14 On infinite lengths and distances

(Draft chapter of the book *Infinity Put to the Test* by Antonio León[1])

Abstract.-It is proved in this chapter that every line in the Euclidean space $\mathbb{R}^3$ has a finite length and that the distance between any two of its points is always finite. These results are also confirmed by a supertask in which a circle with a finite diameter is translated infinitely many times along a straight line.

Keywords: Infinite lengths, infinite distances, ω-asymmetry.

14.1 Introduction

1. This chapter makes use of a few number of basic concepts of Euclidean geometry. Some of them, as point or line, are primitive concepts while other, as segment or distance, are formally defined in terms of these primitive concepts. I assume all of them are well known to the reader.

2. The ω-ordered sequence $\langle x_n \rangle$ of points within the real interval $(0, 1)$ defined by:

$$x_n = (2^n - 1)/2^n \tag{1}$$

is an example of ω-partition of a finite line segment. Each pair of successive points x_n, x_{n+1} defines a part of the partition. The successive parts are disjoint and adjacent, so that the right end of any of them coincides with the left end of the following one:

$$(x_1, x_2](x_2, x_3](x_3, x_4] \dots \tag{2}$$

3. As is well known, at least since the 18th century, ω-partitions of finite line segments are only possible if the successive adjacent parts of the ω-partition are of a decreasing length, otherwise the length of the line would have to be infinite [1]. This inevitable restriction originates a huge asymmetry in the partition. Indeed, whatever be the length of the ω-partitioned segment AB and whatever be the ω-partition, all its parts, except a finite number of them, will necessarily lie within a final interval CB arbitrarily small.

4. For the sake of illustration, consider an ω-partition of a straight line segment AB whose length is $9{,}3 \times 10^{10}$ light years, the assumed diameter of

[1] Next publication

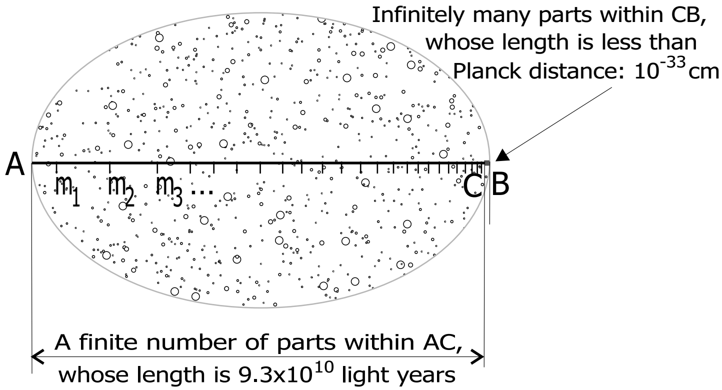

Figura 14.1 – Spatial ω-asymmetry in the ω-partition of a line segment AB whose length is the diameter of the visible universe.

the visible universe. Whatever be the ω-partition of this enormous segment all its infinitely many parts, except a finite number of them, will inevitably lie within a final interval CB inconceivable less than, for instance, Planck length ($\sim 10^{-33}$ cm). There is no way of performing a less asymmetric partition if the partition has to be ω-ordered, the smallest of the infinite partitions (Figure 14.1). Thus, ω-partitions are ω-asymmetrical. And being ω the smallest infinite ordinal, any transfinite partition has to contain at least one ω-ordered partition (Theorem of the ωth Element, Chapter **??**).

5. The unaesthetic consequence of the above asymmetry becomes a little more controversial if the parts of the partition are not of a decreasing length.

14.2 Euclidean lines and distances

6. In this section, only lines that do no intersect themselves will be considered. The length of a line AB whose endpoints are A and B will be denoted by AB. If AB is a straight line, AB is also the distance between A and B.

7. Let $f(x)$ be a real valued function whose graph is any curved line AB with two endpoints A and B in the Euclidean space $\mathbb{R}^3$. Let $\mathbf{P}$ be te sequence of all points P_1, P_2, $P_3\dots$ of AB defined according to: $\forall P_{i\geq 1}$: iff $P_iB \geq AP_1$, take a point P_{i+1} separated from P_i by a distance AP_1. Consider the closed segment QB whose length is AP_1. It holds: $\forall P_\alpha \in \mathbf{P}$ and $P_\alpha \in AQ$, there is at least one point $Q' \in QB$ such that $P_\alpha Q' \geq AP_1$. In consequence, there must be in QB one point P_ϕ of $\mathbf{P}$, otherwise $\mathbf{P}$ would not contain all points P_i of AB such that $P_{i-1}P_i = AP_1$, which is not the case. So, the sequence **P** has a last element P_ϕ. The endpoints A and B and the sequence **P** define in AB a sequence **S** of successive adjacent segments: AP_1, P_1P_2, $P_2P_3 \ldots P_\phi B$ of the same length AP_1, except at most the last one $P_\phi B \leq AP_1$, all of them left-open and right-closed, except AP_1 that is closed. In the ordering **O** of **S**, there is a first element AP_1; a last element $P_\phi B$; each element P_iP_{i+1} has an immediate predecessor $P_{i-1}P_i$ (or AP_1), except AP_1, and an immediate successor $P_{i+1}P_{i+2}$ (or $P_\phi B$), except $P_\phi B$; no element exists between any two of its successive elements; and any non empty subsequence **S'** of **S**, containing for instance P_vP_{v+1}, will also contain an element that precedes in the ordering **O** of **S** all elements of **S'** except itself: one of the elements AP_1, P_1P_2, $P_2P_3 \ldots P_vP_{v+1}$. So, **S** is a well ordered sequence, to which an ordinal number ϕ can be assigned [3, p. 152]. In addition **S** cannot be non-denumerable [2].

8. The ordinal ϕ of **S** cannot be the least transfinite ordinal ω because the sequences whose ordinal is ω (as the sequence of all finite ordinals 1, 2, 3,...) have not a last element, which is not the case of **S**. So, if the ordinal of **S** were infinite, it would be greater than ω, in which case there would be a first element succeeding all elements AP_1, P_1P_2, $P_2P_3 \ldots$ indexed by the sequence of all finite ordinals 1, 2, 3,... which can only be the limit of all them $P_\omega P_{\omega+1}$ [3, Theorem I, p. 158]. Take in AB a point R at any given distance from P_ω less than AP_1, and in the direction from P_ω to A. R could only belong to a segment P_vP_ω immediately preceding $P_\omega P_{\omega+1}$ (or of $P_\omega B$). But P_vP_ω is impossible because there is not a last finite ordinal v whose immediate successor $v+1$ is ω. Hence, the ordinal of **S** cannot be infinite but finite. **S** can only have a finite number of elements. And being finite the sum of any finite number of finite lengths, AB has a finite length.

9. The above argument P7-P8 proves the following:

Theorem of the Finite Segments.-*In the Euclidean space* $\mathbb{R}^3$*, any line with two endpoints has a finite length.*

And from this theorem it is immediate to prove the following:

Corollary of the Closed Lines.-*In the Euclidean space* $\mathbb{R}^3$*, any closed line has a finite length.*

Proof.-Let L be any closed line in the Euclidean space $\mathbb{R}^3$, and A and B any two of its points. A and B define in L and in the same direction of rotation two adjacent segments AB and BA whose lengths sum the length of the whole line L. According to the Theorem of the finite segments AB and BA are finite. So, L is also finite.

Corollary of the Finite Distances.-In the Euclidean space $\mathbb{R}^3$

the distance between any two of its points is always finite.
Proof Let A and B be any two points of the Euclidean space $\mathbb{R}^3$. Join them by a straight line AB. According to the theorem of the finite segments AB has a finite length. So, the distance from A to B, which is the length of AB, is finite.

Corollary of the Finite Lines.-*In the Euclidean space $\mathbb{R}^3$ straight lines of infinite length are impossible.*
Proof.-Let L be a straight line in the Euclidean space $\mathbb{R}^3$, and S the set of all couples of points of L separated by an infinite distance. According to the Theorem of the finite length, $S = \emptyset$. This conclusion is not an indeterminacy regarding the determination of a couple of points separated by an infinite distance; this conclusion is an impossibility: no couple of points exists in L separated by an infinite distance. So, L cannot have an infinite length.

14.3 A geometrical supertask

10. Let r be a straight line and c a circle of a finite diameter d and whose center is a points x_a of r. Assume that, in the direction from x_a to the right, r has an infinite length.

11. Let $\langle t_n \rangle = t_1, t_2, t_3 \ldots$ be an ω-ordered and strictly increasing sequence of instants within the finite interval (t_a, t_b), whose limit is t_b. And assume that at each instant t_i of $\langle t_n \rangle$, and only at each instant t_i of $\langle t_n \rangle$, the circle c is translated along the line r in the same direction from left to right and by a distance equal to its diameter d, so that its center its placed on a point x_i of the line r (Figure 14.2).

12. At t_b the circle c will have been translated infinitely many times in the same direction and by the same distance d along the straight line r. So, at t_b, and wherever it is, the circle c will continue to be a circle whose center will be a certain point x_b of r (Principle of Invariance).

13. We can consider, therefore, two points on the straight line r: the center x_a of c at instant t_a and the center x_b of c_b at instant t_b, after having performed its infinitely many translations along the straight line r. According to the theorem of the finite segments, the length L of the segment $x_a x_b$ will be finite. And being d and L two finite numbers, the number $k = L/d$ is also finite.

14. So, at t_b, and after being translated an infinite number of times, the circle c has only been translated a finite number k of times.This contradiction proves the inconsistency of the initial assumption on the infinite

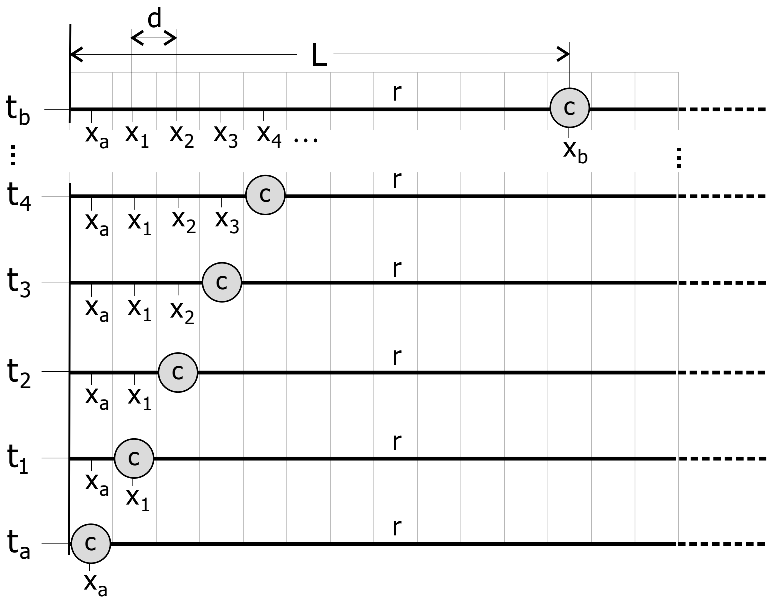

Figura 14.2 – Translating a circle along an infinite straight line

length of r.

15. The above argument can be applied to any type of line and figure. We must therefore conclude that all distances, lines and figures we can consider within a given space will always be finite, which suggests that space itself is also finite in all of its dimensions.

Chapter References

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