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Multi-Dimensional Dot And Cross Product Of Complex Vectors

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Abstract. A normal vector $\mathbf{r}$ is that whose components (x, y, z) are scalars. This work shows that the components in a complex vector are instead complex numbers, of the form (a_i+jb_i), for $j = \sqrt{-1}$. So, each component has a real and an imaginary part and conceptually both are conceived perpendicular to each other. This work finds the way to take into account these aspects and introduces a new approach to construct and define N-dimensional complex vectors and how to obtain correctly their properties and characteristics. Obtained multidimensional complex vectors allowed calculating the N-dimensional Dot and Cross product. Complex Vector's properties and their differences with scalars, complex numbers and pure vectors are discussed.

1. Introduction

Using the criterion of solving equations of lines that are not cut, a "cutoff point" was obtained for instance, in the case of the two-dimensional equations of the parabola with its directrix. By expressing its complex roots in vector form, they required a four-dimensional space for its representation (Section **2**). The same problem was then posed inside a three-dimensional space and Its complex vector roots required in this case six-dimensions, confirming the 2N-dimensional structure previously obtained. With this result, a formal definition of complex vectors was proposed in sections **3** & **4**, showing that this vector definition of the complex roots, simultaneously contained some properties of both complex numbers and vectors. In Section **5** the dot and cross products were developed for two complex vectors in spaces of two, three and N dimensions. A Discussion on these results is given in section **6**. Section **7** has a summary of the main characteristics and properties of complex vectors.

2. Cut of a Parabola with its Directrix

2.1 Two-Dimensional Cut of a Parabola with its Directrix

Let's calculate the "cut" of a parabola with its directrix (they never cut each other). A *parabola* is the locus of all points equidistant from a given point called *focus* and a given straight line called its *directrix*, also located in the plane of the curve. Let's start with a two-dimensional space. The distances from any point $\mathrm{P}(x, y)$ of the curve to the focus $\mathrm{F}(g,\ h)$ and to the directrix are: $d^2 = (x-g)^2 + (y-h)^2$ and $\frac{m(x-x_0)-l(y-y_0)}{l^2+m^2}$, respectively; where l and m define the direction of the directrix, which passes through the point $\mathrm{P_0(x_0,\ y_0)}$, also a point of the axis of the parabola. The points of the directrix meet: $\frac{x-x_0}{l} = \frac{y-y_0}{m}$, which implies that its equation is $m(x-x_0) - l(y-y_0) = 0$. By equaling these distances under the last equation, it becomes:

$$(x-g)^2 + (y-h)^2 = \frac{\begin{vmatrix} l & m \\ x-x_0 & y-y_0 \end{vmatrix}^2}{l^2+m^2}; \text{ for } m(x-x_0) - l(y-y_0) = 0 \tag{1}$$

Substituting and simplifying, we obtain:

$$(x-g)^2 + \left(\frac{m}{l}x - \frac{1}{l}(mx_0 - ly_0) - h\right)^2 = 0 \tag{2}$$

$$\begin{Bmatrix} p_x = lg \\ q_x = mx_0 - l(y_0 - h) \end{Bmatrix} \rightarrow \begin{Bmatrix} (lx-p_x)^2 + (mx-q_x)^2 = 0 \\ x^2(l^2+m^2) - 2(lp_x + mq_x)x + (p_x^2 + q_x^2) = 0 \end{Bmatrix} \tag{3}$$

$$x = \frac{(lp_x + mq_x) \pm j\sqrt{(lq_x - mp_x)^2}}{(l^2 + m^2)} \quad (4)$$

$$x = \frac{(l^2+m^2)x_0 - l(l(x_0-h)+m(y_0-k)) \pm jl\sqrt{(l^2+m^2)[(x_0-h)^2+(y_0-k)^2] - [(l(x_0-h)) + m(y_0-k)]^2}}{(l^2+m^2)} \quad (5)$$

Note: The expression in the square root is obtained, as indicated below, by adding and subtracting to the right side the same term between brackets:

$$\begin{aligned}(lq_x - mp_x)^2 &= l^2(m(x_0-g) - l(y_0-h))^2 \\ &= l^2\{m^2(x_0-g)^2 + l^2(y_0-h)^2 - 2m(x_0-g)l(y_0-h] + [\boldsymbol{l^2(x_0-g)^2 + m^2(y_0-h)^2}] - [\boldsymbol{l^2(x_0-g)^2} \\ &\quad + \boldsymbol{m^2(y_0-h)^2}]\} \\ &= l^2\{(l^2+m^2)[(x_0-g)^2 + (y_0-h)^2] - \left[\left(l(x_0-g)\right) + m(y_0-h)\right]^2\} \quad (6)\end{aligned}$$

Doing the same for the y's, where $x = \frac{l}{m}y - \left(\frac{l}{m}y_0 - x_0 + g\right)$:

$$(ly - (ly_0 - m(x_0+g)))^2 + (my - mh)^2 = 0 \quad \begin{Bmatrix} p_y = ly_0 - m(x_0-g) \\ q_y = mh \end{Bmatrix} \quad (7)$$

$$y = \frac{(lp_y + mq_y) \pm j\sqrt{(lq_y - mp_y)^2}}{(l^2+m^2)} \quad (8)$$

$$y = \frac{(l^2+m^2)y_0 - m(l(x_0-g)+m(y_0-h)) \pm jm\sqrt{(l^2+m^2)[(x_0-g)^2+(y_0-h)^2] - [(l(x_0-g)) + m(y_0-h)]^2}}{(l^2+m^2)} \quad (9)$$

The expression, $l(x_0 - g) + m(y_0 - h)$, appearing two times in the numerator, can be set as the dot product of two vectors: $(l\mathbf{i} + m\mathbf{j}) \cdot [(x_0 - g)\mathbf{i} + m(y_0 - h)\mathbf{j}]$. Since they are perpendicular to each other, due to they are the directrix and axis directions of the parabola, then its dot product is null. By defining the segment $\overline{\mathrm{P_0F}}$ between the point $\mathrm{P_0}$ and the focus F in the axis direction, as $a_0 = \sqrt{(x_0 - g)^2 + (y_0 - h)^2}$, and the unitary directrix direction $\sqrt{l^2 + m^2} = 1$, the solution to the "cut" of the curve with its directrix is:

$$x = x_0 \pm jla_0 \;; \qquad y = y_0 \pm jma_0 \quad (10)$$

Given that imaginary parts behave as perpendicular vectors to the real ones, let's construct them as vectors.

2.2 Construction and Definition of the perpendicularity of imaginary parts in Complex Vectors

It can be observed that the perpendicularity between the imaginary and real parts is an intuitive characteristic that works and is completely accepted by scientists worldwide. Also, in this accepted feature of complex vectors the particle $j = \sqrt{-1}$ does not appear affecting anything in the diagram of the imaginary and real vectors. By considering these characteristics and then in the construction of the complex vectors, a mathematical interpretation of this phenomenon could be that the imaginary vector comes from the product of the particle, j, and the unitary vector along the coordinate involved, i.e.: the multiplication of the complex component on the x axis by the unit vector $\mathbf{i}$, and that of the y axis by $\mathbf{j}$ would define the products $j\mathbf{i} = \boldsymbol{\hat{\imath}_\perp}$ and $j\mathbf{j} = \boldsymbol{\hat{\jmath}_\perp}$ as imaginary vectors perpendicular to $\mathbf{i}, \mathbf{j}$, and between them too, $\boldsymbol{\hat{\imath}_\perp}$, $\boldsymbol{\hat{\jmath}_\perp}$; Then, by doing these actions, after multiplying and reordering, we will obtain as final result a four-dimensional complex vector $\mathbf{r}$ composed by a real vector $\mathbf{r_0}$ and an imaginary one $\boldsymbol{\widehat{a_0}}$, in the way indicated below:

$$\mathbf{r} = [x_0\mathbf{i} + la_0(j\mathbf{i})] + (y_0\mathbf{j} + ma_0(j\mathbf{j})) = (x_0\mathbf{i} + y_0\mathbf{j}) + a_0[j(l\mathbf{i} + m\boldsymbol{j})] \quad (11)$$

$$\mathbf{r} = (x_0\mathbf{i} + y_0\mathbf{j}) + a_0(l\boldsymbol{\hat{\imath}_\perp} + m\boldsymbol{\hat{\jmath}_\perp}) = \mathbf{r_0} + \boldsymbol{\widehat{a_0}}$$

$$\text{For: } |\mathbf{r_0}| = \sqrt{x_0^2 + y_0^2}; \qquad |\boldsymbol{\widehat{a_0}}| = a_0\sqrt{l^2 + m^2} = a_0 \quad (12)$$

Since the real $(\mathbf{i}, \mathbf{j})$ and imaginary vectors$(\hat{\boldsymbol{\imath}}_\perp,\ \hat{\boldsymbol{\jmath}}_\perp)$ define planes that are perpendicular to each other, with the origin O as their only common point, then both resultant vectors, $\mathbf{r_0}$ and $\widehat{\boldsymbol{a_0}}$, are also perpendicular. **Proof:**

$$\widehat{\boldsymbol{a_0}} = a_0(l\hat{\boldsymbol{\imath}}_\perp + m\hat{\boldsymbol{\jmath}}_\perp) = a_0(lj\mathbf{i} + mj\mathbf{j}) = a_0[j(l\mathbf{i} + m\mathbf{j})] \tag{13}$$

Namely, the product $j(l\mathbf{i} + m\mathbf{j})$ indicates that the resultant vector $\widehat{\boldsymbol{a_0}} = a_0(l\hat{\boldsymbol{\imath}}_\perp + m\hat{\boldsymbol{\jmath}}_\perp)$, in the axis direction is perpendicular to vector $(l\mathbf{i} + m\mathbf{j})$, the directrix direction. Or also: since the direction (l,m) is perpendicular to the segment $a_0 = \overline{\mathrm{P_0F}}$, then $\widehat{\boldsymbol{a_0}}$ is simultaneously perpendicular to the directrix line and to the axis of the parabola $\overline{\mathrm{P_0F}}$. Thus, $\widehat{\boldsymbol{a_0}}$ is perpendicular to the plane formed by these two directions, and also to the direction $\mathbf{r_0}$, which is located in this plane (and to any direction in, or parallel, to this plane). Thus, we can write that modulus of $\mathbf{r}$ becomes $|\mathbf{r}| = \sqrt{r_0^2 + a_0^2} = \sqrt{x_0^2 + y_0^2 + a_0^2} = \sqrt{x_0^2 + y_0^2 + (x_0 - g)^2 + (y_0 - h)^2}$.

2.3 Cut of a Parabola with its Directrix in a Three-Dimensional Space

The same problem is now presented in a three-dimensional space. We will continue considering the directrix line having a **unitary** direction vector (l, m, n) and doing the interpretation of the imaginary vector defined above. As previously mentioned, the directrix passes through $\mathrm{P_0}(x_0, y_0, z_0)$, its cutoff point with the axis of the parabola. The equation of the parabola in space (defined as that with equal distances from a point of the parabola to its directrix, and to the focus, $\mathrm{F}(g, h, p)$, is:

$$(x-g)^2 + (y-h)^2 + (z-k)^2 = \frac{\begin{vmatrix} m & n \\ y-y_0 & z-z_0 \end{vmatrix}^2 + \begin{vmatrix} n & l \\ z-z_0 & x-x \end{vmatrix}^2 + \begin{vmatrix} l & m \\ x-x_0 & y-y_0 \end{vmatrix}^2}{l^2+m^2+n^2}; \tag{14}$$

For $$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}; \qquad y = \frac{m}{l}x - \left(\frac{m}{l}x_0 - y_0\right); \quad z = \frac{n}{l}x - \left(\frac{n}{l}x_0 - z_0\right) \tag{15}$$

We obtain: $$\begin{vmatrix} m & n \\ y-y_0 & z-z_0 \end{vmatrix}^2 + \begin{vmatrix} n & l \\ z-z_0 & x-x_0 \end{vmatrix}^2 + \begin{vmatrix} l & m \\ x-x_0 & y-y_0 \end{vmatrix}^2 = 0^2 + 0^2 + 0^2 = 0$$

So, we can express it as follows: $$(x-g)^2 + \left(\frac{m}{l}x - \left(\frac{m}{l}x_0 - y_0 + h\right)\right)^2 + \left(\frac{n}{l}x - \left(\frac{n}{l}x_0 - z_0 + k\right)\right)^2 = 0$$

$$\left\{\begin{matrix} p_x = lg \\ q_x = mx_0 - l(y_0 - h) \\ r_x = nx_0 - l(z_0 - k) \end{matrix}\right\} \left\{\begin{matrix} (lx - lg)^2 + (mx - (mx_0 - l(y_0 - h)))^2 + (nx - (nx_0 - l(z_0 - k)))^2 = 0 \\ (lx - p_x)^2 + (mx - q_x)^2 + (nx - r_x)^2 = 0 \\ x^2(l^2 + m^2 + n^2) - 2(lp_x + mq_x + nr_x)y + (p_x^2 + q_x^2 + r_x^2) = 0 \end{matrix}\right\} \tag{16}$$

The solution for this case is again: $x = x_0 \pm jla_0$, where $a_0 = \overline{\mathrm{P_0F}} = \sqrt{(x_0 - g)^2 + (y_0 - h)^2 + (z_0 - p)^2}$; Doing the same for y and z, and constructing the corresponding complex vector, $\mathbf{r} = \mathbf{r_0} + \widehat{\boldsymbol{a_0}}$, we find:

$$y = y_0 \pm jma_0; \quad z = z_0 \pm jna_0 \ \rightarrow\ \mathbf{r} = (x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}) + a_0(l\hat{\boldsymbol{\imath}}_\perp + m\hat{\boldsymbol{\jmath}}_\perp + n\hat{\boldsymbol{k}}_\perp) = \mathbf{r_0} + \widehat{\boldsymbol{a_0}} \tag{17}$$

For: $$|\mathbf{r_0}| = \sqrt{x_0^2 + y_0^2 + z_0^2}; \ \ |\widehat{\boldsymbol{a_0}}| = a_0\sqrt{l^2 + m^2 + n^2} = a_0; \ \ j = \sqrt{-1}; \ \ j\mathbf{i} = \hat{\boldsymbol{\imath}}_\perp; \ \ j\mathbf{j} = \hat{\boldsymbol{\jmath}}_\perp; \ \ j\mathbf{k} = \hat{\boldsymbol{k}}_\perp \tag{18}$$

$$|\mathbf{r}| = \sqrt{r_0^2 + a_0^2} = \sqrt{x_0^2 + y_0^2 + z_0^2 + a_0^2} = \sqrt{x_0^2 + y_0^2 + z_0^2 + (x_0 - g)^2 + (y_0 - h)^2 + (z_0 - p)^2}.$$

2.4 Cut of a Parabola with its Directrix in an N-Dimensional Space

Similarly and in a general way, the cut of a parabola with its directrix line inside an N-dimensional space produces 2N-dimensional complex roots that can be expressed as the following complex vector:

$$\mathbf{r} = (x_0\mathbf{i} + y_0\mathbf{j} + \cdots + z_0\mathbf{k}) + a_0(l\hat{\boldsymbol{\imath}}_\perp + m\hat{\boldsymbol{\jmath}}_\perp + \cdots + n\hat{\boldsymbol{k}}_\perp) = \mathbf{r_0} + \widehat{\boldsymbol{a_0}} \tag{19}$$

$$|\mathbf{r_0}| = \sqrt{x_0^2 + y_0^2 + \cdots + z_0^2}; \ |\widehat{\boldsymbol{a_0}}| = a_0\sqrt{l^2 + m^2 + \cdots + n^2} = a_0; \ \ j = \sqrt{-1}; \ j\mathbf{i} = \hat{\boldsymbol{\imath}}_\perp; \ \ j\mathbf{j} = \hat{\boldsymbol{\jmath}}_\perp; \ \ldots \ ; j\mathbf{k} = \hat{\boldsymbol{k}}_\perp$$

Where, as before, given that all unit vectors are perpendicular to each other, so are $\mathbf{r_0}$ and $\widehat{\boldsymbol{a_0}}$:

$$\widehat{\boldsymbol{a_0}} = a_0(l\hat{\boldsymbol{\imath}}_\perp + m\hat{\boldsymbol{\jmath}}_\perp + \cdots + n\hat{\boldsymbol{k}}_\perp) = a_0(lj\mathbf{i} + mj\mathbf{j} + \cdots + nj\mathbf{k}) = ja_0(l\mathbf{i} + m\mathbf{j} + \cdots + n\mathbf{k}) \tag{20}$$

Since the direction $(l, m, \ldots n)$ of the directrix line is perpendicular to the segment $a_0 = \overline{\mathrm{FP_0}}$ which has the axis direction, then $\widehat{\boldsymbol{a_0}}$ is simultaneously perpendicular to the directrix line and to the axis of the parabola. To be precise, $\widehat{\boldsymbol{a_0}}$ is perpendicular to the plane formed by these two directions; then, it is also perpendicular to the direction $\mathbf{r_0}$, as well as to any direction in, or parallel to the plane. So,

$$|\mathbf{r}| = \sqrt{r_0^2 + a_0^2} = \sqrt{x_0^2 + y_0^2 + \cdots + z_0^2 + a_0^2}.$$

3. Complex Vector Definition based on the imaginary perpendicular vector definition

We have seen that interpreting the product of the imaginary number j and a basis vector $\mathbf{v}$ as an authentic new vector $\widehat{\boldsymbol{v}}_\perp$, perpendicular to $\mathbf{v}$, in which the particle $j = \sqrt{-1}$, is eliminated, and so those characteristics of conjugate numbers inside complex vectors. The following examples of complex vectors are constructed.

3.1 "One-dimensional" Complex Vector

The unit vector $\mathbf{i}$ multiplied by the complex expression in the component $x = a + jb$ creates two vectors: a real vector and an imaginary one, perpendicular to each other. So, a "one-dimensional" complex vector $\mathbf{r}$ turns out to be represented as a two-dimensional vector in the following form:

$$\mathbf{r} = \mathbf{x} = x\mathbf{i} = (a + jb)\mathbf{i} = a\mathbf{i} + \mathrm{b}(j\mathbf{i}) = a\mathbf{i} + \mathrm{b}\hat{\boldsymbol{\imath}}_\perp; \quad \text{with a vectorial magnitude } r = (a^2 + b^2)^{1/2} \qquad (21)$$

where $\hat{\boldsymbol{\imath}}_\perp$ and $\mathbf{i}$ have a common origin at O(0,0). The resultant vector $\mathbf{r}$ can be represented, in general, as the sum of two perpendicular vectors, $\boldsymbol{\rho} + \widehat{\boldsymbol{\gamma}}$, in the following way:

$$\mathbf{r} = \boldsymbol{\rho} + \widehat{\boldsymbol{\gamma}} \quad \text{for} \quad \boldsymbol{\rho} = a\mathbf{i}; \quad \widehat{\boldsymbol{\gamma}} = b\hat{\boldsymbol{\imath}}_\perp; \qquad r = (\rho^2 + \gamma^2)^{1/2} = (a^2 + b^2)^{1/2} \qquad (22)$$

3.2 "Two-dimensional" Complex Vector

Similarly, a "two-dimensional" vector $\mathbf{r} = (a + jb)\mathbf{i} + (c + jd)\mathbf{j}$, needs four dimensions (two real and two imaginary), all of them perpendicular to each other, namely:

$$\mathbf{r} = (a + jb)\mathbf{i} + (c + jd)\mathbf{j} = (a\mathbf{i} + b\hat{\boldsymbol{\imath}}_\perp) + (c\mathbf{j} + d\hat{\boldsymbol{\jmath}}_\perp) = (a\mathbf{i} + c\mathbf{j}) + (b\hat{\boldsymbol{\imath}}_\perp + d\hat{\boldsymbol{\jmath}}_\perp); \qquad (23)$$

$$\mathbf{r} = \boldsymbol{\rho} + \widehat{\boldsymbol{\gamma}} \quad \text{for} \quad \boldsymbol{\rho} = a\mathbf{i} + c\mathbf{j} \quad \text{and} \quad \widehat{\boldsymbol{\gamma}} = b\hat{\boldsymbol{\imath}}_\perp + d\hat{\boldsymbol{\jmath}}_\perp; \qquad r = (\rho^2 + \gamma^2)^{1/2} \qquad (24)$$

Since unit vectors $\hat{\boldsymbol{\imath}}_\perp$ and $\hat{\boldsymbol{\jmath}}_\perp$ are perpendicular to vector $\mathbf{i}$ and $\mathbf{j}$, with a common origin at O(0,0,0,0), they are also perpendicular between them; then, $\rho = (a^2 + c^2)^{1/2}$ and $\gamma = (b^2 + d^2)^{1/2}$. So, we can write that, for $\boldsymbol{\rho} = a\mathbf{i} + c\mathbf{j} = \rho\mathbf{u}$, and $\widehat{\boldsymbol{\gamma}} = b\hat{\boldsymbol{\imath}}_\perp + d\hat{\boldsymbol{\jmath}}_\perp = \gamma\widehat{\boldsymbol{u}}_\perp$, where the unit vector $\widehat{\boldsymbol{u}}_\perp$ is parallel to $\widehat{\boldsymbol{\gamma}}$ and perpendicular to $\boldsymbol{\rho}$ and $\mathbf{i}, \mathbf{j}$:

$$\mathbf{r} = \boldsymbol{\rho} + \widehat{\boldsymbol{\gamma}} = \rho\mathbf{u} + \gamma\widehat{\boldsymbol{u}}_\perp \quad r = (\rho^2 + \gamma^2)^{1/2} = (a^2 + c^2 + b^2 + d^2)^{1/2}. \qquad (25)$$

3.3 "N-dimensional" Complex Vector

Thus, an "N-dimensional" complex vector needs 2N dimensions to be represented:

$$\mathbf{r} = (a + jb)\mathbf{i} + (c + jd)\mathbf{j} + .. + (s + jt)\mathbf{n} + = (a\mathbf{i} + c\mathbf{j} + .. + s\mathbf{n}) + [b(j\mathbf{i}) + d(j\mathbf{j}) + .. + t(j\mathbf{n})]$$

$$\mathbf{r} = \boldsymbol{\rho} + \widehat{\boldsymbol{\gamma}}, \text{ for } \boldsymbol{\rho} = a\mathbf{i} + c\mathbf{j} + .. + s\mathbf{n} = \rho\mathbf{u} \text{ and } \widehat{\boldsymbol{\gamma}} = b(j\mathbf{i}) + d(j\mathbf{j}) + .. + t(j\mathbf{n}) = b\hat{\boldsymbol{\imath}}_\perp + d\hat{\boldsymbol{\jmath}}_\perp + .. + t\widehat{\boldsymbol{n}}_\perp. \qquad (26)$$

$$\mathbf{r} = \rho\mathbf{u} + \gamma j\mathbf{u} = \rho\mathbf{u} + \gamma\widehat{\boldsymbol{u}}_\perp; \; r = (\rho^2 + \gamma^2)^{1/2}; \text{ for } \rho = (a^2 + c^2 + .. + s^2)^{1/2} \text{ and } \gamma = (b^2 + d^2 + .. + t^2)^{1/2} \qquad (27)$$

$$r = (\rho^2 + \gamma^2)^{1/2} = (a^2 + c^2 + .. + s^2 + b^2 + d^2 + .. + t^2)^{1/2} \qquad (28)$$

Where, the perpendicular unit vectors: i and $\hat{\boldsymbol{\imath}}_\perp$, j and $\hat{\boldsymbol{\jmath}}_\perp$, .. , and, n and $\widehat{\boldsymbol{n}}_\perp$, constitute a 2N-dimensional space, N real dimensions and N perpendicular imaginary ones, with a common origin O(0,0, .. ,0,0, .. ,0). Likewise, given the perpendicularity between the real and imaginary n-dimensional unit vectors, the real resultant vector $\boldsymbol{\rho}$ and the imaginary vector $\widehat{\boldsymbol{\gamma}} = j\boldsymbol{\gamma}$ are perpendicular to each other. This just defines $\mathbf{r} = \boldsymbol{\rho} + \widehat{\boldsymbol{\gamma}}$ as a complex vector, with both the real and imaginary parts perpendicular to each other. Let's recall some of the properties of complex numbers and those of the dot and cross product that can almost be extended to complex vectors. From now on, the imaginary vector will be written without the subscript that implied perpendicularity, $\hat{\boldsymbol{\imath}}_{\perp} \rightarrow \hat{\boldsymbol{\imath}}$ leaving it only in the italic format for a simplification of its writing. So, Let's summarize and recall the properties of complex numbers (one-dimensional vectors or "scalar" plus its conjugate property) and the complex vectors' properties (the same of vectors' properties).

4. Characteristics of Complex Numbers and Vector Dot and Cross products

4.1 Complex Numbers

A complex number z is a number of the form $z = a + jb$, where a and b are scalars and $j = \sqrt{-1}$, Satisfying $j^2 = -1$. The complex conjugate of the number $z = a + jb$ is defined to be $\bar{z} = a - jb$, satisfying $z\bar{z} = |z|^2 = a^2 + b^2 = r^2$, where $a = r.\cos\theta$ and $b = r.\sin\theta$.

4.1.1 *Properties of Complex Numbers*

1. $\overline{z \pm w} = \bar{z} \pm \bar{w}$; $\overline{z\,w} = \bar{z}\,\bar{w}$; $\overline{z / w} = \bar{z} / \bar{w}$
2. $(a + jb) \pm (c + jd) = (a \pm c) + (b \pm d)i$
3. $(a + jb)(c + jd) = (c + jd)(a + jb) = (ac - bd) + (bc + ad)i$
4. $\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{(a+jb)\,(c-jd)}{(c+jd)\,(c-jd)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + j\,\frac{bc-ad}{c^2+d^2}$
5. $e^{jx} = \cos\theta + j\,r.\sin\theta$;
6. $z = r.\cos\theta + j\,(r.\sin\theta)$; $z^n = r^n\,e^{jnx}$; $z^n = r^n.(\cos n\theta + j.r.\sin n\theta)$
7 $\sqrt[n]{z} = \sqrt[n]{r}e^{j\frac{\theta+2k\pi}{n}x} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + j\sin\frac{\theta+2k\pi}{n}\right)$

4.2 Scalar or Dot Product of Vectors

The scalar product of two vectors **a** and **b** is defined as the product of their magnitudes multiplied by the cosine of the angle φ between the referred vectors:

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\phi. \qquad (29)$$

4.2.1 *Properties of the scalar or dot product of two vectors*

1. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
2. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$
3. $(\lambda\,\mathbf{a})\cdot\mathbf{b} = \lambda\,(\mathbf{a}\cdot\mathbf{b})$
4. The magnitude of a vector is expressed via the scalar product as: $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{\mathrm{a}^2}$
5. The scalar product of basis vectors is defined as:
   $$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0; \qquad \mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1. \qquad (30)$$
6. Vectors **a** and **b** are perpendicular if and only if $\mathbf{a}\cdot\mathbf{b} = 0$.
7. Expressing vectors by their coordinates, $\mathbf{a} = (a_x, a_y, a_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$:
   $$\mathbf{a}\cdot\mathbf{b} = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\cdot(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}) = a_xb_x + a_yb_y + a_zb_z = |\mathbf{a}||\mathbf{b}|.\cos\phi$$
   With magnitudes $|\mathbf{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$ and $|\mathbf{b}| = \sqrt{b_x^2 + b_y^2 + b_z^2}$.
8. The angle θ between vectors **a** and **b** is determined from: $\cos\phi = \mathbf{a}\cdot\mathbf{b}/|\mathbf{a}||\mathbf{b}|$. (31)

4.3 Cross or Vector Product

The cross or vector product of two vectors **a** and **b** is defined as the vector **c** denoted by $\mathbf{a}\times\mathbf{b}$:

1. $|\mathbf{c}| = |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\varphi$ or $\mathbf{c} = \mathbf{a}\times\mathbf{b} = a\,b\sin\varphi\,\mathbf{n}$. (32)
2. The vector **c** is perpendicular to the plane formed by **a, b**; i.e., $\mathbf{c}\perp\mathbf{a},\mathbf{b}$.
3. The vector **c** points perpendicular from the side in which the sense of the shortest rotation from **a** to **b** is counterclockwise.

4.3.1 Properties of the cross product of two vectors

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2 $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
3. Vectors **a** and **b** are collinear or parallel if and only if $\mathbf{a} \times \mathbf{b} = 0$. (33)
4. $(\lambda\, \mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\lambda\, \mathbf{b}) = \lambda\, (\mathbf{a} \times \mathbf{b})$
5. $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$; $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. (34)
6. $$\begin{cases}\mathbf{a} = (a_x, a_y, a_z)\\ \mathbf{b} = (a_x, a_y, a_z)\end{cases} \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}. \quad (35)$$

5. Applying the concept of Dot and Cross product to Complex Vectors

5.1 Dot and cross product of two Complex Vectors in a space of "one dimension"

We have seen that complex vector viewed in one dimension actually require really two dimensions or two perpendicular axes. Let two one-dimensional complex vectors be $\mathbf{r_1} = (x_1 + jy_1)\mathbf{i}$ and $\mathbf{r_2} = (x_2 + jy_2)\mathbf{i}$, where:

$$x_1 = r_1 \cos\theta_1;\ y_1 = r_1 \sin\theta_1; \qquad r_1 = \sqrt{x_1^2 + y_1^2} \quad (36)$$
$$x_2 = r_2 \cos\theta_2;\ y_2 = r_2 \sin\theta_2; \qquad r_2 = \sqrt{x_2^2 + y_2^2} \quad (37)$$

Their dot product, recalling the definitions set in section **3**, can be reordered and represented as:

$$\mathbf{r_1} \cdot \mathbf{r_2} = (x_1 + jy_1)\mathbf{i} \cdot (x_2 + jy_2)\mathbf{i} = [x_1\mathbf{i} + y_1(j\mathbf{i})] \cdot [x_2\mathbf{i} + y_2(j\mathbf{i})] \quad (38)$$

As previously mentioned, the product $j\mathbf{i}$ converts to a new basis vector $\hat{\boldsymbol{\imath}}$, perpendicular to $\mathbf{i}$. So, substituting in (38), we express the result by recalling the dot property for complex basis vectors: $\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}} = \mathbf{1}, \boldsymbol{i} \cdot \boldsymbol{i} = \mathbf{1}$ and $\hat{\boldsymbol{\imath}} \cdot \boldsymbol{i} = \boldsymbol{i} \cdot \hat{\boldsymbol{\imath}} = \mathbf{0}$:

$$\mathbf{r_1} \cdot \mathbf{r_2} = [x_1\mathbf{i} + y_1\hat{\boldsymbol{\imath}}] \cdot [x_2\mathbf{i} + y_2\hat{\boldsymbol{\imath}}] = x_1x_2 + y_1y_2 \quad (39)$$

Note: The result in (39) imposes that the modulus definition of vectors applied to complex vectors be: $|\mathbf{r}|^2 = \mathbf{r} \cdot \mathbf{r} = x^2 + y^2$, and <u>not that of the complex numbers</u> (product of vector by its conjugate, $\mathbf{r} \cdot \bar{\mathbf{r}} = x^2 - y^2 \neq |\mathbf{r}|^2$). Thus, the concept of "conjugate" in complex vectors does not behave in the same way as in complex **numbers**. Complex vectors behave as known authentic vectors. If we work instead with components expressed in polar form we arrive at the known definition of dot product for complex vectors:

$$\mathbf{r_1} \cdot \mathbf{r_2} = [x_1\mathbf{i} + y_1\hat{\boldsymbol{\imath}}] \cdot [x_2\mathbf{i} + y_2\hat{\boldsymbol{\imath}}] = r_1 r_2[\cos\theta_1.\cos\theta_2 + \sin\theta_1.\sin\theta_2] = r_1 r_2\cos(\theta_2 - \theta_1)$$
$$\mathbf{r_1} \cdot \mathbf{r_2} = r_1 r_2 \cos\theta \quad (40)$$

Doing the same for the cross product of two complex vectors, its definition arrives at the known expression:

$$\mathbf{r_1} \times \mathbf{r_2} = [x_1\mathbf{i} + y_1\hat{\boldsymbol{\imath}}] \times [x_2\mathbf{i} + y_2\hat{\boldsymbol{\imath}}] = (x_1y_2(\mathbf{i} \times \hat{\boldsymbol{\imath}}) + x_2y_1(\hat{\boldsymbol{\imath}} \times \mathbf{i})) = (x_1y_2 - x_2y_1)(\mathbf{i} \times \hat{\boldsymbol{\imath}})$$
$$= r_1 r_2[\sin\theta_2.\cos\theta_1 - \sin\theta_1.\cos\theta_2](\mathbf{i} \times \hat{\boldsymbol{\imath}})$$
$$\mathbf{r_1} \times \mathbf{r_2} = r_1 r_2 \sin(\theta_2 - \theta_1)(\mathbf{i} \times \hat{\boldsymbol{\imath}}) = r_1 r_2 \sin\theta\, (\mathbf{i} \times \hat{\boldsymbol{\imath}}) \quad (41)$$

Where, $(\mathbf{i} \times \hat{\boldsymbol{\imath}})$ is the vector $\mathbf{n}$, normal to both complex vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ and to the basis vectors $\mathbf{i}$ and $\hat{\boldsymbol{\imath}}$.

So, we can imagine a matrix pattern for this cross product: $\mathbf{r_1} \times \mathbf{r_2} = \begin{vmatrix} \mathbf{n} & \mathbf{i} & \hat{\boldsymbol{\imath}} \\ \mathbf{0} & x_1 & y_1 \\ \mathbf{0} & x_2 & y_2 \end{vmatrix} = (x_1y_2 - x_2y_1)\mathbf{n}.$ (42)

Another way to represent this result is through the cross and dot product relationship, which comes out from their definitions. See (31) and (32):

$$\mathbf{r_1} \times \mathbf{r_2} = r_1r_2 \sin\theta\, \mathbf{n} = r_1r_2\sqrt{1 - \cos^2\theta}\, \mathbf{n} = r_1r_2\sqrt{1 - \left(\frac{\mathbf{r_1} \cdot \mathbf{r_2}}{r_1r_2}\right)^2}\, \mathbf{n} = \sqrt{(r_1r_2)^2 - (\mathbf{r_1} \cdot \mathbf{r_2})^2}\, \mathbf{n}$$

Substituting terms in the last expression of this problem, we obtain easily the same result (42) as previously:

$$\mathbf{r_1} \times \mathbf{r_2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2) - (x_1x_2 + y_1y_2)^2}\mathbf{n} = \sqrt{(x_1y_2 - x_2y_1)^2}\mathbf{n} = (x_1y_2 - x_2y_1)\mathbf{n}$$

5.2 Dot and cross product of two Complex Vectors in a space of "Two dimensions"

As we have previously seen in the introduction and in the cut of a parabola with its directrix line calculation, a complex vector of two dimensions (2D) requires really four dimensions (4D): two real and two imaginary ones. Let the two complex vectors be $\mathbf{r_1} = [(x_{11} + jy_{11})\mathbf{i} + [(x_{12} + jy_{12})\mathbf{j}]$ and $\mathbf{r_2} = [(x_{21} + jy_{21})\mathbf{i} + [(x_{22} + jy_{22})\mathbf{j}]$. Its dot product, where all unit vectors are perpendicular, represented and reordered as a 4D dot product becomes:

$$\mathbf{r_1} \cdot \mathbf{r_2} = [(x_{11} + jy_{11})\mathbf{i} + [(x_{12} + jy_{12})\mathbf{j}] \cdot [(x_{21} + jy_{21})\mathbf{i} + [(x_{22} + jy_{22})\mathbf{j}]$$
$$= [x_{11}\mathbf{i} + x_{12}\mathbf{j} + y_{11}\hat{\boldsymbol{\imath}} + y_{12}\hat{\boldsymbol{\jmath}}] \cdot [x_{21}\mathbf{i} + x_{22}\mathbf{j} + y_{21}\hat{\boldsymbol{\imath}} + y_{22}\hat{\boldsymbol{\jmath}}] = x_{11}x_{21} + x_{12}x_{22} + y_{11}y_{21} + y_{12}y_{22} \quad (43)$$

Thus, the cross or vector product of two complex vectors represented in 2D and reordered as 4D, become:

$$\begin{aligned}\mathbf{r_1} \times \mathbf{r_2} &= [(x_{11} + jy_{11})\mathbf{i} + [(x_{12} + jy_{12})\mathbf{j}] \times [(x_{21} + jy_{21})\mathbf{i} + [(x_{22} + jy_{22})\mathbf{j}] \\ &= [x_{11}\mathbf{i} + x_{12}\mathbf{j} + y_{11}\hat{\boldsymbol{\imath}} + y_{12}\hat{\boldsymbol{\jmath}}] \times [x_{21}\mathbf{i} + x_{22}\mathbf{j} + y_{21}\hat{\boldsymbol{\imath}} + y_{22}\hat{\boldsymbol{\jmath}}] \\ &= (x_{11}x_{22} - x_{12}x_{21})\mathbf{i} \times \mathbf{j} + (x_{11}y_{21} - x_{12}y_{11})\mathbf{i} \times \hat{\boldsymbol{\imath}} + (x_{11}y_{22} - x_{21}y_{12})\mathbf{i} \times \hat{\boldsymbol{\jmath}} + \\ &\quad + (x_{12}y_{21} - y_{11}x_{22})\mathbf{j} \times \hat{\boldsymbol{\imath}} + (x_{12}y_{22} - y_{12}x_{22})\mathbf{j} \times \hat{\boldsymbol{\jmath}} + (y_{11}y_{22} - y_{21}y_{12})\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}\end{aligned} \quad (44)$$

Where $\mathbf{i}, \mathbf{j}$, are real and $\hat{\boldsymbol{\imath}} = j\mathbf{i}$, $\hat{\boldsymbol{\jmath}} = j\mathbf{j}$, are imaginary axes, all perpendicular among them. Expressing the following products as unit vectors, perpendicular to each other: $\mathbf{i} \times \mathbf{j} = \mathbf{u_{ij}}$, $\mathbf{i} \times \hat{\boldsymbol{\imath}} = \mathbf{u}_{\mathbf{i}\hat{\imath}}$, $\mathbf{i} \times \hat{\boldsymbol{\jmath}} = \mathbf{u}_{\mathbf{i}\hat{\jmath}}$, $\mathbf{j} \times \hat{\boldsymbol{\imath}} = \mathbf{u}_{\mathbf{j}\hat{\imath}}$, $\mathbf{j} \times \hat{\boldsymbol{\jmath}} = \mathbf{u}_{\mathbf{j}\hat{\jmath}}$, $\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = \mathbf{u}_{\hat{\imath}\hat{\jmath}}$, the resultant complex vector $\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2}$, reorders to the following expression:

$$\begin{aligned}\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} &= [x_{11}\mathbf{i} + x_{12}\mathbf{j} + y_{11}\hat{\boldsymbol{\imath}} + y_{12}\hat{\boldsymbol{\jmath}}] \times [x_{21}\mathbf{i} + x_{22}\mathbf{j} + y_{21}\hat{\boldsymbol{\imath}} + y_{22}\hat{\boldsymbol{\jmath}}] \\ &= (x_{11}x_{22} - x_{12}x_{21})\mathbf{u_{ij}} + j(x_{11}y_{21} - x_{12}y_{11})\mathbf{u}_{\mathbf{i}\hat{\imath}} + j(x_{11}y_{22} - x_{21}y_{12})\mathbf{u}_{\mathbf{i}\hat{\jmath}} \\ &\quad + j(x_{12}y_{21} - y_{11}x_{22})\mathbf{u}_{\mathbf{j}\hat{\imath}} + j(x_{12}y_{22} - y_{12}x_{22})\mathbf{u}_{\mathbf{j}\hat{\jmath}} + (y_{11}y_{22} - y_{21}y_{12})\mathbf{u}_{\hat{\imath}\hat{\jmath}}\end{aligned}$$

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = \sqrt{\begin{aligned}&(x_{11}x_{22} - x_{12}x_{21})^2 + (x_{11}y_{21} - x_{12}y_{11})^2 + (x_{11}y_{22} - x_{21}y_{12})^2 + \\ &+(x_{12}y_{21} - y_{11}x_{22})^2 + (x_{12}y_{22} - y_{12}x_{22})^2 + (y_{11}y_{22} - y_{21}y_{12})^2\end{aligned}}\;\mathbf{n} \quad (45)$$

Where $\mathbf{c}$ is by definition perpendicular to $\mathbf{r_1}$ and $\mathbf{r_2}$ and **n** is a unit vector in the direction **c**, perpendicular to all basis vectors $\mathbf{u}_{mn}$ for $\boldsymbol{m,n}$ = **i**, **j**, $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$. A way to remember and construct the terms between parentheses in (45), inside the square root, corresponding to each a 2x2 matrix, is to use the following arrangement:

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} \\ x_{11} & x_{12} & y_{11} & y_{12} \\ x_{21} & x_{22} & y_{21} & y_{22}\end{vmatrix} \quad (46)$$

The "2D" cross product reordered as a 4D product becomes equal to the number of combinations, $C_{4,2} = 6$, put alike in matrix form, where the first row indicates the referred unit vector and the involved columns:

$$\begin{aligned}\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} &= \begin{vmatrix}\mathbf{u_{ij}} & \mathbf{i} & \mathbf{j} \\ \mathbf{0} & x_{11} & x_{12} \\ \mathbf{0} & x_{21} & x_{22}\end{vmatrix} + \begin{vmatrix}\mathbf{u}_{\mathbf{i}\hat{\imath}} & \mathbf{i} & \hat{\boldsymbol{\imath}} \\ \mathbf{0} & x_{11} & y_{11} \\ \mathbf{0} & x_{21} & y_{21}\end{vmatrix} + \begin{vmatrix}\mathbf{u}_{\mathbf{i}\hat{\jmath}} & \mathbf{i} & \hat{\boldsymbol{\jmath}} \\ \mathbf{0} & x_{11} & y_{12} \\ \mathbf{0} & x_{21} & y_{22}\end{vmatrix} + \\ &+ \begin{vmatrix}\mathbf{u}_{\mathbf{j}\hat{\imath}} & \mathbf{j} & \hat{\boldsymbol{\imath}} \\ \mathbf{0} & x_{12} & y_{11} \\ \mathbf{0} & x_{22} & y_{21}\end{vmatrix} + \begin{vmatrix}\mathbf{u}_{\mathbf{j}\hat{\jmath}} & \mathbf{j} & \hat{\boldsymbol{\jmath}} \\ \mathbf{0} & x_{12} & y_{12} \\ \mathbf{0} & x_{22} & y_{22}\end{vmatrix} + \begin{vmatrix}\mathbf{u}_{\hat{\imath}\hat{\jmath}} & \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} \\ \mathbf{0} & y_{11} & y_{12} \\ \mathbf{0} & y_{21} & y_{22}\end{vmatrix}\end{aligned} \quad (47)$$

Let's check equations (45), (46) and (47). According to the definition of the cross product, $\mathbf{r_1} \times \mathbf{r_2} = (r_1r_2\sin\theta)\mathbf{n}$, for $\mathbf{r_1} = [x_{11}\mathbf{i} + x_{12}\mathbf{j} + y_{11}\hat{\boldsymbol{\imath}} + y_{12}\hat{\boldsymbol{\jmath}}]$ and $\mathbf{r_2} = [x_{21}\mathbf{i} + x_{22}\mathbf{j} + y_{21}\hat{\boldsymbol{\imath}} + y_{22}\hat{\boldsymbol{\jmath}}]$, reordering we have:

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = \sqrt{(r_1r_2)^2 - (\mathbf{r_1} \cdot \mathbf{r_2})^2}\;\mathbf{n} =$$
$$= \sqrt{(x_{11}^2 + x_{12}^2 + y_{11}^2 + y_{12}^2)(x_{21}^2 + x_{22}^2 + y_{21}^2 + y_{22}^2) - (x_{11}x_{21} + x_{12}x_{22} + y_{11}y_{21} + y_{12}y_{22})^2}\mathbf{n} \quad (48)$$

After some work, we arrive at the expression inside the square root that was previously obtained in (45).

It is worth noticing that vector $\mathbf{c}$, has two components, one of them, $a\mathbf{u_R}$, is a real vector composed by the terms multiplying the real basis vectors $\mathbf{u_{ij}} = \mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{u}_{\hat{\imath}\hat{\jmath}} = \hat{\imath} \times \hat{\jmath} = \hat{\hat{\mathbf{k}}}$. The last unit vector becomes a real vector perpendicular to $\hat{\imath}, \hat{\jmath}, \hat{k}$ and of course to $\mathbf{i}, \mathbf{j}, \mathbf{k}$, because it does not contain the factor j; the other component $b\widehat{u_I}$ is constituted by the remaining four imaginary terms that contained the factor j. Thus,

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = a\mathbf{u_R} + b\widehat{\mathbf{u}_I} = a\mathbf{u_R} + b(j\mathbf{u_I}) = a\mathbf{u_R} + jb\mathbf{u_I}, \tag{49}$$

We have used $\mathbf{c} = a\mathbf{u_R} + jb\mathbf{u_I}$, instead of $\mathbf{c} = a\mathbf{u_R} + b\widehat{\mathbf{u}_I}$, only to emphasize the perpendicularity of $\mathbf{u_R}$ and $\widehat{\mathbf{u}_I} = j\mathbf{u_I}$.) So, the cross product is also a complex vector $\mathbf{c} = a\mathbf{u_R} + b\widehat{\mathbf{u}_I}$, with $\mathbf{u_R}$ perpendicular to $\widehat{\mathbf{u}_I}$.

$$a\mathbf{u_R} = (x_{11}x_{22} - x_{12}x_{21})\mathbf{u_{ij}} + (y_{11}y_{22} - y_{21}y_{12})\mathbf{u}_{\hat{\imath}\hat{\jmath}},$$
$$b\widehat{\mathbf{u}_I} = (x_{11}y_{21} - x_{12}y_{11})\mathbf{u}_{\mathbf{i}\hat{\imath}} + (x_{11}y_{22} - x_{21}y_{12})\mathbf{u}_{\mathbf{i}\hat{\jmath}} + (x_{12}y_{21} - y_{11}x_{22})\mathbf{u}_{\mathbf{j}\hat{\imath}} + (x_{12}y_{22} - y_{12}x_{22})\mathbf{u}_{\mathbf{j}\hat{\jmath}}.$$

Notice that in the case of pure vectors, the imaginary axes, $\hat{\imath}$ and $\hat{\jmath}$, are null and the expression reduces to:

$$\mathbf{r_1} \times \mathbf{r_2} = \begin{vmatrix} \mathbf{k} & \mathbf{i} & \mathbf{j} \\ \mathbf{0} & x_{11} & x_{12} \\ \mathbf{0} & x_{21} & x_{22} \end{vmatrix} = (x_{11}x_{22} - x_{21}x_{12})\mathbf{k} \tag{50}$$

Namely, it simplifies to the known case of the cross product of two pure vectors. It is apparent this configures a general procedure that determines the dot or cross product of complex vectors for “three” (six) or “N” (2N) dimensions, which in turn can also be reduced to those of pure vectors with null imaginary axes.

5.3 Dot and Cross product of two Complex Vectors in a space of “Three dimensions”

Let two complex vectors be: $\mathbf{r_1} = [(x_{11} + jy_{11})\mathbf{i} + [(x_{12} + jy_{12})\mathbf{j} + [(x_{13} + jy_{13})\mathbf{k}]$ and $\mathbf{r_2} = [(x_{21} + jy_{21})\mathbf{i} + [(x_{22} + jy_{22})\mathbf{j} + [(x_{23} + jy_{33})\mathbf{k}]$. The dot product can be written and reordered as:

$$\begin{aligned}\mathbf{r_1} \cdot \mathbf{r_2} = {} & [(x_{11} + jy_{11})\mathbf{i} + [(x_{12} + jy_{12})\mathbf{j} + [(x_{13} + jy_{13})\mathbf{k}] \cdot [(x_{21} + jy_{21})\mathbf{i} + [(x_{22} + jy_{22})\mathbf{j} + [(x_{23} \\ & + jy_{23})\mathbf{k}] \\ & = [x_{11}\mathbf{i} + x_{12}\mathbf{j} + x_{13}\mathbf{k} + y_{11}\hat{\imath} + y_{12}\hat{\jmath} + y_{13}\hat{k}] \cdot [x_{21}\mathbf{i} + x_{22}\mathbf{j} + x_{23}\mathbf{k} + y_{21}\hat{\imath} + y_{22}\hat{\jmath} + y_{23}\hat{k}]\end{aligned}$$

$$\mathbf{r_1} \cdot \mathbf{r_2} = x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23} + y_{11}y_{21} + y_{12}y_{22} + y_{13}y_{23} \tag{51}$$

The cross or vector product of two complex vectors in three dimensions (actually six: three real, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$; and three imaginary axes, $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ as it was in the dot product) becomes:

$$\mathbf{r_1} \times \mathbf{r_2} = [(x_{11} + jy_{11})\mathbf{i} + (x_{12} + jy_{12})\mathbf{j} + (x_{13} + jy_{13})\mathbf{k}] \times [(x_{21} + jy_{21})\mathbf{i} + (x_{22} + jy_{22})\mathbf{j} + (x_{23} + jy_{23})\mathbf{k}]$$
$$\mathbf{r_1} \times \mathbf{r_2} = [x_{11}\mathbf{i} + x_{12}\mathbf{j} + x_{13}\mathbf{k} + y_{11}\hat{\imath} + y_{12}\hat{\jmath} + y_{13}\hat{k}] \times [x_{21}\mathbf{i} + x_{22}\mathbf{j} + x_{23}\mathbf{k} + y_{21}\hat{\imath} + y_{22}\hat{\jmath} + y_{23}\hat{k}] \tag{52}$$

We see that order in the cross product is achieved from the following arrangement: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_{11} & x_{12} & x_{13} & y_{11} & y_{12} & y_{13} \\ x_{21} & x_{22} & x_{23} & y_{21} & y_{22} & y_{23} \end{vmatrix}$.

$$\begin{aligned}\mathbf{r_1} \times \mathbf{r_2} = {} & \begin{vmatrix} \mathbf{u_{ij}} & \mathbf{i} & \mathbf{j} \\ \mathbf{0} & x_{11} & x_{12} \\ \mathbf{0} & x_{21} & x_{22} \end{vmatrix} + \begin{vmatrix} \mathbf{u_{ik}} & \mathbf{i} & \mathbf{k} \\ \mathbf{0} & x_{11} & x_{13} \\ \mathbf{0} & x_{21} & x_{23} \end{vmatrix} + \begin{vmatrix} \mathbf{u}_{\mathbf{i}\hat{\imath}} & \mathbf{i} & \hat{\imath} \\ \mathbf{0} & x_{11} & y_{11} \\ \mathbf{0} & x_{21} & y_{21} \end{vmatrix} + \begin{vmatrix} \mathbf{u}_{\mathbf{i}\hat{\jmath}} & \mathbf{i} & \hat{\jmath} \\ \mathbf{0} & x_{11} & y_{12} \\ \mathbf{0} & x_{21} & y_{22} \end{vmatrix} + \begin{vmatrix} \mathbf{u}_{\mathbf{i}\hat{k}} & \mathbf{i} & \hat{k} \\ \mathbf{0} & x_{11} & y_{13} \\ \mathbf{0} & x_{21} & y_{23} \end{vmatrix} + \\ & \begin{vmatrix} \mathbf{u_{jk}} & \mathbf{j} & \mathbf{k} \\ \mathbf{0} & x_{12} & x_{13} \\ \mathbf{0} & x_{22} & x_{23} \end{vmatrix} + \begin{vmatrix} \mathbf{u}_{\mathbf{j}\hat{\imath}} & \mathbf{j} & \hat{\imath} \\ \mathbf{0} & x_{12} & y_{11} \\ \mathbf{0} & x_{22} & y_{21} \end{vmatrix} + \begin{vmatrix} \mathbf{u}_{\mathbf{j}\hat{\jmath}} & \mathbf{j} & \hat{\jmath} \\ \mathbf{0} & x_{12} & y_{12} \\ \mathbf{0} & x_{22} & y_{22} \end{vmatrix} + \begin{vmatrix} \mathbf{u}_{\mathbf{j}\hat{k}} & \mathbf{j} & \hat{k} \\ \mathbf{0} & x_{12} & y_{13} \\ \mathbf{0} & x_{22} & y_{23} \end{vmatrix} + \begin{vmatrix} \mathbf{u}_{\mathbf{k}\hat{\imath}} & \mathbf{k} & \hat{\imath} \\ \mathbf{0} & x_{13} & y_{11} \\ \mathbf{0} & x_{23} & y_{21} \end{vmatrix} + \\ & \begin{vmatrix} \mathbf{u}_{\mathbf{k}\hat{\jmath}} & \mathbf{k} & \hat{\jmath} \\ \mathbf{0} & x_{13} & y_{12} \\ \mathbf{0} & x_{23} & y_{22} \end{vmatrix} + \begin{vmatrix} \mathbf{u}_{\mathbf{k}\hat{k}} & \mathbf{k} & \hat{k} \\ \mathbf{0} & x_{13} & y_{13} \\ \mathbf{0} & x_{23} & y_{23} \end{vmatrix} + \begin{vmatrix} \mathbf{u}_{\hat{\imath}\hat{\jmath}} & \hat{\imath} & \hat{\jmath} \\ \mathbf{0} & y_{11} & y_{12} \\ \mathbf{0} & y_{21} & y_{22} \end{vmatrix} + \begin{vmatrix} \mathbf{u}_{\hat{\imath}\hat{k}} & \hat{\imath} & \hat{k} \\ \mathbf{0} & y_{11} & y_{13} \\ \mathbf{0} & y_{21} & y_{23} \end{vmatrix} + \begin{vmatrix} \mathbf{u}_{\hat{\jmath}\hat{k}} & \hat{\jmath} & \hat{k} \\ \mathbf{0} & y_{12} & y_{13} \\ \mathbf{0} & y_{22} & y_{23} \end{vmatrix}\end{aligned} \tag{53}$$

Note: The number of terms is given by the combinations of dimensions taken in pairs, i.e.: $C_{6,2}. = 15$

$$\begin{aligned}\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = {} & (x_{11}x_{22} - x_{21}x_{12})\mathbf{u_{ij}} + (x_{11}x_{23} - x_{21}x_{13})\mathbf{u_{ik}} + (x_{11}y_{21} - x_{21}y_{11})\mathbf{u_{i\hat{\imath}}} + (x_{11}y_{22} - x_{21}y_{12})\mathbf{u_{i\hat{\jmath}}} \\ & + (x_{11}y_{23} - x_{21}y_{13})\mathbf{u_{i\hat{k}}} + (x_{12}x_{23} - x_{22}x_{13})\mathbf{u_{jk}} + (x_{12}y_{21} - x_{22}y_{11})\mathbf{u_{j\hat{\imath}}} \\ & + (x_{12}y_{22} - x_{22}y_{12})\mathbf{u_{j\hat{\jmath}}} - (x_{12}y_{23} - x_{22}y_{13})]\mathbf{u_{j\hat{k}}} + (x_{13}y_{21} - x_{23}y_{11})\mathbf{u_{k\hat{\imath}}} \\ & + (x_{13}y_{22} - x_{23}y_{12})\mathbf{u_{k\hat{\jmath}}} - (x_{13}y_{23} - x_{23}y_{13})\mathbf{u_{k\hat{k}}} + (y_{11}y_{22} - y_{21}y_{12})\mathbf{u_{\hat{\imath}\hat{\jmath}}} \\ & + (y_{11}y_{23} - y_{21}y_{13})\mathbf{u_{\hat{\imath}\hat{k}}} + (y_{12}x_{23} - y_{22}x_{13})\mathbf{u_{\hat{\jmath}\hat{k}}}\end{aligned}$$

Where, as we know, the direction of the resultant vector $\mathbf{n}$ is simultaneously perpendicular to all unit vectors $\mathbf{u}_{mn}$ for $\boldsymbol{m}, \boldsymbol{n} = \mathbf{i}, \mathbf{j}, \mathbf{k}, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$, to all basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$ and to the vectors $\mathbf{r_1}$ and $\mathbf{r_2}$.

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = \sqrt{\begin{aligned}&(x_{11}x_{22} - x_{21}x_{12})^2 + (x_{11}x_{23} - x_{21}x_{13})^2 + (x_{11}y_{21} - y_{11}x_{21})^2 \\ &+ (x_{11}y_{22} - y_{12}x_{21})^2 + (x_{11}y_{23} - y_{13}x_{21})^2 + (x_{12}x_{23} - x_{22}x_{13})^2 \\ &+ (x_{12}y_{21} - x_{22}y_{11})^2 + (x_{12}y_{22} - x_{22}y_{12})^2 + (x_{12}y_{23} - x_{22}y_{13})^2 \\ &+ (x_{13}y_{21} - x_{23}y_{11})^2 + (x_{13}y_{22} - x_{23}y_{12})^2 - (x_{13}y_{23} - x_{23}y_{13})^2 \\ &+ (y_{11}y_{22} - y_{21}y_{12})^2 + (y_{11}y_{23} - y_{21}y_{13})^2 + (y_{12}x_{23} - y_{22}x_{13})^2\end{aligned}}\ \mathbf{n} \tag{54}$$

Checking this output through $\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = (r_1 r_2 \sin\theta)\mathbf{n} = \sqrt{(r_1 r_2)^2 - (\mathbf{r_1} \cdot \mathbf{r_2})^2}\,\mathbf{n}$, for $\mathbf{r_1} = [x_{11}\mathbf{i} + x_{12}\mathbf{j} + x_{13}\mathbf{k} + y_{11}\hat{\boldsymbol{\imath}} + y_{12}\hat{\boldsymbol{\jmath}} + y_{13}\hat{\boldsymbol{k}}]$ and $\mathbf{r_2} = [x_{21}\mathbf{i} + x_{22}\mathbf{j} + x_{23}\mathbf{k} + y_{21}\hat{\boldsymbol{\imath}} + y_{22}\hat{\boldsymbol{\jmath}} + y_{23}\hat{\boldsymbol{k}}]$ we obtain the expression:

$$\begin{aligned}\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} &= \sqrt{(r_1 r_2)^2 - (\mathbf{r_1} \cdot \mathbf{r_2})^2}\,\mathbf{n} = \\ &= \sqrt{\begin{aligned}&(x_{11}^2 + x_{12}^2 + x_{13}^2 + y_{11}^2 + y_{12}^2 + y_{13}^2)(x_{21}^2 + x_{22}^2 + y_{23}^2 + y_{21}^2 + y_{22}^2 + y_{23}^2) \\ &- (x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23} + y_{11}y_{21} + y_{12}y_{22} + y_{13}y_{23})^2\end{aligned}}\ \mathbf{n}\end{aligned} \tag{55}$$

As expected, this is the same result as in (54), and the same comment at the end of section **2.3.3** resulting in the expression of the complex vector $\mathbf{c}$ as a sum of a real and an imaginary term, is repeated here:

$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = a\mathbf{u_R} + b\widehat{\boldsymbol{u_I}}$, Where:

$$a\mathbf{u_R} = (x_{11}x_{22} - x_{21}x_{12})\mathbf{u_{ij}} + (x_{11}x_{23} - x_{21}x_{13})\mathbf{u_{ik}} + (x_{12}x_{23} - x_{22}x_{13})\mathbf{u_{jk}} + (y_{11}y_{22} - y_{21}y_{12})\mathbf{u_{\hat{\imath}\hat{\jmath}}} + (y_{11}y_{23} - y_{21}y_{13})\mathbf{u_{\hat{\imath}\hat{k}}} + (y_{12}x_{23} - y_{22}x_{13})\mathbf{u_{\hat{\jmath}\hat{k}}} \tag{56}$$

$$b\widehat{\boldsymbol{u_I}} = (x_{11}y_{21} - x_{21}y_{11})\mathbf{u_{i\hat{\imath}}} + (x_{11}y_{22} - x_{21}y_{12})\mathbf{u_{i\hat{\jmath}}} + (x_{11}y_{23} - x_{21}y_{13})\mathbf{u_{i\hat{k}}} + (x_{12}y_{21} - x_{22}y_{11})\mathbf{u_{j\hat{\imath}}} + (x_{12}y_{22} - x_{22}y_{12})\mathbf{u_{j\hat{\jmath}}} - (x_{12}y_{23} - x_{22}y_{13})]\mathbf{u_{j\hat{k}}} + (x_{13}y_{21} - x_{23}y_{11})\mathbf{u_{k\hat{\imath}}} + (x_{13}y_{22} - x_{23}y_{12})\mathbf{u_{k\hat{\jmath}}} - (x_{13}y_{23} - x_{23}y_{13})\mathbf{u_{k\hat{k}}}. \tag{57}$$

After this parenthesis, let's continue. If it had been the case of pure vectors, the imaginary axes, $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$ **a**nd $\hat{\boldsymbol{k}}$, would be null. Then, the previous expression for vector $\mathbf{c}$, simplifies to the known result for two 3D vectors:

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = \begin{vmatrix}\mathbf{u_{ij}} & \mathbf{i} & \mathbf{j} \\ \mathbf{0} & x_{11} & x_{12} \\ \mathbf{0} & x_{21} & x_{22}\end{vmatrix} + \begin{vmatrix}\mathbf{u_{ik}} & \mathbf{i} & \mathbf{k} \\ \mathbf{0} & x_{11} & x_{13} \\ \mathbf{0} & x_{21} & x_{23}\end{vmatrix} + \begin{vmatrix}\mathbf{u_{jk}} & \mathbf{j} & \mathbf{k} \\ \mathbf{0} & x_{12} & x_{13} \\ \mathbf{0} & x_{22} & x_{23}\end{vmatrix} \tag{58}$$

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = \sqrt{(x_{11}x_{22} - x_{21}x_{12})^2 + (x_{11}x_{23} - x_{21}x_{13})^2 + (x_{12}x_{23} - x_{22}x_{13})^2}\ \mathbf{n} \tag{59}$$

Effectively, for $\mathbf{r_1} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{r_2} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$; for $\mathbf{u_{ij}} = \mathbf{k}$, $\mathbf{u_{ki}} = \mathbf{j}$, $\mathbf{u_{jk}} = \mathbf{i}$, and substituting and applying related changes, we obtain the cross product we were used to (30):

$$\begin{aligned}\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} &= \\ &= \begin{vmatrix}\mathbf{k} & \mathbf{i} & \mathbf{j} \\ \mathbf{0} & a_x & a_y \\ \mathbf{0} & b_x & b_y\end{vmatrix} + \begin{vmatrix}-\mathbf{j} & \mathbf{i} & \mathbf{k} \\ \mathbf{0} & a_x & a_z \\ \mathbf{0} & b_x & b_z\end{vmatrix} + \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{0} & a_y & a_z \\ \mathbf{0} & b_y & b_z\end{vmatrix} = \begin{vmatrix}\mathbf{k} & \mathbf{i} & \mathbf{j} \\ \mathbf{0} & a_x & a_y \\ \mathbf{0} & b_x & b_y\end{vmatrix} + \begin{vmatrix}\mathbf{j} & \mathbf{k} & \mathbf{i} \\ \mathbf{0} & a_z & a_x \\ \mathbf{0} & b_z & b_x\end{vmatrix} + \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{0} & a_y & a_z \\ \mathbf{0} & b_y & b_z\end{vmatrix}\end{aligned}$$

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z\end{vmatrix} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k} \tag{60}$$

$$\mathbf{c} = \mathbf{r_1} \times \mathbf{r_2} = \sqrt{(a_y b_z - a_z b_y)^2 + (a_z b_x - a_x b_z)^2 + (a_x b_y - a_y b_x)^2}\, \mathbf{n} \tag{61}$$

5.4 Dot and Cross product of two Complex Vectors in a space of "N dimensions"

This case actually has 2N dimensions, N real and N imaginary, and originates 2N terms in the dot product:

$$\mathbf{r_1} \cdot \mathbf{r_2} = \sum_{i=1}^{N} (x_{1i} x_{2i} + y_{1i} y_{2i}) \tag{62}$$

And $C_{2N,2}$ vector terms in the cross product, using the ordered arrangement: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \cdots & \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} & \cdots \\ x_{11} & x_{12} & x_{13} & \cdots & y_{11} & y_{12} & y_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots & y_{21} & y_{22} & y_{23} & \cdots \end{vmatrix}$

and constructing matrices 2x2, of four individuals of the type $\begin{vmatrix} x_{1j} & x_{1,j+k} \\ x_{2j} & x_{2,j+k} \end{vmatrix}$, $\begin{vmatrix} x_{1j} & y_{1,j+k} \\ x_{2j} & y_{2,j+k} \end{vmatrix}$ or $\begin{vmatrix} y_{1j} & y_{1,j+k} \\ y_{2j} & y_{2,j+k} \end{vmatrix}$; formed starting by the first column followed by the second column, then the same first followed by the third, and so on until reaching the last column. Then, the procedure is repeated starting with the second column followed by the third and after the same second column followed by the fourth and so on until reaching the last column. The process continues until forming the second last column followed by the last one. After that, by solving the matrices we obtain terms of the form $(x_{1i} x_{2j} - x_{2i} x_{1j})\mathbf{u}_{ij}$, or of the form $(x_{1i} y_{2j} - x_{2i} y_{1j})\mathbf{u}_{ij}$ or of the form $(y_{1i} y_{2j} - y_{2i} y_{1j})\mathbf{u}_{ij}$, where the subindex *i,j* of the terms x_{ij} and y_{ij} refers to the position of the specified column of the real or imaginary axes, and the subindex $\boldsymbol{ij}$ of the unit vectors $\mathbf{u}_{ij}$ refers to the involved product of the basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}, \ldots, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}, \ldots$ With these terms constructed, the Cross Product can be calculated as shown for "one" (two), "two" (four) and "N" (2N) dimensions. Changing the $y_{1 \text{ or } 2,k}$ to $x_{1 \text{ or } 2,N+k}$ and then reverting the changes.

$$\mathbf{r_1} \times \mathbf{r_2} = \sum_{i=1,j=1}^{i+j=2N} (x_{1,i} x_{2,i+j} - x_{2,i} x_{1,i+j})^2 \mathbf{n} \quad \begin{cases} if\ i > N\ then\ x_{1,i}, x_{2,i} \rightarrow y_{1,i-N}, y_{2,i-N}, \text{ and/or:} \\ if\ i + j > N\ then\ x_{2,i+j},\ x_{1,i+j} \rightarrow y_{2,i+j-N},\ y_{1,i+j-N} \end{cases} \tag{63}$$

6. Discussion.

The interpretation that the product of a real base vector, **i, j, k,..** by the particle *j* originates, each one another basis vector, *j***i** = $\hat{\boldsymbol{\imath}}$, *j***j** = $\hat{\boldsymbol{\jmath}}$, *j***k** = $\hat{\boldsymbol{k}}$,.. imaginary and perpendicular to that involved, makes it easy to define the magnitude of a complex number as that of an authentic vector with a real and an imaginary vector part. This also explains in a natural way the perpendicularity of the imaginary component with the real component (always noticed with this property in all the studies related to this subject), but not concretized in its clear and authentic vector notation and definition, in the way it is done in this work. Such interpretation, for example, allowed to obtain and define in a natural way the scalar product without the influence of the imaginary particle $j = \sqrt{-1}$, which disappears within the definition of the imaginary vector, $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$,.. . It also allowed the magnitude, or modulus, of a complex vector to be expressed via a scalar product in the same way as for a pure vector: $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{\mathrm{a}^2}$. Such definition of the imaginary vector, without the particle *j*, is one of the aspects not conceived in the works of W. R. Hamilton[3] in 1844 and of J. W. Gibbs[2], 40 years later in 1884, in which the particle *j* was preserved within the complex vector, forcing the module of a complex vector to be expressed through the scalar product of the complex "vector" itself by its "conjugate", in the same way as it was achieved for a complex number: $|\mathbf{z}| = \sqrt{\mathbf{z} \cdot \bar{\mathbf{z}}}$. The definition of the complex vector module, done by these authors in this way, introduces the contradiction with the known definition of the module of a pure vector.

After the pioneer work of these authors on complex vectors, carrying this conceptual error, all those who addressed this issue used the same mistake in their research, uncritically. May be this has been the main reason for not having developed, for more than one dimension, the scalar and vector product of two complex vectors. And the new definition given in this work, allowed obtaining the natural definitions of the vector module and the dot and cross product for two vectors in a space of any number of dimensions.

7. Complex Vector characteristics

From our previous results, and the new definitions in section **3,** can be derived the following properties for the complex vectors, additional to those of pure vectors.

1. $\mathbf{r} = (a + jb)\mathbf{i} + (c + jd)\mathbf{j} + .. + (s + jt)\mathbf{n} + = (a\mathbf{i} + c\mathbf{j} + .. + s\,\mathbf{n}) + [b(j\mathbf{i}) + d(j\mathbf{j}) + .. + t(j\mathbf{n})]$
   A basis vector $\mathbf{v}$ multiplied by $j = \sqrt{-1}$ creates a new basis vector $\hat{\boldsymbol{v}} = j\mathbf{v}$ perpendicular to $\mathbf{v}$.
   $\mathbf{r} = \boldsymbol{\rho} + \hat{\boldsymbol{\gamma}}$; $\boldsymbol{\rho} = a\mathbf{i} + c\mathbf{j} + .. + s\mathbf{n}$ and $\hat{\boldsymbol{\gamma}} = b(j\mathbf{i}) + d(j\mathbf{j}) + .. + t(j\mathbf{n}) = b\hat{\boldsymbol{\imath}} + d\hat{\boldsymbol{\jmath}} + .. + t\hat{\boldsymbol{n}}$
2. $\mathbf{r} = \rho\mathbf{u} + \gamma\widehat{\boldsymbol{u}_\gamma}$, where, $\mathbf{u}$ and $\widehat{\boldsymbol{u}_\gamma}$ are perpendicular. By doing $\hat{\boldsymbol{u}}\&\widehat{\boldsymbol{u}_\gamma}$ parallel and since they are unitary, then $\hat{\boldsymbol{u}} = \widehat{\boldsymbol{u}_\gamma}$. So, $\mathbf{r} = \rho\mathbf{u} + \gamma\hat{\boldsymbol{u}} = r\mathbf{u_r}$; for $r = \sqrt{\rho^2 + \gamma^2}$; $\mathbf{u_r} = (\rho\mathbf{u} + \gamma\hat{\boldsymbol{u}})/\sqrt{\rho^2 + \gamma^2}$; $\rho = (a^2 + c^2 + .. + s^2)^{1/2}$ and $\gamma = (b^2 + d^2 + .. + t^2)^{1/2}$; So, $r = \sqrt{a^2 + c^2 + .. + s^2 + b^2 + d^2 + .. + t^2}$
3. $\mathbf{r} = \boldsymbol{\rho} + j\boldsymbol{\gamma} = \boldsymbol{\rho} + \hat{\boldsymbol{\gamma}} = \rho\mathbf{u} + \gamma\hat{\boldsymbol{u}} \rightarrow \mathbf{r} = r(\cos\theta\mathbf{u} + \sin\theta\hat{\boldsymbol{u}}) = r.e^{j\theta}\mathbf{u_r}$, where, $\mathbf{r}$, inside a 2N-dimensional space, is located in the plane formed by the perpendicular vectors $\boldsymbol{\rho}$ and $\hat{\boldsymbol{\gamma}}$; and θ is the angle between $\mathbf{r}$ and $\boldsymbol{\rho}$. In this way, we can also say that $\rho = r.\cos\theta$ and $\gamma = r.\sin\theta$.
4. Dot product for basis vectors: $\mathbf{p}\cdot\mathbf{q} = 0$ for $\mathbf{p} \neq \mathbf{q}$; $\mathbf{p}\cdot\mathbf{q} = 1$ for $\mathbf{p} = \mathbf{q}$; $\mathbf{p},\mathbf{q} = \mathbf{i}, \mathbf{j}, .. , \mathbf{n}, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, .., \hat{\boldsymbol{n}}$.
5. "Conjugate" complex vector: $\bar{\mathbf{r}} = (\rho - \gamma j)\mathbf{u} = \rho\mathbf{u} - \gamma j\mathbf{u} = \rho\mathbf{u} - \gamma\hat{\boldsymbol{u}} = \boldsymbol{\rho} - \hat{\boldsymbol{\gamma}}$. See 10, below.
6. $\mathbf{r}' = z(k(\mathbf{r})) = z(k(\boldsymbol{\rho} + j\boldsymbol{\gamma})) = kz(\boldsymbol{\rho} + j\boldsymbol{\gamma})$ where k is a scalar and $\mathrm{z} = |\mathrm{z}|e^{j\varphi}$, a complex number. In addition to multiplying by k, a rotation φ of the vector $\mathbf{r}$ is produced by z: i.e.: $\mathbf{r} = re^{j\theta}\mathbf{u_r}$ and $\mathrm{z} = |\mathrm{z}|e^{j\varphi} \rightarrow \mathbf{r}' = kr|z|e^{j(\theta+\varphi)}\mathbf{u_{r'}} = kr|z|[\cos(\theta+\varphi)\mathbf{u} + \sin(\theta+\varphi)\hat{\boldsymbol{u}}]$.

Dot Product of Two Complex Vectors

7. $\mathbf{r_1}\cdot\mathbf{r_2} = (\boldsymbol{\rho_1} + \widehat{\boldsymbol{\gamma_1}})\cdot(\boldsymbol{\rho_2} + \widehat{\boldsymbol{\gamma_2}}) = (\rho_1\rho_2 + \gamma_1\gamma_2)$
8. $\mathbf{r_1}\cdot\mathbf{r_2} = (a_1\mathbf{i} + c_1\mathbf{j} + .. + s_1\mathbf{n} + b_1\hat{\boldsymbol{\imath}} + d_1\hat{\boldsymbol{\jmath}} + .. + t_1\hat{\boldsymbol{n}})\cdot(a_2\mathbf{i} + c_2\mathbf{j} + .. + s_2\mathbf{n} + b_2\hat{\boldsymbol{\imath}}$ $+ d_2\hat{\boldsymbol{\jmath}} + .. + t_2\hat{\boldsymbol{n}}) = (a_1a_2 + c_1c_2 + .. + s_1s_2 + b_1b_2 + d_1d_2 + .. + t_1t_2)$
9. $\mathbf{r}\cdot\mathbf{r} = (a\mathbf{i} + c\mathbf{j} + .. + s\mathbf{n} + b\hat{\boldsymbol{\imath}} + d\hat{\boldsymbol{\jmath}} + .. + t\hat{\boldsymbol{n}})\cdot(a\mathbf{i} + c\mathbf{j} + .. + s\mathbf{n} + b\hat{\boldsymbol{\imath}} + d\hat{\boldsymbol{\jmath}} + .. + t\hat{\boldsymbol{n}}) =$
   $= (a^2 + c^2 + .. + s^2 + b^2 + d^2 + .. + t^2)$.
10. $\mathbf{r_1}\cdot\bar{\mathbf{r}}_2 = (\boldsymbol{\rho_1} + \widehat{\boldsymbol{\gamma_1}})\cdot(\boldsymbol{\rho_2} - \widehat{\boldsymbol{\gamma_2}}) = (\rho_1\rho_2 - \gamma_1\gamma_2) \rightarrow \mathbf{r}\cdot\bar{\mathbf{r}} = (\boldsymbol{\rho} + \hat{\boldsymbol{\gamma}})\cdot(\boldsymbol{\rho} - \hat{\boldsymbol{\gamma}}) = \rho^2 - \gamma^2$ (!)
11. $\mathbf{r}\cdot\mathbf{r} = \bar{\mathbf{r}}\cdot\bar{\mathbf{r}} = (\rho\mathbf{u}\pm\gamma\hat{\boldsymbol{u}})\cdot(\rho\mathbf{u}\pm\gamma\hat{\boldsymbol{u}}) = \rho^2 + \gamma^2 = |\mathbf{r}|^2 = |\bar{\mathbf{r}}|^2$ (Same modulus)
12. $(\boldsymbol{\rho_1} + j\boldsymbol{\gamma_1})\pm(\boldsymbol{\rho_2} + j\boldsymbol{\gamma_2}) = (\boldsymbol{\rho_1} \pm \boldsymbol{\rho_2}) + j(\boldsymbol{\gamma_1} \pm \boldsymbol{\gamma_2}) = \boldsymbol{\rho} + j\boldsymbol{\gamma}$; $\boldsymbol{\rho} = \boldsymbol{\rho_1} \pm \boldsymbol{\rho_2}$; $\boldsymbol{\gamma} = \boldsymbol{\gamma_1} \pm \boldsymbol{\gamma_2}$

Cross Product of Two Complex Vectors

13 For $\mathbf{r_1} = (x_{1,1} + jx_{1,m+1})\mathbf{i} + (x_{1,2} + jx_{1,m+1})\mathbf{j} + .. + (x_{1,m} + jx_{1,2m})\mathbf{m}$
   and $\mathbf{r_2} = (x_{2,1} + jx_{2,m+1})\mathbf{i} + (x_{2,2} + jx_{2,m+1})\mathbf{j} + .. + (x_{2,m} + jx_{2,2m})\mathbf{m}$

$$\mathbf{r_1}\times\mathbf{r_2} = \sum_{i=1,j=1}^{i+j=2m}\left[\left(x_{1,i}x_{2,i+j} - x_{2,i}x_{1,i+j}\right)^2\right]\mathbf{n},$$

$$\begin{Bmatrix} if\ i > m\ then\ x_{1,i}, x_{2,i} \rightarrow y_{1,i-m}, y_{2,i-m}, \quad \text{and/or:} \\ if\ i + j > m\ then\ x_{2,i+j}, x_{1,i+j} \rightarrow y_{2,i+j-m}, y_{1,i+j-m} \end{Bmatrix}$$

   Where, the unit vector $\mathbf{n}$, is normal to $\mathbf{r_1}$ & $\mathbf{r_2}$ and to all basis vectors.
14. $\mathbf{r_1}\times\mathbf{r_2} = (\boldsymbol{\rho_1} + \widehat{\boldsymbol{\gamma_1}})\times(\boldsymbol{\rho_2} + \widehat{\boldsymbol{\gamma_2}}) = \boldsymbol{\rho_1}\times\widehat{\boldsymbol{\gamma_2}} - \boldsymbol{\rho_2}\times\widehat{\boldsymbol{\gamma_1}} =$
   $= \rho_1\mathbf{u}\times\gamma_2\hat{\boldsymbol{u}} - \rho_2\mathbf{u}\times\gamma_1\hat{\boldsymbol{u}} = (\rho_1\gamma_2 - \rho_2\gamma_1)\mathbf{u}\times\hat{\boldsymbol{u}}$

8. Conclusion

This work has defined an expression of a Multidimensional Complex Vector, different to that given in [2], [3], and [4]. It was demonstrated how this new expression can be applied as a "solution" of the cutoff problem of a parabola with its directrix (they never cut each other) in order to obtain complex roots. We also found a way to calculate the Multidimensional Dot and Cross product of two complex vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ based on our definition of complex vectors consistent with the Dot Product of two complex vectors, as a real number, and their Cross Product as another Vector (Complex), perpendicular to $\mathbf{r_1}$ and $\mathbf{r_2}$. In section **5** we explain how the definitions of the Multidimensional Dot and Cross Product emerge as a natural consequence of the multidimensional complex vector concept developed in Section **3**. Finally, we demonstrate that the "conjugate" in complex vectors does not have the meaning and properties found in complex numbers.

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