Quaternion fields

The six dimensions of space-time

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Abstract

It is explored some possibilities of mathematical symmetry between physical space and time. We define the space plus time with (three) Pauli matrices plus Quaternions. Indicated the possibility of extending the ideas of Lorentz rotation to general method of transformation coordinates, for arbitrary force field. The theoretical possibility of inversion of coordinates of space and time was observed.

Quaternions are number systems that extend the set of complex numbers. They were discovered by Irish mathematician William Rowan Hamilton in 1843 who has applied it to the mechanics. Hamilton defined quaternion as the quotient of two directed lines, or two vectors in three-dimensional space.

1 Algebra

Algebra in the narrow sense is the branch of mathematics that studies the equation. In a broader sense, *algebra* is the part of mathematics that uses numbers and other quantities represented by letters and symbols for studying formulas and equations.

1.1 Group Theory

A set S together with a binary operation $\circ$, which satisfies the first three (a1-a3) of the next axioms, is called a *group* and denoted by $(S, \circ)$. if it meets the fourth axiom (a4), it is called the commutative or *Abelian group*. For all $\sigma \in S$ in Abel group $(S, \circ)$ is valid:

a1. associativity: $(\sigma_1 \circ \sigma_2) \circ \sigma_3 = \sigma_1 \circ (\sigma_2 \circ \sigma_3)$,

a2. exist neutral $e \in G$ that is $e \circ \sigma = \sigma \circ e = \sigma$,

a3. each $\sigma \in G$ has inverse $\sigma^{-1} \in G$ such that $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = e$,

a4. commutativity: $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1$.

For example, a set of integers ($\mathbb{Z}$), rational ($\mathbb{Q}$), or real ($\mathbb{R}$) numbers is the group together with the common relation plus ($+$) of which the zero (0) is neutral, and the number $-\sigma$ is the inverse element to the number σ. The set of rational or real numbers is the group together with the common multiplication ($\times$) of which the one (1) is neutral, and the number σ^{-1} is the inverse to the number $\sigma \neq 0$.

A *division algebra* $(S, +, \times)$, or division *ring*, is the set S together with the two binary operators, addition and multiplication, respectively $+$ and $\times$, such that $(S, +)$ is Abelian group, $(S, \times)$ is group and apply the law of left and right distributivity:

a5. $\sigma_1 \times (\sigma_2 + \sigma_3) = \sigma_1 \times \sigma_2 + \sigma_1 \times \sigma_3$, and $(\sigma_1 + \sigma_2) \times \sigma_3 = \sigma_1 \times \sigma_3 + \sigma_2 \times \sigma_3$.

In the future we will use dot for the multiplication, or we will omit it as usual.

In the ring $(S, +, \cdot)$, the group $(S, \cdot)$ is *anti-commutative*, if for all $\sigma \in S$ is $\sigma_1 \cdot \sigma_2 = -\sigma_2 \cdot \sigma_1$. Frobenius and Peirce in 1878, and Mishchenko and Solovyov in 2000, proved that the only associative real division algebras are real numbers, complex numbers and quaternions.

Lemma 1. *There is only one neutral element of the group.*

Proof. By contradiction. Suppose that $e_1, e_2 \in S$ are neutral elements and $e_1 \neq e_2$. Than $e_1 \circ e_2 = e_2$ since e_1 is neutral, and $e_1 \circ e_2 = e_1$ since e_2 is neutral. Therefore $e_1 = e_2$ which is contradictions with the assumption. □

Lemma 2. *If $\sigma \circ x = \sigma \circ y$, then $x = y$.*

Proof. There exists inverse $\sigma^{-1} \in S$ for each σ. Therefore:

$$x = e \circ x = (\sigma^{-1} \circ \sigma) \circ x = \sigma^{-1} \circ (\sigma \circ x) = \sigma^{-1} \circ (\sigma \circ y) = e \circ y = y.$$

□

The reducing the equations, so that from $3x = 3y$ follows $x = y$. We can see it as left multiplication to the both sides by the inverse element 3^{-1}.

Lemma 3. *Inverse of inverse is the starting element.*

Proof. It is $\sigma^{-1} \circ (\sigma^{-1})^{-1} = e = \sigma^{-1} \circ \sigma$. For lemma 2 follows $(\sigma^{-1})^{-1} = \sigma$. □

That's why the minus of minus is plus.

Lemma 4. *It is always $(\sigma_1 \circ \sigma_2)^{-1} = \sigma_2^{-1} \circ \sigma_1^{-1}$.*

Proof. It is

$$\sigma_1 \circ (\sigma_2 \circ (\sigma_1 \circ \sigma_2)^{-1}) = (\sigma_1 \circ \sigma_2) \circ (\sigma_1 \circ \sigma_2)^{-1} = e.$$

Left-multiplying the both sides by σ_1^{-1} gives successively:

$$\sigma_1^{-1} \circ (\sigma_1 \circ (\sigma_2 \circ (\sigma_1 \circ \sigma_2)^{-1})) = \sigma_1^{-1},$$

$$(\sigma_1^{-1} \circ \sigma_1) \circ (\sigma_2 \circ (\sigma_1 \circ \sigma_2)^{-1}) = \sigma_1^{-1},$$
$$e \circ (\sigma_2 \circ (\sigma_1 \circ \sigma_2)^{-1}) = \sigma_1^{-1},$$
$$\sigma_2 \circ (\sigma_1 \circ \sigma_2)^{-1} = \sigma_1^{-1}.$$

Left-multiplying the both sides by σ_2^{-1} gives:

$$\sigma_2^{-1} \circ (\sigma_2 \circ (\sigma_1 \circ \sigma_2)^{-1}) = \sigma_2^{-1} \circ \sigma_1^{-1},$$
$$(\sigma_2^{-1} \circ \sigma_2) \circ (\sigma_1 \circ \sigma_2)^{-1} = \sigma_2^{-1} \circ \sigma_1^{-1},$$
$$e \circ (\sigma_1 \circ \sigma_2)^{-1} = \sigma_2^{-1} \circ \sigma_1^{-1},$$
$$(\sigma_1 \circ \sigma_2)^{-1} = \sigma_2^{-1} \circ \sigma_1^{-1}.$$

□

And so on. That is the method of building the abstract algebra whose one fundament is the group theory.

1.2 Matrix

The square matrix of the second order is a group in relation to the *matrix multiplication*:

$$\hat{\mu}\hat{\nu} = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix} \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} = \hat{\sigma},$$

where all matrix coefficients μ_{ab}, ν_{ab} and $\sigma_{ab} = \sum_{\gamma=1}^{2} \mu_{a\gamma}\nu_{\gamma b}$ are complex numbers. The neutral element of that group is the *unit matrix*:

$$\hat{e} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

From linear algebra we know that the matrices multiplied by a complex number (scalar) gives the matrix of the same order with each coefficient of the matrix multiplied by that number. The sum of two matrices (of the second order) is the matrix of the same order whose coefficients are the sum of the corresponding coefficients. Matrixes are commutative group in relation to the addition of the matrix.

Lemma 5 (Pauli matrices)**.** *Solutions of the matrix equation $\hat{\sigma}^2 = \hat{e}$ are $\hat{e}$ plus the three Pauli matrices:*

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

where $i = \sqrt{-1}$ is imaginary unit.

These matrices are known from quantum mechanics, from the *Pauli equation* which takes into account the interaction of the *spin* of a particle with an external electromagnetic field. The proof of this lemma follows from direct multiplication Pauli matrices. These are the only *Hermitian*[1] and *unitary*[2] solutions.

Lemma 6 (Quaternion matrices)**.** *Solutions of the matrix equation $\hat{\sigma}^2 = -\hat{e}$ are the Quaternion matrices:*

$$\hat{\sigma}_i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \hat{\sigma}_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \hat{\sigma}_k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{3}$$

with $i = \sqrt{-1}$.

Quaternion matrices are used in quantum mechanics too, and in the treatment of multi body problems. The assertion of Lemma 6 is easy to check by direct matrix multiplication.

For example, multiplying the Pauli matrices we obtain:

$$\hat{\sigma}_x^2 = \hat{\sigma}_y^2 = \hat{\sigma}_z^2 = \hat{e}, \quad \hat{\sigma}_x\hat{\sigma}_y\hat{\sigma}_z = i\hat{e}, \tag{4}$$

$$i\hat{\sigma}_x = \hat{\sigma}_y\hat{\sigma}_z = -\hat{\sigma}_z\hat{\sigma}_y, \quad i\hat{\sigma}_y = \hat{\sigma}_z\hat{\sigma}_x = -\hat{\sigma}_x\hat{\sigma}_z, \quad i\hat{\sigma}_z = \hat{\sigma}_x\hat{\sigma}_y.$$

Multiplying the Quaternion matrices:

$$\hat{\sigma}_i^2 = \hat{\sigma}_j^2 = \hat{\sigma}_k^2 = -\hat{e}, \quad \hat{\sigma}_i\hat{\sigma}_j\hat{\sigma}_k = -\hat{e}, \tag{5}$$

$$\hat{\sigma}_i = \hat{\sigma}_j\hat{\sigma}_k = -\hat{\sigma}_k\hat{\sigma}_j, \quad \hat{\sigma}_j = \hat{\sigma}_k\hat{\sigma}_i = -\hat{\sigma}_i\hat{\sigma}_k, \quad \hat{\sigma}_k = \hat{\sigma}_i\hat{\sigma}_j = -\hat{\sigma}_j\hat{\sigma}_i.$$

Together with the unitary matrix, each of these groups makes the base of the four-dimensional vector space. Indeed, the linear system $\hat{e}\xi_0 + \hat{\sigma}_1\xi_1 + \hat{\sigma}_2\xi_2 + \hat{\sigma}_3\xi_3 = 0$ has only the trivial solution $(0,0,0,0)$ in both cases, Pauli or quaternions $\hat{\sigma}_\gamma$, $\gamma = 1,2,3$. In the both cases, the determinant of the said system is 2.

Example 1. *Represent an arbitrary matrix of the second order by:*
a. Pauli, b. Quaternion matrices.

Result. a.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{a_{11} + a_{22}}{2}\hat{e} + \frac{a_{12} + a_{21}}{2}\hat{\sigma}_x + \frac{i(a_{12} - a_{21})}{2}\hat{\sigma}_y + \frac{a_{11} - a_{22}}{2}\hat{\sigma}_z.$$

b.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{a_{11} + a_{22}}{2}\hat{e} - \frac{i(a_{11} - a_{22})}{2}\hat{\sigma}_i + \frac{a_{12} - a_{21}}{2}\hat{\sigma}_j - \frac{i(a_{12} + a_{21})}{2}\hat{\sigma}_k.$$

□

[1]The Hermitian matrix (or self-adjoint matrix) is a square matrix with complex entries that is equal to its own conjugate transpose.

[2]The complex square matrix $\hat{u}$ is unitary if $\hat{u}^*\hat{u} = \hat{u}\hat{u}^* = e$. The analogue of a unitary matrix in real is an orthogonal matrix.

Returning to the complex numbers, imaginary unit will be indicated by $\sigma_i, \sigma_j, \sigma_k$, or simply by $i, j, k = \sqrt{-1}$. So $\sqrt{-16} = 4i$ or $4j$ or $4k$. Quaternion is then complex numbers of the form $q = q_0 + iq_i + jq_j + kq_k$, where $q_0, q_i, q_j, q_k \in \mathbb{R}$ and properties similar to (5).

1.3 Versor

Simply said a *versor* is a unit quaternion. The word is from Latin *versus* (turned) and was introduced by Hamilton in the context of his quaternion theory [2]. The versor of a Cartesian axis is also known as a normalized vector or standard basis vector.

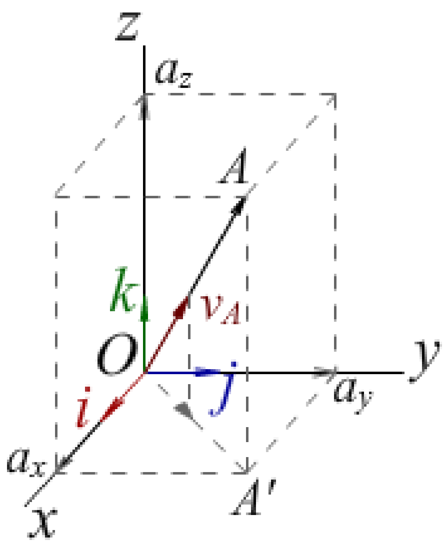

Figure 1: Versors i, j, k.

In a 3-dimensional Euclidean space ($\mathbb{R}^3$) in a Cartesian coordinate system $Oxyz$ the versors of the axes are

$$\begin{cases} x: & \mathbf{i} = (1,0,0), \\ y: & \mathbf{j} = (0,1,0), \\ z: & \mathbf{k} = (0,0,1), \end{cases} \tag{6}$$

as shown in Figure 1. They define the direction of the axes and represent any vector $\overrightarrow{OA}$ or

$$\vec{a} = \vec{a}_x + \vec{a}_y + \vec{a}_z$$

as

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}.$$

Versor of the vector $\overrightarrow{OA}$ has the form

$$\mathbf{v}_A = \cos\alpha_x\mathbf{i} + \cos\alpha_y\mathbf{j} + \cos\alpha_z\mathbf{k},$$

where $\cos\alpha_n = a_n/a$ for $n \in \{x, y, z\}$. The length $a = \|\vec{a}\| = \sqrt{a_x^2 + a_y^2 + a_z^2}$ is also called *magnitude* or tensor of the vector $\vec{a}$. The both projections of the vector $\mathbf{v}_A$ and $\mathbf{A}$ to the plane Oxy lie on the same straight line OA', at Figure 1.

Generally, each versor has the form

$$\mathbf{v} = \cos\alpha + \mathbf{r}\sin\alpha, \quad \mathbf{r}^2 = -1 \tag{7}$$

and represents the rotation for the angle α (in radian) about the axis $\mathbf{r}$. Now let us try to explain.

The above Cartesian system intersect by the plane containing the origin O. In the plane let's define the orthogonal coordinates $Ox'y'$. Thus, we define a complex coordinate plane $\mathbb{C}$. Arbitrary complex number $z' = x' + iy'$ multiply by versor (7):

$$z'\mathbf{v} = (x' + \mathbf{r}y')(\cos\alpha + \mathbf{r}\sin\alpha)$$

$$= (x'\cos\alpha - y'\sin\alpha) + \mathbf{r}(x'\sin\alpha + y'\cos\alpha) = z''.$$

The result is the rotation of complex number z' to $z'' = x'' + \mathbf{r}y''$ about the origin O by the angle α, in the complex plane $\mathbb{C}$:

$$\begin{cases} x'' = x' \cos\alpha - y' \sin\alpha, \\ y'' = x' \sin\alpha + y'' \cos\alpha. \end{cases} \tag{8}$$

Instead of $\mathbf{r}$ we also can use the imaginary unit $i = \sqrt{-1}$. Next let us resume.

A *scalar* is defined as the ratio between the lengths of two parallel vectors. For the given parallel vectors $\vec{a}||\vec{b}$ at Figure 2 left, the scalar s represent the relative length of one vector with respect to the other. It is the quotient between two parallel vectors, $s = \frac{\vec{a}}{\vec{b}}$ or $\vec{b} = s\vec{a}$. The scalar alone is an *operator* that changes the scale of the vector and keeps its orientation unchanged.

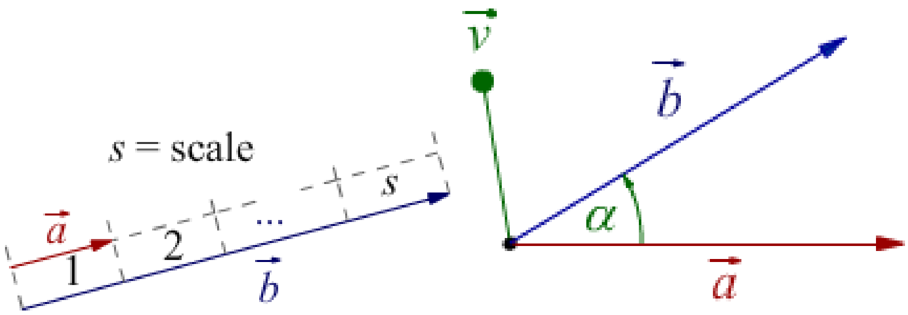

Figure 2: Scalar and versor.

A *versor* is defined as the quotient, $\mathbf{v} = \frac{\vec{a}}{\vec{b}} = e^{i\alpha}$, or $\vec{b} = e^{i\alpha}\vec{a}$, between two non-parallel vectors of equal length. It represents the relative orientation of one vector with respect to the other.

A versor is an operator that changes the orientation of the vector and keeps its length unchanged. A *right versor* is a versor that applies a rotation of $\pi/2$ or $90°$. On the Figure 3, the right versor inverts direction $\vec{b}$ into $\vec{a}$, and the double application of the right versor changes the direction of vector $\vec{b}$ into opposite $-\vec{b}$. The inversion operator -1 inverts the direction of a vector too.

Versor composition is not (generally) commutative. For example, let's see the right versors $\mathbf{v}_x = e^{\frac{i\pi}{2}}$ and $\mathbf{v}_y = e^{\frac{j\pi}{2}}$ on a vector at z-axis:

$$\mathbf{v}_x\mathbf{v}_y(0,0,z) = \mathbf{v}_x(z,0,0) = (z,0,0), \quad \mathbf{v}_y\mathbf{v}_x(0,0,z) = \mathbf{v}_y(0,-z,0) = (0,-z,0).$$

whereas $(z,0,0) \neq (0,-z,0)$, it is $\mathbf{v}_x\mathbf{v}_y \neq \mathbf{v}_y\mathbf{v}_x$. However, the rotations in the same plane are commutative.

The second example is matrices (3), that the quaternions itself are not mutually commutative:

$$\hat{\sigma}_i\hat{\sigma}_j = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

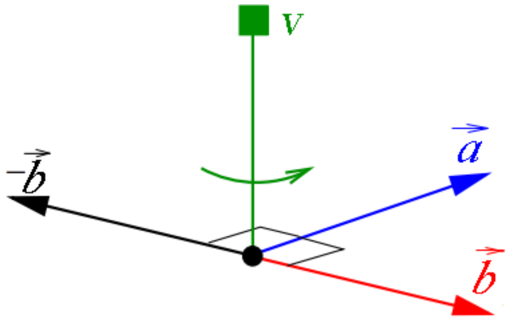

Figure 3: The double right versor.

$$\hat{\sigma}_j\hat{\sigma}_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

because they are anti-commutative.

A *quaternion* is the geometrical quotient of two vectors $\mathbf{q} = \frac{\vec{A}}{\vec{B}}$. A quaternion is an operator that changes the orientation of the vector and changes the length of the vector.

2 Force Field

Using the Pauli matrices Lemma 5, we define a matrix $d\hat{l} = \hat{e}dl$, which is an infinitesimal line element of space in Cartesian coordinates and do the same with the element of time $d\hat{t}$ using quaternions, Lemma 6:

$$\begin{cases} d\hat{l} = \hat{\sigma}_x dx + \hat{\sigma}_y dy + \hat{\sigma}_z dz, \\ id\hat{t} = \hat{\sigma}_i dt_i + \hat{\sigma}_j dt_j + \hat{\sigma}_k dt_k. \end{cases} \tag{9}$$

Because the pairs of matrices in each of the groups are anticommutative, after squaring we found for all four corresponding coefficient of the matrices:

$$\begin{cases} dl^2 = dx^2 + dy^2 + dz^2, \\ -dt^2 = -dt_i^2 - dt_j^2 - dt_k^2. \end{cases} \tag{10}$$

In this way, we obtain an interval ds of 6-dim space-time $ds^2 = dl^2 - c^2dt^2$ or, which is the same as:

$$(ds)^2 = (dl)^2 + (icdt)^2. \tag{11}$$

It is an element of Minkowski space, in compliance with the article [3].

Note that Pauli matrices and quaternions are neither mutually all commutative, nor anti-commutative:

$$\begin{cases} \hat{\sigma}_i\hat{\sigma}_x = -\hat{\sigma}_x\hat{\sigma}_i = -\hat{\sigma}_y, & \hat{\sigma}_i\hat{\sigma}_y = -\hat{\sigma}_y\hat{\sigma}_i = \hat{\sigma}_x, & \hat{\sigma}_i\hat{\sigma}_z = \hat{\sigma}_z\hat{\sigma}_i = i\hat{e}, \\ \hat{\sigma}_j\hat{\sigma}_x = -\hat{\sigma}_x\hat{\sigma}_j = \hat{\sigma}_z, & \hat{\sigma}_j\hat{\sigma}_y = \hat{\sigma}_y\hat{\sigma}_j = i\hat{e}, & \hat{\sigma}_j\hat{\sigma}_z = -\hat{\sigma}_z\hat{\sigma}_j = -\hat{\sigma}_x, \\ \hat{\sigma}_k\hat{\sigma}_x = \hat{\sigma}_x\hat{\sigma}_k = i\hat{e}, & \hat{\sigma}_k\hat{\sigma}_y = -\hat{\sigma}_y\hat{\sigma}_k = -\hat{\sigma}_z & \hat{\sigma}_k\hat{\sigma}_z = -\hat{\sigma}_z\hat{\sigma}_k = \hat{\sigma}_y. \end{cases} \tag{12}$$

That's why we separated space of time coordinates (9).

The second reason for separation (9) to (11) is the nature of the physical fields of (any) force. The force is the one that changes the particle's velocity, and the velocity is a vector. On the other hand, the velocity is a rotation of coordinate pair direction-time.

2.1 Relativity

For example, in Special relativity we have the Lorenz transformation:

$$dl = \cos\psi\, dl - \sin\psi\, icd\bar{t}, \quad icdt = \sin\psi\, d\bar{l} + \cos\psi\, icd\bar{t}, \tag{13}$$

where

$$\sin\psi = \frac{-i\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}, \quad \cos\psi = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}},$$

The system (bar) is moving with constant velocity v in direction dl relative to the reference system, and c is speed of light.

The interval ds is invariant to the Lorenz transformations:

$$ds^2 = dl^2 + (icdt)^2 =$$
$$= (\cos\alpha\, d\bar{l} - \sin\alpha\, icd\bar{t})^2 + (\sin\alpha\, d\bar{l} + \cos\alpha\, icd\bar{t})^2$$
$$= \cos^2\alpha\, d\bar{l}^2 - 2\sin\alpha\cos\alpha\, d\bar{l}\, icdt - \sin^2\alpha\, c^2 d\bar{t}^2 +$$
$$+ \sin^2\alpha\, d\bar{l}^2 + 2\sin\alpha\cos\alpha\, d\bar{l}\, icdt - \cos^2\alpha\, c^2 d\bar{t}^2$$
$$= d\bar{l}^2 + (icd\bar{t})^2.$$

So $ds = d\bar{s}$.

In General Relativity, we have an inertial reference system S_0 somewhere out the gravity field. The observer in it measure the referent length dl_0 and time dt_0. On the other side, in the gravitational field at arbitrary point (event), the corresponding length and time are:

$$dl = dl_0\sqrt{1-\beta^2}, \quad icdt = \frac{icdt_0}{\sqrt{1-\beta^2}}, \tag{14}$$

where $\beta \in (-1, +1)$ is characterized by the strength of the gravitational field at a given point in relation to the reference observer.

According to article [4], it is $\beta^2 = l_s/l$ or $\beta^2 = l/l_u$. In the first case l_s is the Schwarzschild radius of the massive body with dl extending radially ($dl = d\rho$), and in the second case l_u is the radius of the Universe with dl orthogonal to the radius ρ. The mentioned article is working with the metric:

$$(ds)^2 = \left(\frac{dl}{\sqrt{1-\beta^2}}\right)^2 + \left(\sqrt{1-\beta^2}\, icdt\right)^2. \tag{15}$$

By the procedure in the article is easy to find the general transformation:

$$\begin{cases} dl = \lambda\, d\bar{l} - \sqrt{1-\beta^2}\sqrt{1-\lambda^2(1-\beta^2)}\, icd\bar{t}, \\ icdt = \frac{\sqrt{1-\lambda^2(1-\beta^2)}}{\sqrt{1-\beta^2}}\, d\bar{l} + \lambda(1-\beta^2)\, icd\bar{t}, \quad \lambda \in \mathbb{R}. \end{cases} \tag{16}$$

They transform Euclidean metric (11) into non-Euclidean (15), and all other coordinates stay unchanged. Note, at the left side of the second equation (40) or (46) in the article [4] stands cdt instead here's $icdt$.

From the reference system S_0 the same transformation (16) looks simpler:

$$\begin{cases} dl = \lambda\sqrt{1-\beta^2}\, d\bar{l}_0 - \sqrt{1-\lambda^2(1-\beta^2)}\, icd\bar{t}_0, \\ icdt = \sqrt{1-\lambda^2(1-\beta^2)}\, d\bar{l}_0 + \lambda\sqrt{1-\beta^2}\, icd\bar{t}_0. \end{cases} \tag{17}$$

To find them, applies (14). After the substitution:

$$\cos\psi = \lambda\sqrt{1-\beta^2}, \quad \sin\psi = \sqrt{1-\lambda^2(1-\beta^2)}, \tag{18}$$

$$\cos^2\psi + \sin^2\psi = 1, \quad \forall\lambda \in [0, (1-\beta^2)^{-\frac{1}{2}}] \subset \mathbb{R},$$

we find the "rotation":

$$\begin{cases} dl = \cos\psi\, d\bar{l}_0 - \sin\psi\, icd\bar{t}_0, \\ icdt = \sin\psi\, d\bar{l}_0 + \cos\psi\, icd\bar{t}_0. \end{cases} \tag{19}$$

The gravitational field depends only on the parameter $\beta = \beta(l)$, but the rotation (17) defines one more degree of freedom, the parameter $\lambda \in \mathbb{R}$. Why?

The possible answer is in the (hypothetical) 3-dim space of versors. Rotation (19) defines versor perpendicular to the timeline ($icdt_0$) of the viewer from the reference system S_0. However, that versor is not unique. If we assume the versors space has three (time) dimensions, than we have the versors on the circle in the plane perpendicular to the given time axes. Each of them defines by one of rotations (19).

2.2 Inversion

Note, the length and the time are formally symmetric. The first confirmation is the very form of the spherical metrics of Schwarzschild and the Universe, both concise in (15). In the both cases of the metric tensor, the coordinates for length dl and time $icdt$ can be formally replaced.

The second confirmation is in the rotation (17), or the transformation (16). If we chose the parameter $\lambda = 0$, we get $\psi = \frac{\pi}{2}$ or:

$$dl = \sqrt{1-\beta^2}\, icd\bar{t}, \quad icdt = \frac{d\bar{l}}{\sqrt{1-\beta^2}}. \tag{20}$$

The time and space are changing positions and one of the axis directions.

The third, an extreme case, when a material point is moving near the speed of light, it happened such a rotation of the coordinate axes that the timeline (ict) becomes (negative) abscissa (x) and abscissa becomes (positive) timeline. All that in the same plane (see Figure 1 in [3]). The question is whether it is possible the inversion of the axis, that our timeline is abscissa of a particle, and vice versa, our abscissa is its timeline?

Let the particle P moves with uniform velocity v along the x axis, relative to an inertial system. When a clock attached to the particle measure a period of time Δt, then the clock at rest measured period Δt_0, where:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{21}$$

Similarly, consider two extremely fast particles P_1 and P_2, which moves along the x axis by velocities $v_1 < v_2$ close to the speed of light c. They go forward in time at different speeds, so that faster particles have a relatively slower time, due to (21). Their timelines rotates to almost abscissa of the observer in rest, and the faster particle at its timeline lags.

However, the timeline of the relative observer is still the timeline of fast particles. The three spatial dimensions of the observer are still the spatial dimensions of the speedy particles. When the particles collide they change the directions of its movements relative to observer but they stay in the same 3-space. Here arises a theoretical question. Is there such inverse matter, for which timeline is one of the space lines in our 3-dim space, and the spatial dimensions are the three (hypothetical) timelines of which we perceive only one?

In this sense, an inverse system of coordinates would have three time axis $\xi_4 = ict_i$, $\xi_5 = jct_j$, $\xi_6 = kct_k$ ($i, j, k = \sqrt{-1}$) representing space and one timeline $\xi_1 = x$. If the inverse particle of matter is moving by our same geodesic lines ($ds = 0$), then the line element of their 4-dim space of time could be written:

$$ds^2 = dx^2 - c^2 dt_i^2 - c^2 dt_j^2 - c^2 dt_k^2.$$

Consistently, we should expect that the inverse particle apply the same laws of gravity.

Suppose further, that the timeline of such inverse particle is our x-axis, and that it moves through one line in its space, which is our timeline ict. Our abscissa is its time and diversion from that path is not an option. Accordingly, such a particle not at all, or very rarely react with our particles, in a way that could turn its direction of the abscissa.

Due to the impossibility of defining us as the observer relative to the space of such particles, it is logical to assume that the velocity of the particles on our abscissa is equal to the speed of light. The speed of light is the only one which is the same with respect to each of the observer.

Another possibility is that the inverse particle in its space travels so that it does not appear on our timeline at all, or that it cats the timeline only at one moment in one place. In our 4-dim space-time, these particles appeared at a single point in space at one moment, and before and after that moment there is none it. Her appearance is a virtual, or if there is interaction with our matter, it must be consistent with all known laws of physics, as well as conservation laws.

3 Conclusion

By introducing quaternion, the theoretical considerations of space-time assumed a simpler mathematical form. This does not mean the calculations become an elementary, but are removed some asymmetrical concepts of space and time, to formally harmonious way.

Unusually easily fits the story of rotations instead of the Lorentz transformation. The straight line moving, the relativistic translation becomes a formal rotation of the corresponding pairs of coordinates of space-time. It is interesting that each field force can be treated as an action along the (infinitesimal) route, as per the direction of movement, and then as a formal rotation of the pair of coordinates of space-time. Such an idea becomes a simple mathematical method.

It is only open the opportunity to explore the inverse of matter, the inverted spatial and temporal coordinates. It seems that such a theory is formally viable, and that its development is too large for this article.

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