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A naive solution for Navier-Stokes equations

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Abstract – We seek some attempt solutions for the system of Navier-Stokes equations for spatial dimensions n = 2 and n = 3. These solutions have the principal objective to provide a better numerical evaluation of the exact analytical solution, thus contributing to the solution not only from a theoretical mathematical problem, but from a practical problem worldwide.

Keywords – Navier-Stokes equations, numerical solutions, exact solutions, equivalent systems, Millennium Problem, existence, smoothness.

1 – Introduction

The reading the last pages of chapter 10 of the book by Ian Stewart, "Seventeen Equations that Changed the World" [1], reminded me once again of the importance of the Navier-Stokes equations, especially of its solutions. A sense of urgency proved necessary for this issue. It is not equal to seek proof of the Riemann hypothesis, which although it is one of the most difficult problems of mathematics does not seem to bring greater immediate consequences to the world.

The problem of the Navier-Stokes equations described in the millennium problems[2] was solved for the case (C), the breakdown of the solutions[3], [4], based on the acceptance of known theorems of existence and uniqueness solutions, theorems I believed fail only when the nonlinear terms are equal to zero, or not have all the terms, although I recognize that the cases more interesting and useful to solve are the cases (A) and (B), the proof of existence and smoothness of their solutions for all initial velocity $u^0(x)$ obeying determinate conditions.

The world is running a serious heating problem, either by natural or human causes. The more likely they are combined causes, of course. The northern hemisphere is heating up more (much more...) that the southern hemisphere, so we cannot rule out the human influence in this heat. Evidently the northern hemisphere is the most industrialized hemisphere of the world, which produces more heat due to their machines, and thus would be more likely to contribute to this warming.

The problem of global warming is not only the increase in temperature, the feeling of discomfort, but also in the disasters that it is able to produce, as the melting ice of the poles, the corresponding increase in sea levels, as well as torrential rains, storms, fires and the most destructive hurricanes.

According Ian Stewart in the mencioned book, two climate vital aspects are the atmosphere and the oceans. Both are fluid, and both can be treated using the Navier-Stokes equation. The secrets of the climate system are closed in the Navier-Stokes equation. He said, referring to a research council document in physical sciences and engineering (EPSRC – Engineering & Physical Sciences Research Council, from United Kingdom), published in 2010: "The secrets of the climate system are closed in the Navier-Stokes equation, but it is too complex to be solved directly". Instead, researchers of climate models are using numerical methods to calculate the fluid flow at the point of a three-dimensional grid covering the globe from the depths of the oceans to the highest points of the atmosphere. The horizontal grid spacing is 100 km; anything less makes your computation impractical. Faster computers will not serve much, then the best way forward is to think harder. Mathematicians are developing more efficient means to numerically solve the Navier-Stokes equation.

Then that's it. The purpose of this paper is to find a solution to the system of Navier-Stokes equations, given the initial condition $u(x,0) = u^0(x),\ x \in \mathbb{R}^n, n = 2$ and $n = 3$, for both the cases that must be obeyed the equation of incompressibility, $\nabla \cdot u = \nabla \cdot u^0 = 0$, as also for the general case, any values of $\nabla \cdot u$ and $\nabla \cdot u^0$. When $\nabla \cdot u \neq 0$ must be added the term $\frac{1}{3}\,\nu\nabla(\nabla \cdot u)$ to $\nu\nabla^2 u$ on the right side of the Navier-Stokes equations[5], which for simplicity will be omitted here. Obviously this method can be used for the numerical solution of these equations, which I hope to have your accuracy greatly increased (1 m or less instead of 100 km would be excellent). A grid with width cell 100 km is absolutely unreliable.

2 – Solutions for $n = 2$

The system of Navier-Stokes equations in spatial dimension $n = 2$ is

$$(2.1)\qquad \begin{cases} \dfrac{\partial p}{\partial x} + \dfrac{\partial u_1}{\partial t} + u_1\dfrac{\partial u_1}{\partial x} + u_2\dfrac{\partial u_1}{\partial y} = \nu\nabla^2 u_1 + f_1 \\ \dfrac{\partial p}{\partial y} + \dfrac{\partial u_2}{\partial t} + u_1\dfrac{\partial u_2}{\partial x} + u_2\dfrac{\partial u_2}{\partial y} = \nu\nabla^2 u_2 + f_2 \end{cases}$$

or in vectorial form

$$(2.2)\qquad \nabla p + \frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu\nabla^2 u + f,$$

where $u(x,y,t) = (u_1(x,y,t), u_2(x,y,t))$, $u: \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}^2$, is the velocity of the fluid, of components u_1, u_2, p is the pressure, $p: \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}$, and $f(x,y,t) = (f_1(x,y,t), f_2(x,y,t))$, $f: \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}^2$, is the density of external force applied in the fluid in point (x,y) and at the instant of time t, for example, gravity force per mass unity, with $x,y,t \in \mathbb{R}$, $t \geq 0$. The coefficient $\nu \geq 0$ is the viscosity coefficient, and in the special case that $\nu = 0$ we have the Euler equations. $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is the nabla operator and $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \equiv \Delta$ is the Laplacian operator.

If u_1 and u_2 are solutions of system (1) then are valid the following equalities:

$$(2.3) \qquad u_2 = \frac{\nu \nabla^2 u_1 + f_1 - \left(\frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x}\right)}{\frac{\partial u_1}{\partial y}}, \text{ if } \frac{\partial u_1}{\partial y} \neq 0,$$

and

$$(2.4) \qquad u_1 = \frac{\nu \nabla^2 u_2 + f_2 - \left(\frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial y}\right)}{\frac{\partial u_2}{\partial x}}, \text{ if } \frac{\partial u_2}{\partial x} \neq 0.$$

The equation (2.3) says that u_2 is a function of u_1, as well as the equation (2.4) says that u_1 is a function of u_2. Therefore, if we have the correct value of u_1 we can get the value of u_2, and vice versa, need for this too that the pressure can be obtained. The equations (2.3) and (2.4) can not contradict each other, i.e., the obtaining u_2 given u_1 in (2.3) must be verified next by the use of the equation (2.4), confirming it, and vice versa. If the pressure p is not a given function for the problem, both equations (2.3) and (2.4) need be solved to the complete obtainment of p. Thus, in principle, the velocity and pressure can be obtained completely following this method, since that $\frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x} \neq 0$. In this case, the systems (2.3)–(2.4) and (2.1) are equivalent.

The solutions (2.3) and (2.4) are valid for all $t \geq 0$ on condition that $\frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x} \neq 0$, and in this case, in $t = 0$, defining $f(x,y,0) = f^0(x,y)$ and $p(x,y,0) = p^0(x,y)$, we come to

$$(2.5) \qquad u_2^0 = \frac{\nu \nabla^2 u_1^0 + f_1^0 - \left(\frac{\partial p^0}{\partial x} + \frac{\partial u_1}{\partial t}|_{t=0} + u_1^0 \frac{\partial u_1^0}{\partial x}\right)}{\frac{\partial u_1^0}{\partial y}}, \text{ if } \frac{\partial u_1^0}{\partial y} \neq 0,$$

and

(2.6) $$u_1^0 = \frac{\nu\nabla^2 u_2^0 + f_2^0 - \left(\frac{\partial p^0}{\partial y} + \frac{\partial u_2}{\partial t}|_{t=0} + u_2^0\frac{\partial u_2^0}{\partial y}\right)}{\frac{\partial u_2^0}{\partial x}}, \text{ if } \frac{\partial u_2^0}{\partial x} \neq 0,$$

i.e., u_1^0 and u_2^0 are related by (2.5) and (2.6), beyond the incompressibility condition, $\nabla \cdot u^0 = 0$, if this is a condition imposed.

The equations (2.5) and (2.6) can be used to calculate $\frac{\partial u_1}{\partial t}|_{t=0}$ and $\frac{\partial u_2}{\partial t}|_{t=0}$, supposing that the pressure or its respective spatial derivatives are provided at least at time $t = 0$.

For other values of $t, t > 0$, through the value of $\frac{\partial u}{\partial t}$, held fixed position (x, y), it is possible to calculate the value of $u(x, y, t)$, obviously by integrating with respect to time the local acceleration $\frac{\partial u}{\partial t}$, i.e.,

(2.7) $$u = \int \frac{\partial u}{\partial t} dt + v(x, y),$$

where $v(x, y)$ may be encountered by given initial conditions.

Numerically, we have

(2.8) $$u^{T+\Delta T} = u^T + \frac{\partial u}{\partial t}|_{t=T}\, \Delta T,$$

where u^T is the fluid velocity in the position (x, y) at time $t = T$. ΔT is a positive not null small constant, the increment in time to each step calculation for u^T.

Using (2.1) in (2.8) comes

(2.9) $$u_1^{T+\Delta T} = u_1^T + \left(\nu\nabla^2 u_1^T + f_1^T - \frac{\partial p^T}{\partial x} - u_1^T \frac{\partial u_1^T}{\partial x} - u_2^T \frac{\partial u_1^T}{\partial y}\right)\Delta T,$$

(2.10) $$u_2^{T+\Delta T} = u_2^T + \left(\nu\nabla^2 u_2^T + f_2^T - \frac{\partial p^T}{\partial y} - u_1^T \frac{\partial u_2^T}{\partial x} - u_2^T \frac{\partial u_2^T}{\partial y}\right)\Delta T,$$

where f^T and p^T are the external force and pressure, respectively, in the position (x, y) at time $t = T$, supposing given $p^T \equiv p(x, y, T)$. Numerically and algorithmically, we need to use the approximations (among other that knows in the literature about numerical methods[6])

(2.11) $$\frac{\partial u_1^T}{\partial x} \approx \frac{u_1^T(x+\Delta x, y, T) - u_1^T(x, y, T)}{\Delta x},$$

(2.12) $$\frac{\partial u_1^T}{\partial y} \approx \frac{u_1^T(x, y+\Delta y, T) - u_1^T(x, y, T)}{\Delta y},$$

(2.13) $$\frac{\partial u_2^T}{\partial x} \approx \frac{u_2^T(x+\Delta x,y,T) - u_2^T(x,y,T)}{\Delta x},$$

(2.12) $$\frac{\partial u_2^T}{\partial y} \approx \frac{u_2^T(x,y+\Delta y,T) - u_2^T(x,y,T)}{\Delta y},$$

(2.13) $$\nabla^2 u_1^T \approx \frac{u_1^T(x+2\Delta x,y,T) - 2u_1^T(x+\Delta x,y,T) + u_1^T(x,y,T)}{(\Delta x)^2} + \\ + \frac{u_1^T(x,y+2\Delta y,T) - 2u_1^T(x,y+\Delta y,T) + u_1^T(x,y,T)}{(\Delta y)^2},$$

(2.14) $$\nabla^2 u_2^T \approx \frac{u_2^T(x+2\Delta x,y,T) - 2u_2^T(x+\Delta x,y,T) + u_2^T(x,y,T)}{(\Delta x)^2} + \\ + \frac{u_2^T(x,y+2\Delta y,T) - 2u_2^T(x,y+\Delta y,T) + u_2^T(x,y,T)}{(\Delta y)^2},$$

where $\Delta x \times \Delta y$ is the grid cell size.

This numerical-algorithmic approach, which resulted in the equations (2.9) to (2.14), it shows that we can calculate approximately the system solution (2.1) from $t = 0$ up to any $t = T_{max}$, and the same method can be used in $n = 3$. When greater T_{max} value, however, the greater the accumulation of numerical errors to the correct result. It will be very convenient if it is possible to obtain an exact solution (the great dream) to this problem, at least in certain situations, eliminating thus to the maximum the occurrence of numerical errors. Our naive solution, or better, our first naive attempt solution, will be described to follow.

The smaller the value of T, the closest correct value of u are the results obtained with (2.9) and (2.10). Therefore, considering t a small value, in the first order approximation in time the solution to the u components will be

(2.15) $$u_1 = u_1^0 + \left(\nu \nabla^2 u_1^0 + f_1 - \frac{\partial p}{\partial x} - u_1^0 \frac{\partial u_1^0}{\partial x} - u_2^0 \frac{\partial u_1^0}{\partial y} \right) t,$$

(2.16) $$u_2 = u_2^0 + \left(\nu \nabla^2 u_2^0 + f_2 - \frac{\partial p}{\partial y} - u_1^0 \frac{\partial u_2^0}{\partial x} - u_2^0 \frac{\partial u_2^0}{\partial y} \right) t,$$

which shows the possibility of infinite solutions to velocity, given only the initial velocity, since each different pressure can, in principle, imply a different velocity. Unfortunately, in general the above solution is not limited to the increased time, and therefore in general there is not here a case of velocity belonging to Schwartz space, space of fast decreasing functions. This time t in (2.15) and (2.16) corresponds exactly to the ΔT value that appears in (2.9) and (2.10).

Defining $x_1 := x, x_2 := y$, for an arbitrary value of t, we can try a solution to the system (2.1) in the form

(2.17) $$u_i = u_i^0 + X_i\left(u_1^0, u_2^0, f_i, \frac{\partial p}{\partial x_i}\right) T_i(t),$$

with

(2.18) $$T_i(0) = 0,\ \ T_i{}'(0) = 1,$$

in special

(2.19) $$X_i = \nu \nabla^2 u_i^0 + f_i - \frac{\partial p}{\partial x_i} - u_1^0 \frac{\partial u_i^0}{\partial x} - u_2^0 \frac{\partial u_i^0}{\partial y},$$

or else, for example,

(2.20) $$u_i = u_i^0 + X_i(u_1^0, u_2^0)t + \int \left(f_i - \frac{\partial p}{\partial x_i}\right) dt + v_i(x, y),$$

(2.21) $$X_i = \nu \nabla^2 u_i^0 - u_1^0 \frac{\partial u_i^0}{\partial x} - u_2^0 \frac{\partial u_i^0}{\partial y},$$

solutions based on (2.15) and (2.16), with

(2.22) $$\int \left(f_i - \frac{\partial p}{\partial x_i}\right) dt|_{t=0} + v_i(x, y) = 0.$$

Differentiating (2.20) in relation to time, obviously, we obtain

(2.23) $$\frac{\partial u_i}{\partial t} = X_i(u_1^0, u_2^0) + f_i - \frac{\partial p}{\partial x_i},$$

or, using (2.21),

(2.24) $$\frac{\partial u_i}{\partial t} = \nu \nabla^2 u_i^0 - u_1^0 \frac{\partial u_i^0}{\partial x} - u_2^0 \frac{\partial u_i^0}{\partial y} + f_i - \frac{\partial p}{\partial x_i}.$$

To the equation (2.24) to be equivalent to the system (2.1) for all u_i we need to have

(2.25) $$\nu \nabla^2 u_i - u_1 \frac{\partial u_i}{\partial x} - u_2 \frac{\partial u_i}{\partial y} = \nu \nabla^2 u_i^0 - u_1^0 \frac{\partial u_i^0}{\partial x} - u_2^0 \frac{\partial u_i^0}{\partial y},$$

therefore, trying

(2.26) $$u_i(x, y, t) = u_i^0(x, y) + w_i(t),\ \ w_i(0) = 0,$$

and substituting (2.26) in (2.25), it is necessary that

(2.27) $$w_1(t) \frac{\partial u_i^0}{\partial x} + w_2(t) \frac{\partial u_i^0}{\partial y} = 0.$$

The trivial solutions of (2.27) are $w_1(t) = w_2(t) = 0$ and $u_i^0 = cte$. A more general condition is

(2.28) $$\frac{w_1(t)}{w_2(t)} = -\frac{\partial u_i^0/\partial y}{\partial u_i^0/\partial x} = k,\ k \in \mathbb{R}^*,\ i = 1, 2.$$

Well, the solution (2.26) there is not the same form that (2.20)–(2.21), except if

(2.29) $$\begin{cases} f_i - \dfrac{\partial p}{\partial x_i} = v_i(x, y) = 0 \\ w_i(t) = k_i t,\ k_i \in \mathbb{R}^* \\ X_i = \nu \nabla^2 u_i^0 - u_1^0 \dfrac{\partial u_i^0}{\partial x} - u_2^0 \dfrac{\partial u_i^0}{\partial y} = k_i,\ k_i \in \mathbb{R}^* \end{cases}$$

and, according (2.28),

(2.30) $$\frac{\partial u_i^0}{\partial y} = -k \frac{\partial u_i^0}{\partial x},\ k = \frac{k_1}{k_2},\ k, k_1, k_2 \in \mathbb{R}^*.$$

For this reason, the attempt solution (2.20)–(2.21) correctly solved the system (2.1) for some initial velocities, in special when (2.29) and (2.30) are obeyed. Another case of solution when (2.20)–(2.21) is valid, using trivial solution of (2.27), is

(2.31) $$\begin{cases} f_i - \dfrac{\partial p}{\partial x_i} = v_i(x, y) = 0 \\ u_i^0 = u = cte. \end{cases}$$

The dependence of f in relation to p, related in (2.29) and (2.31), or

(2.32) $$\nabla p = f,$$

shows that it's necessary f be a gradient function, and thus p is a potential function for f (see, for example, [7]). An example for f is a constant gravity acceleration, like $f = (0, -g)$, assuming a two-dimensional world, and in this case we have $p = -gy$.

For more generic initial velocity, the form given by (2.26) is our next attempt solution,

(2.33) $$u_i(x, y, t) = u_i^0(x, y) + w_i(t),\ w_i(0) = 0.$$

Applying (2.33) in (2.1) comes

(2.34) $$\frac{\partial p}{\partial x_i} + \frac{d}{dt} w_i(t) + u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y} + w_i(t) \left[\frac{\partial u_i^0}{\partial x} + \frac{\partial u_i^0}{\partial y}\right] = \\ = \nu \nabla^2 u_i^0 + f_i,$$

using $x_1 := x,\ \ x_2 := y$.

A consistent initial velocity also needs to be (2.1) solution, for $t = 0$. In $t = 0$ the equation (2.34) is equivalent to

$$(2.35)\qquad \frac{\partial p^0}{\partial x_i} + w_i{}'(0) + u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y} = \nu \nabla^2 u_i^0 + f_i^0,$$

using $w_i(0) = 0$, so

$$(2.36)\qquad u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y} = \nu \nabla^2 u_i^0 + f_i^0 - \frac{\partial p^0}{\partial x_i} - w_i{}'(0),$$

the superior symbol 0 meaning the respective function value at time $t = 0$.

Substituting (2.36) in (2.34) we obtain

$$(2.37)\qquad \left(\frac{\partial p}{\partial x_i} - \frac{\partial p^0}{\partial x_i}\right) + \left(w_i'(t) - w_i'(0)\right) + \left[w_1(t)\frac{\partial u_i^0}{\partial x} + w_2(t)\frac{\partial u_i^0}{\partial y}\right] =$$
$$= f_i(x,y,t) - f_i^0(x,y),$$

a equality that allow us to solve the system (2.1) in many situations, for any u^0 (or better, $\forall u^0 \in C^2(\mathbb{R}^2)$), making u_i^0 (and therefore u_i) a function of p, w_j, f_i, or by contrary, making p a function of u_i^0, w_j, f_i, being $u_i(x,y,t) = u_i^0(x,y) + w_i(t)$, according (2.33). But for this reason we cannot to accept any external force and pressure in the system, or model, except when (2.37) is true and the pressure can be calculated. Between numerically solve the system (2.37) or (2.1) seems to me that (2.37) is faster to solve, in special when the pressure is given, $p \in C^1(\mathbb{R}^2 \times [0,\infty))$, and the velocity is the unique unknown variable.

The next and last attempt solution is

$$(2.38)\qquad u_i(x,y,t) = u_i^0(x,y)\, w_i(t),\ \ w_i(0) = 1,$$

where $u_i: \mathbb{R}^2 \times [0,\infty) \to \mathbb{R},\ u_i^0: \mathbb{R}^2 \to \mathbb{R},\ \ w_i: [0,\infty) \to \mathbb{R},\ \ i = 1,2.$

Repeating the steps from (2.33) to (2.37) with (2.38), applying (2.38) in (2.1) we come to

$$(2.39)\qquad \frac{\partial p}{\partial x_i} + u_i^0 \frac{d}{dt} w_i + w_1 w_i u_1^0 \frac{\partial u_i^0}{\partial x} + w_2 w_i u_2^0 \frac{\partial u_i^0}{\partial y} =$$
$$= \frac{\partial p}{\partial x_i} + u_i^0 w_i' + w_i\left[w_1 u_1^0 \frac{\partial u_i^0}{\partial x} + w_2 u_2^0 \frac{\partial u_i^0}{\partial y}\right] = \nu w_i \nabla^2 u_i^0 + f_i.$$

As we have said, a consistent initial velocity also needs to be (2.1) solution, for $t = 0$. In $t = 0$ the equation (2.39) is equivalent to

(2.40) $$\frac{\partial p^0}{\partial x_i} + u_i^0 w_i'^0 + w_1^0 w_i^0 u_1^0 \frac{\partial u_i^0}{\partial x} + w_2^0 w_i^0 u_2^0 \frac{\partial u_i^0}{\partial y} = \\ = \nu w_i^0 \nabla^2 u_i^0 + f_i^0,$$

defining $w_i'^0 = \frac{dw_i}{dt}|_{t=0}$ and $w_i^0 = w_i(0) = 1$, so

(2.41) $$w_1^0 w_i^0 u_1^0 \frac{\partial u_i^0}{\partial x} + w_2^0 w_i^0 u_2^0 \frac{\partial u_i^0}{\partial y} - \nu w_i^0 \nabla^2 u_i^0 = \\ = \left[u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y} - \nu \nabla^2 u_i^0\right] = \\ = f_i^0 - \frac{\partial p^0}{\partial x_i} - u_i^0 w_i'^0.$$

Supposing $w_1 = w_2 = w$ and therefore $w_1^0 = w_2^0 = w^0 = 1$, $w_1' = w_2' = w'$, $w_1'^0 = w_2'^0 = w'^0$, we have from (2.39) and (2.41), respectively,

(2.42) $$\frac{\partial p}{\partial x_i} + u_i^0\, w' + w^2 \left[u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y}\right] = \nu w \nabla^2 u_i^0 + f_i$$

and

(2.43) $$\left[u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y}\right] = \nu \nabla^2 u_i^0 + f_i^0 - \frac{\partial p^0}{\partial x_i} - u_i^0 w'^0.$$

Taking the factor $\left[u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y}\right]$ in (2.43) and leading it in (2.42) we obtain

(2.44) $$\left(\frac{\partial p}{\partial x_i} - \alpha \frac{\partial p^0}{\partial x_i}\right) = \nu\, (w - \alpha) \nabla^2 u_i^0 - u_i^0 (w' - \alpha w'^0) + \\ + \left(f_i - \alpha f_i^0\right),$$

with $\alpha = w^2(t) \neq 0$. This relation (2.44) shows us that there are many possibilities to solve the system of Navier-Stokes equations, for an infinite set of initial velocities, external forces and pressure, thereby eliminating the non-linear term.

The integration of (2.44) is

(2.45) $$p - \alpha p^0 = \int_L \; S \cdot dl, \\ S = \nu\, (w - \alpha) \nabla^2 u^0 - u^0 (w' - \alpha w'^0) + (f - \alpha f^0),$$

where L is any continuous path linking a point (x_0, y_0) to (x, y), supposing that the integrand S is a gradient field[7], without singularities.

3 – Solutions for $n = 3$

Similar to what we saw in section 2 for $n = 2$, now we solve the Navier-Stokes equations for spatial dimension $n = 3$. As we know, it can be put in the form of a system of three nonlinear partial differential equations, as follows:

$$(3.1) \qquad \begin{cases} \frac{\partial p}{\partial x} + \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} = \nu \nabla^2 u_1 + f_1 \\ \frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z} = \nu \nabla^2 u_2 + f_2 \\ \frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z} = \nu \nabla^2 u_3 + f_3 \end{cases}$$

where $u(x, y, z, t) = \big(u_1(x, y, z, t), u_2(x, y, z, t), u_3(x, y, z, t)\big)$, $u: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$, is the velocity of the fluid, of components u_1, u_2, u_3, p is the pressure, $p: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$, and $f(x, y, z, t) = \big(f_1(x, y, z, t), f_2(x, y, z, t), f_3(x, y, z, t)\big)$, $f: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3$, is the density of external force applied in the fluid in point (x, y, z) and at the instant of time t, for example, gravity force per mass unity, with $x, y, z, t \in \mathbb{R}$, $t \geq 0$. The coefficient $\nu \geq 0$ is the viscosity coefficient, and in the special case that $\nu = 0$ we have the Euler equations. $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ is the nabla operator and $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv \Delta$ is the Laplacian operator.

Writing u_1 as a function of u_2 and u_3 we have by the system (3.1) above,

$$(3.2) \qquad u_1 = \frac{\nu \nabla^2 u_2 + f_2 - \left(\frac{\partial p}{\partial y} + \frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z}\right)}{\frac{\partial u_2}{\partial x}}, \text{ if } \frac{\partial u_2}{\partial x} \neq 0,$$

$$(3.3) \qquad u_1 = \frac{\nu \nabla^2 u_3 + f_3 - \left(\frac{\partial p}{\partial z} + \frac{\partial u_3}{\partial t} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z}\right)}{\frac{\partial u_3}{\partial x}}, \text{ if } \frac{\partial u_3}{\partial x} \neq 0,$$

$$(3.4) \qquad \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} = \nu \nabla^2 u_1 + f_1 - \frac{\partial p}{\partial x},$$

therefore valid system when $\frac{\partial u_2}{\partial x} \frac{\partial u_3}{\partial x} \neq 0$.

Similarly to u_1, we obtain the following equations for u_2 and u_3, in index notation, defining $x_1 \coloneqq x$, $x_2 \coloneqq y$, $x_3 \coloneqq z$, and index 4 = index 1, index 5 = index 2, with $1 \leq j \leq 3$,

$$(3.5) \qquad u_i = \frac{\nu \nabla^2 u_j + f_j - \left(\frac{\partial p}{\partial x_j} + \frac{\partial u_j}{\partial t} + u_{i+1} \frac{\partial u_j}{\partial x_{i+1}} + u_{i+2} \frac{\partial u_j}{\partial x_{i+2}}\right)}{\frac{\partial u_j}{\partial x_i}}, \text{ if } \frac{\partial u_j}{\partial x_i} \neq 0,$$

$$(3.6) \qquad \frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z} = \nu \nabla^2 u_i + f_i - \frac{\partial p}{\partial x_i},$$

therefore valid systems when $\frac{\partial u_{i+1}}{\partial x_i}\frac{\partial u_{i+2}}{\partial x_i} \neq 0$, for $1 \leq i \leq 3$.

All solutions obtained in (3.5) can not contradict each other, as well as (3.6) must be true for each i.

The solutions (3.5) are valid for all $t \geq 0$ on condition that $\frac{\partial u_{i+1}}{\partial x_i}\frac{\partial u_{i+2}}{\partial x_i} \neq 0$, for $1 \leq i \leq 3$, and in this case, in $t = 0$, defining $f(x, y, z, 0) = f^0(x, y, z)$ and $p(x, y, z, 0) = p^0(x, y, z)$ and using index notation, we come to

(3.7) $$u_i^0 = \frac{\nu\nabla^2 u_j^0 + f_j^0 - \left(\frac{\partial p^0}{\partial x_j} + \frac{\partial u_j}{\partial t}|_{t=0} + u_{i+1}^0\frac{\partial u_j^0}{\partial x_{i+1}} + u_{i+2}^0\frac{\partial u_j^0}{\partial x_{i+2}}\right)}{\frac{\partial u_j^0}{\partial x_i}}, 1 \leq j \leq 3,$$

where the superior index 0 means the respective value function at time $t = 0$. The equation (3.7) shows that the sum $\frac{\partial p^0}{\partial x_j} + \frac{\partial u_j}{\partial t}|_{t=0}$ cannot have any arbitrary value, independently of u_i^0 relation (3.7), contradicting it.

Numerically we can solve (3.1) through following iteration algorithm, just like we do for $n = 2$, for each natural i in $1 \leq i \leq 3$:

(3.8) $$u_i^{T+\Delta T} = u_i^T + \left(\nu\nabla^2 u_i^T + f_i^T - \frac{\partial p^T}{\partial x} - \sum_{j=1}^n u_j^T\frac{\partial u_i^T}{\partial x_j}\right)\Delta T,$$

where u^T, f^T and p^T are the velocity, external force and pressure, respectively, in the position (x, y, z) at time $t = T$, supposing given $p^T \equiv p(x, y, z, T)$. ΔT is a positive not null small constant, the increment in time to each step calculation for u^T.

Again, we need to use the approximations (among other that knows in the literature containing numerical methods[6])

(3.9) $$\frac{\partial u_1^T}{\partial x} \approx \frac{u_1^T(x+\Delta x,y,z,T) - u_1^T(x,y,z,T)}{\Delta x},$$

(3.10) $$\frac{\partial u_1^T}{\partial y} \approx \frac{u_1^T(x,y+\Delta y,z,T) - u_1^T(x,y,z,T)}{\Delta y},$$

(3.11) $$\frac{\partial u_1^T}{\partial z} \approx \frac{u_1^T(x,y,z+\Delta z,T) - u_1^T(x,y,z,T)}{\Delta z},$$

(3.11) $$\frac{\partial u_2^T}{\partial x} \approx \frac{u_2^T(x+\Delta x,y,z,T) - u_2^T(x,y,z,T)}{\Delta x},$$

(3.12) $$\frac{\partial u_2^T}{\partial y} \approx \frac{u_2^T(x,y+\Delta y,z,T) - u_2^T(x,y,z,T)}{\Delta y},$$

(3.13) $$\frac{\partial u_2^T}{\partial z} \approx \frac{u_2^T(x,y,z+\Delta z,T) - u_2^T(x,y,z,T)}{\Delta z},$$

(3.14) $$\frac{\partial u_3^T}{\partial x} \approx \frac{u_3^T(x+\Delta x,y,z,T) - u_3^T(x,y,z,T)}{\Delta x},$$

(3.15) $$\frac{\partial u_3^T}{\partial y} \approx \frac{u_3^T(x,y+\Delta y,z,T) - u_3^T(x,y,z,T)}{\Delta y},$$

(3.16) $$\frac{\partial u_3^T}{\partial z} \approx \frac{u_3^T(x,y,z+\Delta z,T) - u_3^T(x,y,z,T)}{\Delta z},$$

(3.17) $$\nabla^2 u_1^T \approx \frac{u_1^T(x+2\Delta x,y,z,T)-2u_1^T(x+\Delta x,y,z,T)+u_1^T(x,y,z,T)}{(\Delta x)^2} + $$
$$+\frac{u_1^T(x,y+2\Delta y,z,T)-2u_1^T(x,y+\Delta y,z,T)+u_1^T(x,y,z,T)}{(\Delta y)^2} + $$
$$+\frac{u_1^T(x,y,z+2\Delta z,T)-2u_1^T(x,y,z+\Delta z,T)+u_1^T(x,y,z,T)}{(\Delta z)^2},$$

(3.18) $$\nabla^2 u_2^T \approx \frac{u_2^T(x+2\Delta x,y,z,T)-2u_2^T(x+\Delta x,y,z,T)+u_2^T(x,y,z,T)}{(\Delta x)^2} + $$
$$+\frac{u_2^T(x,y+2\Delta y,z,T)-2u_2^T(x,y+\Delta y,z,T)+u_2^T(x,y,z,T)}{(\Delta y)^2} + $$
$$+\frac{u_2^T(x,y,z+2\Delta z,T)-2u_2^T(x,y,z+\Delta z,T)+u_2^T(x,y,z,T)}{(\Delta z)^2},$$

(3.19) $$\nabla^2 u_3^T \approx \frac{u_3^T(x+2\Delta x,y,z,T)-2u_3^T(x+\Delta x,y,z,T)+u_3^T(x,y,z,T)}{(\Delta x)^2} + $$
$$+\frac{u_3^T(x,y+2\Delta y,z,T)-2u_3^T(x,y+\Delta y,z,T)+u_3^T(x,y,z,T)}{(\Delta y)^2} + $$
$$+\frac{u_3^T(x,y,z+2\Delta z,T)-2u_3^T(x,y,z+\Delta z,T)+u_3^T(x,y,z,T)}{(\Delta z)^2},$$

where $\Delta x \times \Delta y \times \Delta z$ is the three-dimensional grid cell size.

The greater the value of T, the greater the number of times that need to iterate the solution given in (3.8), more numeric errors are added to the correct solution of system (3.1), is therefore highly desirable to find an exact solution for (3.1).

All attempt solutions seen for the case $n = 2$ can be used for $n = 3$, with obviously adaptations. The simplest (and naive) of these solutions is the similar one to (2.33), (3.20) $$u_i(x,y,z,t) = u_i^0(x,y,z) + w_i(t),\ \ w_i(0) = 0,$$

or

(3.21) $$u(x,y,z,t) = u^0(x,y,z) + w(t),\ \ w(0) = 0,$$

whose direct application in (3.1) and more the correspondent use for $t = 0$ leads to the similar condition (2.37) seen previously, i.e.,

(3.22) $$\left(\frac{\partial p}{\partial x_i} - \frac{\partial p^0}{\partial x_i}\right) + \left(w_i{}'(t) - w_i{}'(0)\right) + \\ + \left[w_1 \frac{\partial u_i^0}{\partial x} + w_2 \frac{\partial u_i^0}{\partial y} + w_3 \frac{\partial u_i^0}{\partial z}\right] = f_i(x,y,z,t) - f_i^0(x,y,z).$$

As we have said for two dimensions, this equality allow us to solve the system (3.1) in many situations, for any u^0 (say, $\forall u^0 \in C^2(\mathbb{R}^3)$), making u_i^0 (and u_i) a function of p, w_j, f_i, or by contrary, making p a function of u_i^0, w_j, f_i, being $u_i(x,y,z,t) = u_i^0(x,y,z) + w_i(t)$, according (3.20). For this reason, when (3.20) is valid we cannot to accept any external force and pressure in the system, or model, except when (3.22) is true and the pressure can be calculated. Again, between numerically solve the system (3.22) or (3.1) seems to me that (3.22) is faster to solve, in special when the pressure is given, $p \in C^1(\mathbb{R}^3 \times [0,\infty))$, and the velocity is the unique unknown variable. By default, however, the initial velocity is the given function and the pressure is an unknown variable to be calculated, and in this manner it is necessary that

(3.23) $$S = \left(f(x,y,z,t) - f^0(x,y,z)\right) - \left(w'(t) - w'(0)\right) - \\ - \left(w_1 \frac{\partial u_i^0}{\partial x} + w_2 \frac{\partial u_i^0}{\partial y} + w_3 \frac{\partial u_i^0}{\partial z}\right)_{1 \le i \le 3}$$

is a gradient vector function[7]. In this general case we have

(3.24) $$p - p^0 = \int_L S \cdot dl,$$

where L is any continuous path linking a point (x_0, y_0, z_0) to (x,y,z).

In the special case when $S = (S_1,\ S_2,\ S_3)$ is equal to zero or an explicit function of time and $S(0) = 0$, the solution for (3.22) is

(3.25) $$p - p^0 = [S_1(t)(x - x_0) + S_2(t)(y - y_0) + S_3(t)(z - z_0)] + \\ + \theta(t),$$

where $\theta(t)$ is a well behaved (or physically reasonable) generic time function with $\theta(0) = 0$, and $p^0(x,y,z) = p(x,y,z,0)$.

The solutions for u, (3.21), and p, (3.24), are not unique, due to infinities different possibilities to construct $w(t)$, $w(0) = 0$. Beyond this, the pressure may be unlimited, when $S(t) \neq 0$, due to linear term $[S_1(t)(x-x_0) + S_2(t)(y - y_0) + S_3(t)(z-z_0)]$, although we can choose $u^0(x, y, z)$ and $w(t)$ that limit the velocity $u(x, y, z, t)$.

The next and last attempt solution for $n = 3$ is

(3.26) $$u_i(x, y, z, t) = u_i^0(x, y, z)\, w(t),\ w(0) = 1,$$

where $u_i: \mathbb{R}^3 \times [0, \infty) \to \mathbb{R},\ u_i^0: \mathbb{R}^3 \to \mathbb{R},\ w: [0, \infty) \to \mathbb{R},\ 1 \leq i \leq 3.$

Applying (3.26) in (3.1), following similarly the steps from (2.38) to (2.44), we come to

(3.27) $$\left(\frac{\partial p}{\partial x_i} - \alpha \frac{\partial p^0}{\partial x_i}\right) = \nu\, (w - \alpha)\nabla^2 u_i^0 - u_i^0(w' - \alpha w'^0) + \\ +\left(f_i - \alpha f_i^0\right),$$

with $\alpha = w^2(t) \neq 0$. Note that this form is no longer necessary to worry about the non-linear term, although the correspondent to (2.43) is implicitly valid for all $i = 1, 2, 3$, of course:

(3.28) $$\left[u_1^0 \frac{\partial u_i^0}{\partial x} + u_2^0 \frac{\partial u_i^0}{\partial y} + u_3^0 \frac{\partial u_i^0}{\partial z}\right] = \nu \nabla^2 u_i^0 + f_i^0 - \frac{\partial p^0}{\partial x_i} - u_i^0 w'^0.$$

The integration of (3.27) is

(3.29) $$p - \alpha p^0 = \int_L S \cdot dl, \\ S = \nu\, (w - \alpha)\nabla^2 u^0 - u^0(w' - \alpha w'^0) + (f - \alpha f^0),$$

where L is any continuous path linking a point (x_0, y_0, z_0) to (x, y, z), supposing again that the integrand S is a gradient field[7], without singularities.

This solution is not always possible, when S is not a gradient field, but it is easily soluble when u^0 is equal to zero and the external force also is $f = f^0 = 0$. In this situation we have $u = 0$ and

(3.30) $$p = w^2(t)p^0 + \theta(t),\ w(0) = 1,\ \theta(0) = 0,$$

again showing the non-uniqueness of the solution for the pressure. $\theta(t)$ is our well behaved (physically reasonable) generic time function.

4 – Conclusion

We do not solve exactly the Navier-Stokes equations in the general case, given any initial velocity, nor proved that this is possible, but we developed some attempt solutions for some initial velocities. Particularly, if we know exactly the value of one ($n = 2$) or two ($n = 3$) velocity components and the pressure we can find the exact value of the component we have not initially, according to equations (2.3), (2.4) and (3.5). In special, for $1 \leq i, j \leq 3$, the exact solution that we seek is

$$(4.1) \qquad u_i = \frac{\nu \nabla^2 u_j + f_j - \left(\frac{\partial p}{\partial x_j} + \frac{\partial u_j}{\partial t} + u_{i+1} \frac{\partial u_j}{\partial x_{i+1}} + u_{i+2} \frac{\partial u_j}{\partial x_{i+2}} \right)}{\frac{\partial u_j}{\partial x_i}}, \text{ if } \frac{\partial u_j}{\partial x_i} \neq 0.$$

Nothing easier than this, although there can be no contradiction, of course. For example, for $n = 3$ the equation (3.6) must continue to be satisfied, i.e.,

$$(4.2) \qquad \frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x} + u_2 \frac{\partial u_i}{\partial y} + u_3 \frac{\partial u_i}{\partial z} = \nu \nabla^2 u_i + f_i - \frac{\partial p}{\partial x_i}.$$

Of the numerical point of view, I think that solve the system (3.22),

$$(4.3) \qquad \left(\frac{\partial p}{\partial x_i} - \frac{\partial p^0}{\partial x_i} \right) + \left(w_i{}'(t) - w_i{}'(0) \right) + \\ + \left[w_1 \frac{\partial u_i^0}{\partial x} + w_2 \frac{\partial u_i^0}{\partial y} + w_3 \frac{\partial u_i^0}{\partial z} \right] = f_i(x, y, z, t) - f_i^0(x, y, z),$$

$1 \leq i \leq 3$, is faster than (3.1), and they are equivalent systems, assuming the validity of

$$(4.4) \qquad u_i(x, y, z, t) = u_i^0(x, y, z) + w_i(t), \;\; w_i(0) = 0.$$

The solution above shows that the velocity u can vary in a same point from the initial velocity u^0 to any other value, adding a convenient time function $w(t)$.

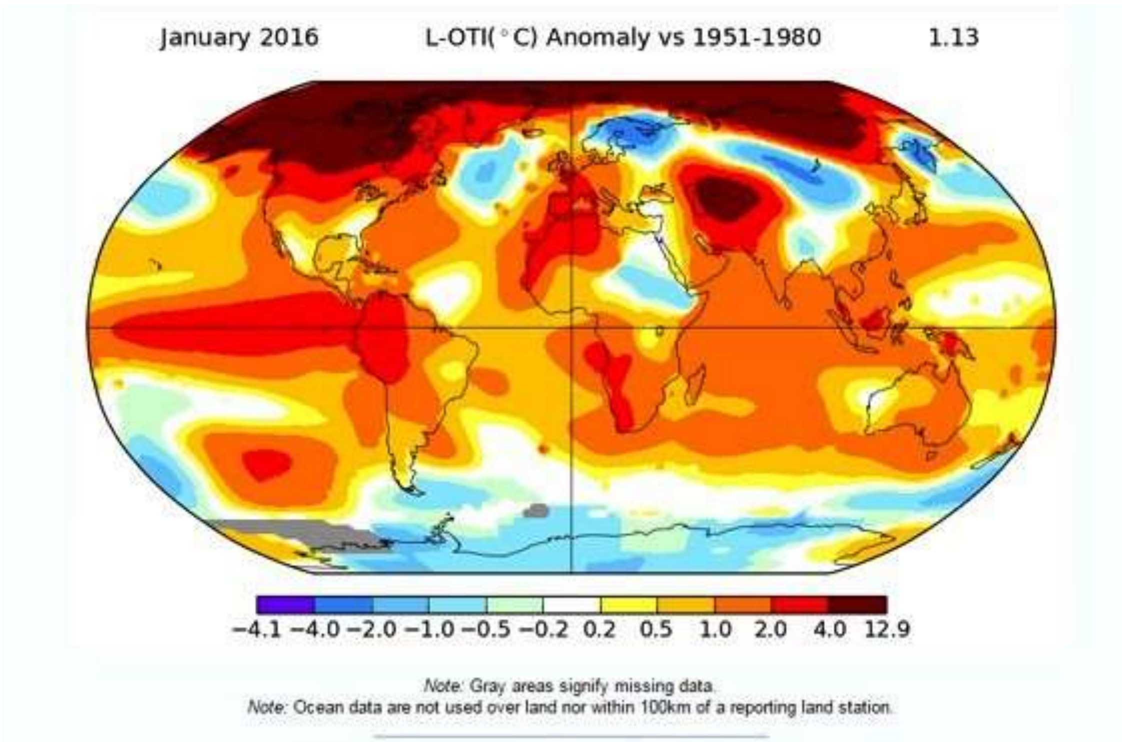
More than this, we reaches to an important conclusion: the solutions of the Navier-Stokes equations may not be unique, even for two spatial dimensions, even for three spatial dimensions, even with all equations with all terms and even for any very small time $t > 0$, and indeed for any value spatial dimension n. We lack at least initial conditions for the pressure, among other requirements, at least in the cases which we analyze. The possibility of infinite solutions, however, even for cases in which all terms are present, leads us to conclude on the need to provide more equations to the models that claim to accurately simulate the atmospheric conditions or fluids in general terms, from simplest cases to the most complex one, or else build more complete Navier-Stokes equations, containing more dependent variable, initial and boundary conditions. What we can see, the same initial velocity can generate both an eternal calm as a giant seaquake.

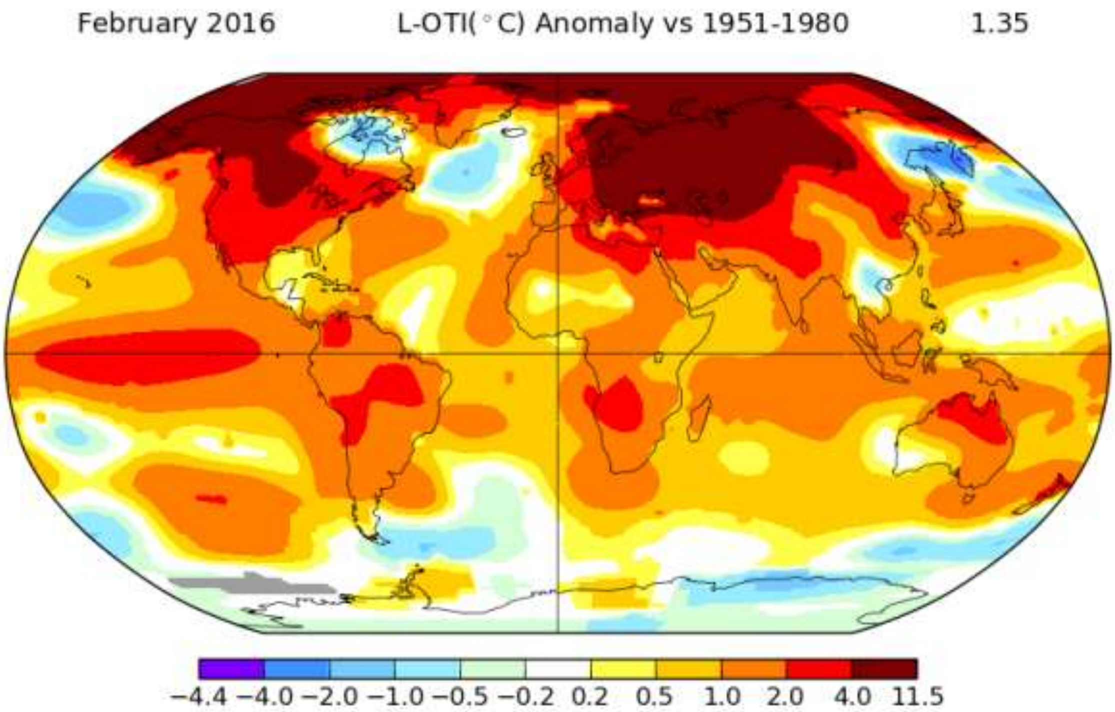
Of course the velocity of a storm, hurricane or a tsunami does not need to be regular, limited, continuous, infinitely differentiable, well behaved and belonging to the Schwartz space, nor obey to the incompressibility condition. This gives us enough freedom to work with these equations.

I think that, in practical terms, the external force can act as a pressure or velocity controller, since it is not only due to the uncontrollable nature, but can also be conveniently constructed by engineering. This is a clear example of Applied Mathematical.

More naive than these solutions is this author…

To world's stability…





Images source: NASA

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