

# CANTOR VERSUS CANTOR

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**Abstract.**—This paper proves a result on the decimal expansion of the rational numbers in the open rational interval  $(0, 1)$ , which is subsequently used to discuss a reordering of the rows of a table  $T$  that is assumed to contain all rational numbers within  $(0, 1)$ , in such a way that the diagonal of the reordered table  $T$  could be a rational number from which different rational antidiagonals (elements of  $(0, 1)$  that cannot be in  $T$ ) could be defined. If that were the case, and for the same reason as in Cantor’s diagonal argument, the open rational interval  $(0, 1)$  would be non-denumerable, and we would have a contradiction in set theory, because Cantor also proved the set of rational numbers is denumerable.

## I. THEOREM OF THE $n$ th DECIMAL

**1.** Let  $\mathbb{Q}_{01}$  be the set of all rational numbers in the open rational interval  $(0, 1)$  expressed in decimal notation and completed, in the cases of finitely many decimal digits, with a denumerable infinite number of 0’s in the right side of their corresponding decimal expansions.

**2.** Let  $d$  be any decimal digit,  $n$  any natural number, and  $q_0$  any element of  $\mathbb{Q}_{01}$  whose  $n$ th decimal digit is just  $d$ , for instance  $q_0 = 0.11^{(n-1)}1d000\dots$ . From  $q_0$  it is possible to define different sequences of different elements of  $\mathbb{Q}_{01}$ , all of them with the same  $n$ th decimal digit  $d$ . For example the sequence  $\langle q_n \rangle$ :

$$\begin{aligned} q_1 &= 0.11^{(n-1)}1d1000\dots \\ q_2 &= 0.11^{(n-1)}1d11000\dots \\ q_3 &= 0.11^{(n-1)}1d111000\dots \\ &\dots \\ q_i &= 0.11^{(n-1)}1d111\dots^{(i)}1000\dots \\ &\dots \end{aligned}$$

The bijection (one to one correspondence)  $f$  between the  $\omega$ -ordered set  $\mathbb{N}$  (the well ordered set of natural numbers in their natural order of precedence, whose ordinal number is  $\omega$ , the least transfinite ordinal [1, §15, Theorem A, p.160]) and  $\langle q_n \rangle$  defined by

$$\forall i \in \mathbb{N} : f(i) = q_i \tag{1}$$

proves the following:

**Theorem of the  $n$ th decimal.** *For any given decimal digit and any given position in the decimal expansion of the elements of  $\mathbb{Q}_{01}$ , there exists a denumerable subset of  $\mathbb{Q}_{01}$ , each of whose elements has the given decimal digit in the given position of its decimal expansion.*

## II. A RATIONAL DIAGONAL ARGUMENT

**3.** Let  $\mathbb{Q}_{d_n}$  be the subset of  $\mathbb{Q}_{01}$  each of whose elements has the same decimal digit  $d_n$  in the same  $n$ th position of its decimal expansion. According to the theorem of the  $n$ th decimal,  $\mathbb{Q}_{d_n}$  is denumerable. So, its superset  $\mathbb{Q}_{01}$  will be infinite, either denumerable or non-denumerable. So, let  $g$  be any injective function between  $\mathbb{N}$  and  $\mathbb{Q}_{01}$ . This function allows to define a table  $T$  whose successive rows  $r_1, r_2, r_3 \dots$  are just the successive images  $g(1), g(2), g(3) \dots$

**4.** According to Cantor, two sets that can be put into a one to one correspondence (*similar* sets in Cantor’s terminology [1, §7, p. 112]) have the same order type [1, §14, p. 152]. Consequently,  $T$  has the same order type as  $\mathbb{N}$  ( $\omega$ -order): it has a first row  $r_1$  but not a last one, and each row  $r_i$  of  $T$  has an immediate preceding row  $r_{i-1}$  (except  $r_1$ ) and an immediate succeeding row  $r_{i+1}$ . Therefore, to assume the existence of the set of all finite natural numbers as a complete infinite totality, as Cantor did in 1883 [1, pp. 103-104], means to assume the rows of  $T$  also exist as a complete infinite totality. According to this Cantor’s assumption (hypothesis of actual infinity subsumed into the Axiom of Infinity in modern set theories), every row  $r_n$  of  $T$  will be preceded by a finite number,  $n - 1$ , of rows and succeeded by an infinite number,  $\aleph_0$  [1, §6, pp. 103-104], of rows. We will now examine a conflicting consequence of this asymmetry.

**5.** The diagonal  $D = 0.d_{11}d_{22}d_{33} \dots$  of  $T$  is a real number within  $(0, 1)$  whose  $n$ th decimal digit  $d_{nn}$  is the  $n$ th decimal digit of the  $n$ th row  $r_n$  of  $T$ . As in Cantor’s diagonal argument [2], it is possible to define another real number  $A$ , said antidiagonal, by replacing each of the infinitely many decimal digits of  $D$  with a different decimal digit. By construction  $A$  cannot be in  $T$  because it differs from each row  $r_i$  of  $T$  at least in its  $i$ th decimal digit. Since  $A$  is a real number within  $(0, 1)$ , it will be either rational or irrational. If it were rational, and for the same reason as in Cantor’s diagonal argument,  $g$  would not be a surjective function.

**6.** A row  $r_i$  of  $T$  will be said  $n$ -modular if its  $n$ th decimal digit is  $n \pmod{10}$ . This means that a row is, for instance, 2348-modular if its 2348th decimal digit is 8; or that it is 45390-modular if its 45390th decimal digit is 0. If a row  $r_n$  is  $n$ -modular (being  $n$  in  $n$ -modular the same number as  $n$  in  $r_n$ ) it will be said  $D$ -modular. For instance, the rows:

$$\begin{aligned} r_1 &= 0.1007647464749943400034577774413\dots \\ r_2 &= 0.2200045667778943000000000000000\dots \\ r_3 &= 0.0030000000000000000000000000000\dots \\ r_7 &= 0.1001007000111111110000000000000\dots \\ r_{19} &= 0.1234567890123456789000000000000\dots \end{aligned}$$

are all of them  $D$ -modular. It is clear that certain rational numbers as  $0.\overline{43}$  or  $0.3353333333$  cannot be  $D$ -modular, whatever be their corresponding rows in  $T$ . As we will see in 14, these rational numbers pose additional problems to the hypothesis of the actual infinity.

**7.** Consider now the following permutation  $\mathbf{D}$  of the rows  $\langle r_n \rangle$  of  $T$ . For each successive row  $r_i$  of  $T$ :

- ▷ If  $r_i$  is  $D$ -modular then let it unchanged.
- ▷ If  $r_i$  is not  $D$ -modular then exchange it with any following  $i$ -modular row  $r_j, j > i$ , provided that at least one of the rows  $r_j, j > i$  be  $i$ -modular. Otherwise let it unchanged.

The exchange of a non- $D$ -modular row  $r_i$  with a following  $i$ -modular row will be referred to as  $D$ -exchange (see Figure 1). Thanks to the condition  $j > i$  (in  $r_j, j > i$ ), once a row  $r_i$  has been  $D$ -exchanged, it becomes  $D$ -modular and will remain  $D$ -modular and unaffected by the subsequent  $D$ -exchanges. On the other hand, the successive  $D$ -exchanges

136900987838344...	136900987838344...
828282828282828...	828282828282828...
103989745600000...	103989745600000...
03296700111111...	65540000000000...
...	...
65540000000000...	03296700111111...
345787352637839...	345787352637839...
...	...

FIGURE 1. The fourth row of  $T$  before being D-exchanged (Left); and after having been D-exchanged (right).

do not change the  $\omega$ -order of  $T$  but the rational numbers indexed by the same successive indexes. Or in other words, D-exchanges interchange the content of some couples of rows of  $T$ , but they do not change the order type of  $T$ .

**8.** The permutation  $\mathbf{D}$  could even be considered as a supertask [3]. Indeed, let  $\langle t_n \rangle$  be an  $\omega$ -ordered sequence of instants within a finite interval of time  $(t_a, t_b)$ , being  $t_b$  the limit of the sequence. Assume that  $\mathbf{D}$  is applied to each row  $r_i$  just at the precise instant  $t_i$ . The bijection  $f(t_i) = r_i$  proves that at  $t_b$  the permutation  $\mathbf{D}$  will have been applied to every row of  $T$ .

**9.** It can be proved that all rows of  $T$  become D-modular as a consequence of the permutation  $\mathbf{D}$ . In effect, assume that a row  $r_n$  did not become D-modular as a consequence of  $\mathbf{D}$ . This means that  $r_n$  is not D-modular and could not be D-exchanged with a  $n$ -modular row  $r_{i, i > n}$ . Now then, all  $n$ -modular rows have the same digit  $n \pmod{10}$  in the same  $n$ th position of its decimal expansion, and according to the theorem of the  $n$ th decimal there are infinitely many rational numbers with the same digit in the same position of its decimal expansion, whatever be the digit and the position. Accordingly, since  $n$  is finite, the row  $r_n$  is preceded by a finite number,  $k$  ( $0 \leq k < n$ ), of  $n$ -modular rows, and succeeded by an infinite number,  $\aleph_o$ , of  $n$ -modular rows. Any of these infinitely many  $n$ -modular rows succeeding  $r_n$  had to be D-exchanged with  $r_n$ . It is then impossible for  $r_n$  not to become D-modular as a consequence of  $\mathbf{D}$ . Therefore, each and every row  $r_n$  of  $T$  becomes D-modular as a consequence of  $\mathbf{D}$ .

**10.** Let us remark the basic formal structure of the above argument **9** (a simple modus tollens). Consider the following two propositions  $P_1$  and  $P_2$  about permutation  $\mathbf{D}$ :

$P_1$ : Not all rows of  $T$  becomes D-modular as a consequence of the permutation  $\mathbf{D}$ .

$P_2$ : At least one non-D-modular row  $r_n$  of  $T$  could not be D-exchanged.

It is quite clear that  $P_1$  implies  $P_2$ : if not all rows of  $T$  becomes D-modular as a consequence of  $\mathbf{D}$ , then at least one non-D-modular row  $r_n$  of  $T$  could not be D-exchanged. Now then, being all natural numbers finite,  $n$  is finite; and taking into account the theorem of the  $n$ th decimal, there is a finite number,  $k$  ( $0 \leq k < n$ ), of  $n$ -modular rows preceding  $r_n$  and an infinite number,  $\aleph_o$ , of  $n$ -modular rows succeeding  $r_n$ , one of which had to be D-exchanged with  $r_n$ . In consequence proposition  $P_2$  is false and so will be  $P_1$ . In symbols:

$$(P_1 \Rightarrow P_2) \wedge (\neg P_2) \Rightarrow \neg P_1 \quad (2)$$

**11.** The result proved in **9** is a formal consequence of both the theorem of the  $n$ th decimal and the fact that every row  $r_n$  of  $T$  is always preceded by a finite number,  $k$  ( $0 \leq k < n$ ), of  $n$ -modular rows and succeeded by an infinite number,  $\aleph_o$ , of such  $n$ -modular rows. As noted in **4**, this asymmetry is an inevitable consequence of assuming, as Cantor did in 1883, the existence of the  $\omega$ -ordered set  $\mathbb{N}$  as a complete infinite totality, an hypothesis subsumed into the Axiom of Infinity.

**12.** Let  $T_D$  be the table resulting from permutation  $\mathbf{D}$ . Since all of its rows are D-modular, its diagonal  $D$  will be the periodic rational number  $0.\overline{1234567890}$ . It is now immediate to define infinitely many rational antidiagonals from  $D$ . Indeed, let us consider periods of ten decimal digits none of which coincide in position with the ten decimal digits of the period  $\overline{1234567890}$  of the diagonal  $D$ . The number of those periods is  $9^{10}$ . From any two of them, for instance,  $q_1 = \overline{0123456789}$  and  $q_2 = \overline{0321456789}$ , it is possible to define different  $\omega$ -ordered sequences of rational antidiagonals  $\langle A_n \rangle$ , for instance:

$$\forall n \in \mathbb{N} : A_n = 0.q_1 q_1 \dots q_1 q_2 \quad (3)$$

whose elements cannot be in  $T_D$  for the same reason as in Cantor's diagonal argument. Since all of them are rational numbers, we must conclude that the injective function  $g$  between  $\mathbb{N}$  and  $\mathbb{Q}_{01}$  defining  $T$ , is not surjective, i.e. it is not a bijection.

**13.** Since the injective function  $g$  defining  $T$  is any injective function between  $\mathbb{N}$  and  $\mathbb{Q}_{01}$  and it cannot be surjective, we must conclude it is impossible to define a bijection between  $\mathbb{N}$  and  $\mathbb{Q}_{01}$ . Consequently,  $\mathbb{Q}_{01}$  is non-denumerable. Although the above inference suffices to reach the conclusion on the non-denumerable nature of  $\mathbb{Q}_{01}$ , it could be (inappropriately) argued, as against Cantor's diagonal argument, that a new table  $T'$  could be defined so that  $r'_1 = A$  and  $r'_{i+1} = r_i$ ,  $r_i \in T$ ,  $\forall i \in \mathbb{N}$ . The new table  $T'$  would be denumerable, but through the same diagonal argument, the same conclusion on the impossibility of a bijection between  $\mathbb{N}$  and  $\mathbb{Q}_{01}$  would be reached. And the same recursive argument could be applied to any table defined in terms of any other previous table and its corresponding antidiagonal, while the new table continue to be denumerable. A bijection between  $\mathbb{N}$  and  $\mathbb{Q}_{01}$  is impossible. So,  $\mathbb{Q}_{01}$  is non-denumerable, and we have a contradiction in set theory because Cantor proved  $\mathbb{Q}$  is denumerable [4].

**14.** Permutation  $\mathbf{D}$  allows to develop other arguments whose conclusions also point to the inconsistency of the hypothesis of the actual infinity. For instance, it is clear that certain elements of  $\mathbb{Q}_{01}$  as,  $\overline{0.21}$ ,  $\overline{0.35421}$ ,  $\overline{0.2111111111}$  and infinitely many others cannot become D-modular<sup>1</sup> if they were in  $T$ . Consider any of them, for instance  $\overline{0.21}$ . If it is not in  $T$ , a new table  $T^*$  could be defined so that  $r^*_1 = \overline{0.21}$ ,  $r^*_{i+1} = r_i$ ,  $r_i \in T$ ,  $\forall i \in \mathbb{N}$ , and permutation  $D$  can also be applied to  $T^*$  to get the reordered table  $T^*_D$ . So, assume  $\overline{0.21}$  is the row  $r_m$  of  $T_D$  (the same would apply to  $T^*_D$ ). Since  $m$  is finite,  $\overline{0.21}$  will be preceded by a finite number,  $k$  ( $0 \leq k < m$ ), of  $m$ -modular rows and succeeded by an infinite number,  $\aleph_o$ , of  $m$ -modular rows, according to the

<sup>1</sup>It is, in fact, possible to prove there are infinitely many of such rows.

theorem of the  $n$ th decimal. In consequence  $0.\overline{21}$ , that is not  $m$ -modular, had to be D-exchanged with any of those  $\aleph_0$   $m$ -modular rows following  $r_m$ . Therefore,  $0.\overline{21}$  cannot be the  $m$ th row  $r_m$  of  $T_D$ . Thus, and being  $m$  any natural number, we must conclude **D** makes  $0.\overline{21}$  disappear from  $T_D$ . Note that what this argument proves is not an indeterminacy regarding the row of  $0.\overline{21}$  in  $T_D$ , but an impossibility: for each  $m \in \mathbb{N}$ ,  $0.\overline{21}$  cannot be the row  $r_m$  of  $T_D$ . Or in other words, the set of possible indexes, and then of rows, for  $0.\overline{21}$  in  $T_D$  is the empty set. So,  $0.\overline{21}$  cannot be in  $T_D$ . Permutation **D** makes it magically disappear from  $T_D$ .

**15.** The above absurdity **14** is a consequence of assuming there can exist a sequence in which *each and every* element has *finitely many* predecessors and *infinitely many* successors. A sequence in which it is impossible to reach an element with infinitely many predecessors, despite the fact that every element has infinitely many successors (what, evidently, makes the above arguments possible).

### III. A FINAL REMARK

**16.** As with all discussions on the hypothesis of the actual infinity, the above one is a conceptual discussion unconcerned, as Cantor's diagonal argument, with the physical possibilities of carrying out all involved operations. The formal inconsistency of a hypothesis does not depend on those possibilities, but on the fact of deducing from it a contradiction. And recall that from an inconsistent hypothesis anything can be deduced, from apparently reasonable assertions to absurdities as the above one deduced in **14**. It seems convenient to end by recalling that an argument cannot be refuted by other different argument simply because it reaches an opposite conclusion. In W. Hodges words [5, p. 4]:

How does anybody get into a state of mind where they persuade themselves that you can criticize an argument by suggesting a different argument which doesn't reach the same conclusion?

This inadmissible strategy is frequently used in the discussions related to the actual infinity hypothesis. But to refute an argument means to indicate where and why that argument fails. If two correct arguments based on the same set of hypotheses lead to contradictory conclusions, they are simply proving the existence of a contradiction. And, therefore, the inconsistency of at least one of the assumed hypotheses. In our case, the hypothesis of the actual infinity, according to which the infinite sets and sequences exist as complete totalities. The alternative is the hypothesis of the potential infinity, according to which only finite sets and sequences can be considered as *complete totalities*, unlimited and as large as wished, but always finite if they have to be considered as complete totalities.

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