_b results from completing the sequence $\langle c_n \rangle$ of clicks: Thomson's lamp becomes S_b just at t_b and at t_b the sequence of clicks has already been completed; t_b is *posterior* to the completion of the sequence $\langle c_n \rangle$ of clicks. Thomson's lamp becomes S_b at the precise instant t_b, but nothing happens at the precise instant t_b for the lamp to become S_b:

1.- At t_b the sequence of clicks $\langle c_n \rangle$ has already been completed.

2.- At t_b the button of the lamp is not clicked.

S_b is then an impossible state, a consequence of assuming that it is possible to complete an uncompletable sequence of actions, uncompletable in the sense that no last element completes the sequence.

22. The fact that the elements of two incompletable sequences can be paired off by a one to one correspondence, as in the case of the above sequences of clicks and of instants, does not prove both sequences exist as complete infinite totalities: they could also be potentially infinite. The possibility of pairing off the elements of two impossible totalities does not make them possible

23. At this point, all that one can expect from infinitists is to be declared incompetent to understand the meaning of the sentence: '*the state of the lamp at t_b is the result of completing the ω-ordered sequence $\langle c_n \rangle$ of clicks, a result that "manifests" for the first time just at t_b*'. But, wait a moment, is not S_b the result of a pressing down the button of the lamp? Don't forget that Thomson's lamp can only change its state if, and only if, you press down its button, if you click it. And that both events, the clicking and the corresponding lamp change of state, are instantaneous and simultaneous by definition.

24. So if S_b appears for the first time at the precise instant t_b and at t_b the button of the lamp is not clicked, what is the cause of S_b? where does S_b come from?

25. In short, S_b must of necessity be originated just at instant t_b, otherwise only a finite number of clicks would have been performed, according to 18-20. But, on the other hand, it cannot be originated at t_b because:

1.- The state of the lamp changes only by clicking its button.

2.- The clicking of the button and the corresponding lamp change of state are instantaneous and simultaneous events that takes place at a definite and precise instant.

3.- Being the clicking of the button and the corresponding lamp change of state instantaneous and simultaneous events, and being the state S_b originated at the precise instant t_b, the button must be clicked at that precise instant t_b.

4.- But at t_b the button of the lamp is not clicked.

26. S_b could only be, therefore, the impossible last state of an ω-ordered sequence of states in which no last state exists. The imprint of an impossibility. The consequence of assuming the hypothesis of the actual infinity from which derives the existence of ω-ordered sequences as *complete totalities*, in spite of the fact that no last element completes them.

27. Thomson's lamp is a theoretical device intentionally invented to facilitate a formal discussion on the actual infinity hypothesis that legitimizes the existence of ω-ordered sequences as complete totalities [9], [10, Theorem 15-A]. Supertasks are an example of such sequences, and contradiction 25 clearly suggests the hypothesis on which they are founded could be inconsistent.

20.3 The counting machine

28. The Counting Machine (CM) we will examine in this section poses a problem similar to the one posed by Thomson's lamp we have just examined. As its name suggests, CM counts natural numbers, and it does it by counting the successive numbers 1, 2, 3... at each of the successive instants t_1, t_2, t_3... of the above sequence $\langle t_n \rangle$. It counts each number n at the precise instants t_n. In addition, the machine has a red LED that turns on when, and only when, the machine counts an even number; and turns off when, and only when, the machine counts an odd number. Obviously, the CM LED is a perfect LED that never fails.

29. The one to one correspondence f:

$$f : \langle t_n \rangle \mapsto \mathbb{N} \tag{2}$$

$$f(t_n) = n \tag{3}$$

proves that at t_b our machine will have counted all natural numbers. All. Thus, if after performing the supertask, our counting machine CM continues to be the same counting machine it was before beginning the supertask, i.e. if performing a supertask does not change the nature of the world nor implies the arbitrary violation of a legitimate formal definition (Principle of Invariance), as that of our CM, then the red LED of CM can only be either *on* or *off*, simply because an LED can only be *on* or *off*, independently of the number of times it has been turned on and off.

30. Assume then that at t_b the red LED is *on* (a similar argument would apply if it were *off*). One of the following two exhaustive and mutually exclusive alternatives must be true:

1.- The red LED is *on* because CM counted a last even number that left it *on*.

2.- The red LED is *on* because of any other reason.

The first alternative is impossible if *all* natural numbers have been in fact counted: each even number has an immediate odd successor and then there is not a last natural number, neither even nor odd. The second alternative would imply the formal definition of CM has been arbitrarily violated: its red LED turns on when, and only when, the machine counts an even number, which excludes the possibility of being turned on by any other reason (Principle of Invariance).

31. If the ω-ordered list of the natural numbers exists as a complete totality in spite of the fact that no last number completes the list, then our modest red LED is and is not *on*. Otherwise, a legitimate definition would be arbitrarily violated with the only purpose of justifying that our LED can be turned on by reasons different from the reason defined as the only reason by which the LED can be turned on, namely by counting an even number and only by counting an even number. In this case anything could be expected from the hypothesis of the actual infinity.

32. Notice again that, as in the case of Thomson's lamp, the above conclusion on the state of the LED once counted all natural numbers is not drawn from the successively performed tasks, but from the fact of being a perfect LED with two definite and precise states (on and off), and so that it turns on if, and only if, *CM* counts an even number.

End note:

> **Principle of Invariance**.-The completion of any finite or infinite sequence of steps of any argument, procedure, definition or proof, as such a completion, is not a new additional step that arbitrarily modify the properties of the intervening objects.

Principle of Autonomy.-The formal consistency of an argument does not depend on the actual existence of the intervening objects

Chapter References

[1] L. V. Allis and T. Koetsierx, *On Some Paradoxes of the infinite*, British Journal for the Philosophy of Science **46** (1991), 235 – 47.

[2] Joseph S. Alper and Mark Bridger, *Mathematics, Models and Zeno's Paradoxes*, Synthese **110** (1997), 143 – 166.

[3] ______, *On the Dynamics of Perez Laraudogotia's Supertask*, Synthese **119** (1999), 325 – 337.

[4] Joseph S. Alper, Mark Bridger, John Earman, and John D. Norton, *What is a Newtonian System? The Failure of Energy Conservation and Determinism in Supertasks*, Synthese **124** (2000), 281 – 293.

[5] Paul Benacerraf, *Tasks, Super-tasks, and Modern Eleatics*, J. Philos. **LIX** (1962), 765–784.

[6] Paul Benacerraf and H. Putnam, *Introduction*, Philosophy of Mathematics: Selected Readings (Cambridge) (P. Benacerraf and H. Putnam, eds.), Cambridge University Press, 1964, pp. 1–27.

[7] M. Black, *Achilles and the Tortoise*, Analysis **XI** (1950 - 51), 91 – 101.

[8] David Bostock, *Aristotle, Zeno, and the potential Infinite*, Proceedings of the Aristotelian Society **73** (1972), 37 – 51.

[9] Georg Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre*, Mathematishe Annalen **XLIX** (1897), 207 – 246.

[10] ______, *Contributions to the founding of the theory of transfinite numbers*, Dover, New York, 1955.

[11] P. Clark and S. Read, *Hypertasks*, Synthese **61** (1984), 387 – 390.

[12] John Earman, *Determinism: What We Have Learned and What We Still Don't Know*, Freedom and Determinism (Michael O'Rourke and David Shier, eds.), MIT Press, Cambridge, 2004, pp. 21–46.

[13] John Earman and John D. Norton, *Forever is a Day: Supertasks in Pitowsky and Malament-hogarth Spacetimes*, Philosophy of Science **60** (1993), 22–42.

[14] ______, *Infinite Pains: The Trouble with Supertasks*, Paul Benacerraf: The Philosopher and His Critics (S. Stich, ed.), Blackwell, New York, 1996.

[15] ______, *Comments on Laraudogoitia's 'classical Particle Dynamics, Indeterminism and a Supertask'*, The British Journal for the Phylosophy of Science **49** (1998), no. 1, 122 – 133.

[16] Robert Goldblatt, *Lectures on the Hyperreals: An Introduction to Nonstandard Analysis*, Springer-Verlag, New York, 1998.

[17] Adolf Grünbaum, *Modern Science and Refutation of the Paradoxes of Zeno*, Zeno's Paradoxes (Wesley C. Salmon, ed.), Hackett Publishing Company, Inc, Indianapolis/Cambridge, 2001, pp. 164 – 175.

[18] ______, *Modern Science and Zeno's Paradoxes of Motion*, Zeno's Paradoxes (Wesley C. Salmon, ed.), Hackett Publishing Company, Inc, Indianapolis/Cambridge, 2001, pp. 200 – 250.

[19] ______, *Zeno's Metrical Paradox of Extension*, Zeno's Paradoxes (Wesley C. Salmon, ed.), Hackett Publishing Company, Inc, Indianapolis/Cambridge, 2001, pp. 176 – 199.

[20] James M. Henle and Eugene M. Kleinberg, *Infinitesimal Calculus*, Dover Publications Inc., Mineola, New York, 2003.

[21] M. L. Hogarth, *Does General Relativity Allow an Observer to view an Eternity in a Finite Time?*, Foundations of Physics Letters **5** (1992), 173 – 181.

[22] H. Jerome Keisler, *Elementary Calculus. An Infinitesimal Approach*, second ed., Author, http://www.wisc.edu/ keisler/keislercalc.pdf, September 2002.

[23] William I. Maclaughlin, *Thomson's Lamp is Dysfunctional*, Synthese **116** (1998), no. 3, 281 – 301.

[24] William I. McLaughlin, *Una resolución de las paradojas de Zenón*, Investigación y Ciencia (Scientifc American) (1995), no. 220, 62 – 68.

[25] William I. McLaughlin and Silvia L. Miller, *An Epistemological Use of non-standard Analysis to Answer Zeno's Objections Against Motion*, Synthese **92** (1992), no. 3, 371 – 384.

[26] Andreas W. Moore, *The Infinite*, Routledge, New York, 2001.

[27] John D. Norton, *A Quantum Mechanical Supertask*, Found. Phys. **29** (1999), 1265 – 1302.

[28] I. Pitowsky, *The Physical Church Thesis and Physical Computational Complexity*, Iyyun **39** (1990), 81 –99.

[29] Jon Pérez Laraudogoitia, *A Beautiful Supertask*, Mind **105** (1996), 49–54.

[30] ______, *Classical Particle Dynamics, Indeterminism and a Supertask*, British Journal for the Philosophy of Science **48** (1997), 49 – 54.

[31] ______, *Infinity Machines and Creation Ex Nihilo*, Synthese **115** (1998), 259 – 265.

[32] ______, *Why Dynamical Self-excitation is Possible*, Synthese **119** (1999), 313 – 323.

[33] ______, *Supertasks*, The Stanford Encyclopaedia of Philosophy (E. N. Zaltax, ed.), Standford University, URL = http://plato.stanford.edu, 2001.

[34] Jon Pérez Laraudogoitia, Mark Bridger, and Joseph S. Alper, *Two Ways of Looking at a Newtonian Supertask*, Synthese **131** (2002), no. 2, 157 – 171.

[35] Wesley C. Salmon, *Introduction*, Zeno's Paradoxes (Wesley C. Salmon, ed.), Hackett Publishing Company, Inc, Indianapolis, Cambridge, 2001, pp. 5 – 44.

[36] Z. K. Silagadze, *Zeno meets modern science*, Philsci-archieve (2005), 1–40.

[37] K. D. Stroyan, *Foundations of Infinitesimal Calculus*, Academic Press, Inc, New York, 1997.

[38] Richard Taylor, *Mr. Black on Temporal Paradoxes*, Analysis **12** (1951 - 52), 38 – 44.

[39] James F. Thomson, *Tasks and Supertasks*, Analysis **15** (1954), 1–13.

[40] John Watling, *The sum of an infinite series*, Analysis **13** (1952 - 53), 39 – 46.

[41] H. Weyl, *Philosophy of Mathematics and Natural Sciences*, Princeton University Press, Princeton, 1949.