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Posolutely and Absitively: The Alternate Yet Universal Definition of the Absolute Value Function

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Abstract

The absolute value function, also known as the modulus function, is a ubiquitous mathematical function in algebra. The absolute value function, in terms of mathematical mapping, is also simple. The absolute value function takes any real number and maps it to either zero or a positive real number. In practice, the absolute value function measures the distance of a real number from zero, which serves as the point of origin.

However, the existing, traditional definition of the absolute value function is flawed. The flaw is in the derivative or differential for the real absolute value function; the absolute value function has a derivative for all real numbers except at zero. The absolute value function is not differentiable at zero. The derivative for all real numbers except zero is defined and given by a step function.

Yet the real absolute value function is a continuous function that has the global minimum where the derivative does not exist and is not differentiable, and thus is discontinuous at zero. This is the flaw and defect of the traditional absolute value function. While this defect or flaw does not have an impact on the mathematical mapping, the defect does create an inconsistency and irregularity in the mathematical properties of the real absolute value function.

An alternative definition of the absolute value function for real numbers, yet a universal definition resolves this defect and flaw. The alternate definition, yet universal real absolute value function, is both continuous and differentiable at zero. The same mathematical mapping is defined, but with uniquely different mathematical properties from the traditional real absolute value function.

Keywords: Absolute value function, composite function, continuous function, derivative, differential, differentiable function, function composition, modulus function, power function, sign function, sub-function

1. Introduction

The absolute value function is approximately over two hundred years old. The concept of the absolute value was first termed a 'module' for a unit by mathematician Jean-Robert Argand (1768-1822) in 1806; he worked with the accepted definition of a complex number and was describing a complex absolute value.

Karl Weierstrass (1815-1897) introduced in 1841 the vertical bar style of $|x|$ notation for absolute value into mathematics. The word ‘module’ was later as a loanword into English as 'modulus' in 1857, and ‘modulus’ is sometimes still used. The terms absolute value and magnitude refer to the same concept. In that time since, the absolute value function definition has been unchanged, immutable in mathematical definition and properties.

1.1 Absolute Value Function

The absolute value function in mathematics is a conceptually simple mathematical function. The absolute value function maps any real number that is negative, zero, or positive in the domain into the range of the positive and zero, or the non-negative real numbers.

$$\mathbb{R} : abs(x \in \mathbb{R}) \rightarrow \mathbb{R}^{+} \cup 0$$

$$abs(x) \rightarrow \mathbb{R}^{+} \cup 0 \text{ where } x \in \mathbb{R}$$

The absolute value function is important in that it provides a mathematical definition and formalism with mathematical properties so that for any real variable x, where x in R, there is the certainty that |x| for x is a non-negative real or x not in R-.

On the real number line, the absolute value function provides a method for measuring of distance between two points that is always non-negative. A negative distance between two points is illogical and nonsensical as a measure of distance between two points.

An important concept in the mathematics of calculus is the limit. The limit definition [Edwar 1986] is given as:

> *We say that the number L is the limit of F(x) as x approaches a provided that, given any number* $\varepsilon > 0$*, there exists a number* $\delta > 0$ *such that:*
>
> $| F(x) - L | < \varepsilon$
>
> *for all x such that*
>
> $0 < |x - a| < \delta$

Note that the absolute value function is used in the definition of the concept of a limit in calculus.

More simply are many instances where what matters is only the absolute value and not the sign associated with the number such as in statistics for the mean value.

1.1.1 Logical Definition If-Then-Else

One mathematical definition of the absolute value is by the piecewise combination of two functions.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad where\ x \in \mathbb{R}$$

Rewriting the definition for an explicit product by multiplication of the real values of -1 and +1 is:

$$|x| = \begin{cases} +1 \cdot x & \text{if } x \geq 0 \\ -1 \cdot x & \text{if } x < 0 \end{cases} \quad where\ x \in \mathbb{R}$$

The functions are determined in logic by a logical if-then-else statement.

This definition of the absolute value function is in pseudo-code:

```
if x <= 0 then
  return -1 * x;
else
  return +1 * x
end if
```

This definition is sufficient and minimal; it is more logic-based than the mathematically based definition.

1.1.2 Mathematical Definition

Two other possible definitions for formally defining the absolute value function are:

1. power function or x^n where $n, x \in \mathbb{R}$
2. sign function or $sgn(x)$ where $x \in \mathbb{R}$

The sign function is often the focus of an alternative definition for the absolute value function.

1.1.2.1 Mathematical Definition with Power Function

Another definition for the absolute value function uses the square of a real value in variable x, and then the square root of that real value. The definition is the functional composition of two functions, for the real value is an identity function, but the sign is an absolute value function—the non-negative sign of the real value.

Stewart [Stewa 2008d] states, “This is equivalent to the definition above, and may be used as an alternative definition of the absolute value of real numbers.”

$$|x| = \sqrt{x^2}$$

1.1.2.2 Mathematical Definition with Sign Function

Yet another mathematical definition uses the sign or `sgn()` function, which takes a real number and returns the sign of the number.

The absolute value function defined with the sign function is:

$$|x| = sgn(x) \cdot x \ where \ x \in R$$

This definition of the absolute value function leads to a definition of the sign or `sgn()` function. Simplifying the equation the sign function is resolved as:

$$sgn(x) = \frac{|x|}{x} \ where \ x \in \mathbb{R}$$

The fraction or ratio of the absolute value for the real value x, and the real value in variable x can be inverted, or the reciprocal. The reciprocal of the fraction is then:

$$sgn(x) = \frac{x}{|x|} \ where \ x \in \mathbb{R}$$

This definition for the sign function in terms of the absolute value function is defined for all real numbers except when the real value in variable x is the real number zero 0.

1.2 Flaws with Absolute Value Function Definition

The conventional, traditional definition of the absolute value function is flawed. One very obvious flaw is that the definition of the sign function from the absolute value function is not defined or indeterminate when the real value is 0. The absolute value is defined using the sign function; the focus is on the sign of a number, and the sign function with its flaws.

The definition using the square root of the square has a fundamental flaw. The square root is often taken of a number for the positive value. Yet consider this simple mathematical equation:

$$x = \sqrt{1} \ where \ x \in \mathbb{R}$$

The real value for the variable x is often given as +1. Yet this is mathematically “sloppy” in that the real value is actually plus or minus the real value of ±1. The sign of the square root of a positive number is dual-signed—positive and negative.

Hence the definition using the square of the square root, while mathematically minimal, and the identity function with the real value, has an invalid sign in that the resulting sign of the real value is positive and negative, not just positive.

1.2.1 Definition If-Then-Else is Logical

The if-then-else definition of the absolute value function is piecewise joining two functions, one identity, one the product of the real value multiplied by real value negative one, -1.0.

```
if x <= 0 then
  return -1 * x;
else
  return x;
end if
```

This is more computational than the mathematical definition of the absolute value of a number.

1.2.2 Mathematical Definition Indeterminate at Zero

The mathematical definition using the sign function, the sign function has the indeterminate or undefined value of 0/0 for the sign of the real value.

For the absolute value function defined with the sign function, consider the sign function at x as the real number zero 0.

$$sgn(x) = \frac{x}{|x|} \; where \; x = 0$$

Substituting the real number zero 0 for the real variable x is:

$$sgn(0) = \frac{0}{|0|}$$

Solving for the equation, the result is:

$$sgn(x) = \frac{x}{|x|} = \frac{0}{|0|} = \frac{0}{0} = indeterminate$$

The result is meaningless for the absolute value function at zero.

1.2.3 Mathematical Definition At Zero

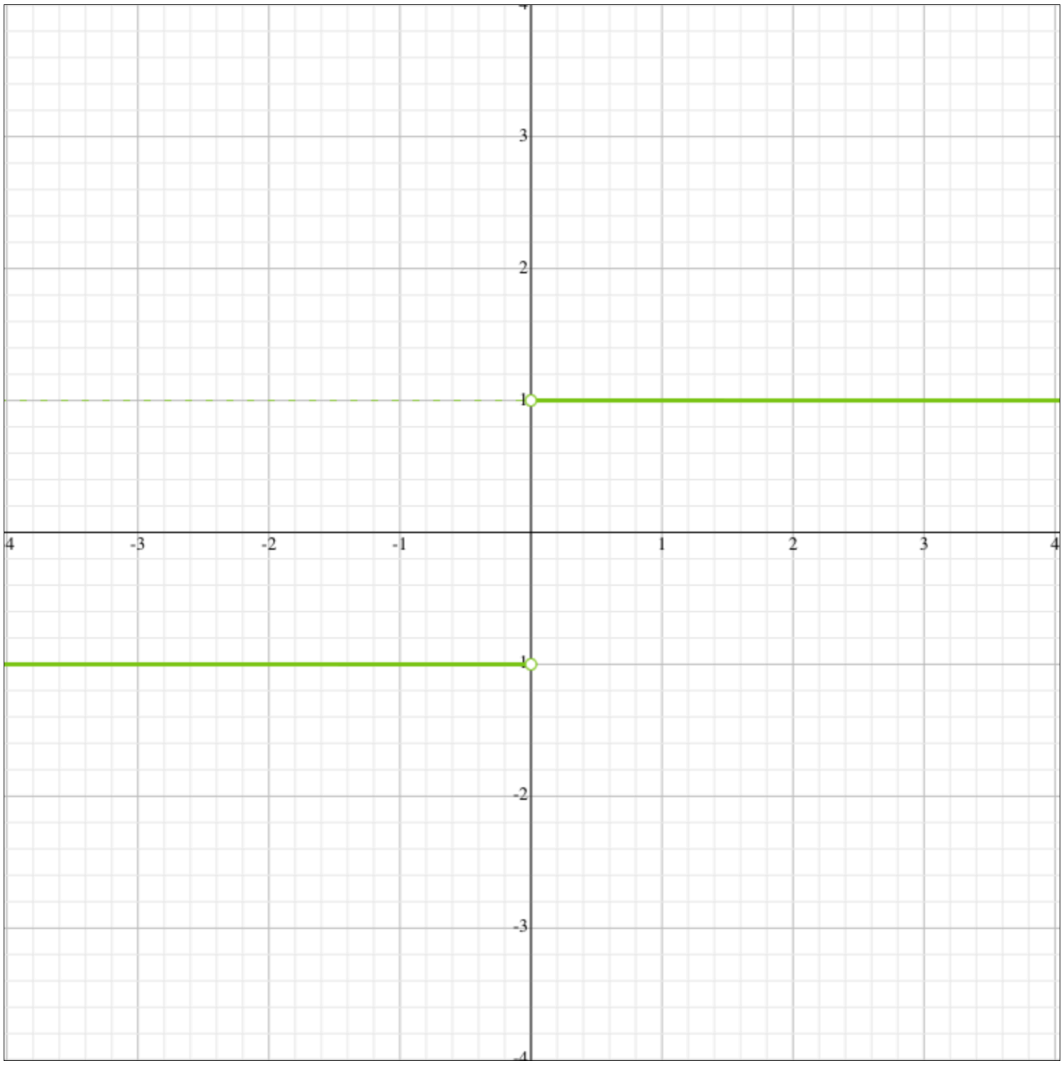
For the mathematical definition of the absolute value function using the sign function, the absolute value function is continuous. Yet the absolute function defined by the sign function is differentiable, but not at the real value is real number zero 0.0.

The derivative of the sign function definition of the absolute value function, is given as:

$$\frac{d}{dx}|x| = \frac{x}{|x|} \; where \; x \in \mathbb{R}$$

For both formal definitions of the derivative, there is a discontinuity at zero 0.

This is a major flaw in this definition of the absolute value function defined using the sign function. The graph of the derivative of the absolute value function illustrates the discontinuity at zero 0.



Graph of Absolute Value Function with Discontinuity at Zero

Thus the variations of the existing definition of the absolute value function will work but have mathematical flaws.

1.3 Alternate Definition for Absolute Value

The accepted, existing definition is not the only possible definition for the absolute value function. There are alternatives by Gilreath, one is the cogent value function.

The cogent value function [Gilre 2018] is an alternate definition of the absolute value function. The cogent value function utilizes two other functions the parabolin and the magnum function to define the absolute value function as the cogent value function.

The image and range of the cogent value function are the same as the absolute value function. The mathematical mapping for the f_{abs} function is:

$$f_{abs} : \mathrm{R} \rightarrow \{-1,0,+1\}$$

All real numbers R are mapped into the set containing the integer numbers $\{-1, 0, +1\}$.

These alternative definitions achieve the same mathematical mapping or the same mathematical result, yet do it via different mathematical functions. With different mathematical properties with the alternative definition from the alternative definition of the absolute value function, some of the problems that arise are avoided completely.

2. Triple Power Quad or TPQ Function

The alternate definition of the absolute value function is termed the “Triple Power Quad Function” or "TriPowQua" or TPQ function. The name of this alternative definition of the absolute value function reflects the definition of the function mathematically.

2.1 Function Composition Property

The formal definition of the alternate definition of the absolute value function, the “TPQ Function” for (triple-power-quad), uses the mathematical concept and property of function composition.

One definition of a mathematical function by Serge [Serge 1971] is:

> *A function defined for all numbers, is an association which to each number associates another number.*

Function composition is creating or building a mathematical function from other mathematical functions. One definition is given by Rosen [Rosen 2012b] who defines function composition as:

> *Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted for all* $a \in A$ *by* $f \circ g$*, is the function from A to C defined by* $(f \circ g)(a) = f(g(a))$.

Function composition is a means of associating values in the domains and ranges of each function, thus creating an association by the composition of the functions from one number to another number.

Function composition allows more complex function definitions in other, simpler functions. The whole function is the composition of its parts or sub-functions.

2.2 Power Function

The TPQ function uses only one function, the power function.

The Encyclopedia of Mathematics defines the power function [Encyc 2020] as:

> *A function f:x ➝-> y with* $y = x^a$ *where a is a constant number.*

The power function `pow()` is a generalization of taking a real number variable and raising it to an integer power or a real fractional power.

The power function pow with an integer value is the same as raising of a real variable x to the power of n or x^2 as:

$$pow(x,n) = x^n \text{ where } x,n \in \mathrm{R}$$

The power function with a fractional value is the same as taking the root of a real variable x to the n root, or $\sqrt[n]{x}$ as:

$$pow(x,\frac{1}{n}) \equiv \sqrt[n]{x} \text{ where } x \in \mathrm{R} \ \wedge n \in \mathrm{Z}^{+}$$

The power function is a generalization of either raising the real variable x to a positive integer power +n as $\frac{n}{1}$ or to a negative integer -n power fractional power $\frac{1}{n}$, where in both cases n is a positive integer.

2.3 Quad Parameters

The “quad” in the TPQ function is from the parameter used, variations of the integer value of integer 2. As the largest parameter is an integer 4, the word “quad” is used.

2.4 TPQ Function Definition

The triple in the TPQ function definition is from the function composition of the power function, nested three times with different integer parameters. The definition of the TPQ function is with pow power function is:

$$tpq(x) = pow(pow(pow(x,2)\frac{1}{4})2) \text{ where } x \in \mathbb{R}$$

The TPQ function is the alternate definition of the absolute value function. In mathematical notation, the TPQ definition is:

$$tpq(x) = \sqrt[4]{x^2}^{\,2} \text{ where } x \in \mathbb{R}$$

By simply using the function composition of the power function, the TPQ function has the same mathematical result as the conventional absolute value function. Yet the TPQ function is without any of the mathematical flaws of any absolute value definition.

3. Analysis by Cases of Real Numbers

The TPQ function is an alternate mathematical definition of the absolute value function. Thus, the mathematical mapping of the TPQ function is defined as:

$$TPQ : \mathbb{R} \rightarrow \{0, +1\}$$

The TPQ function maps the value of the real number to itself; thus the mapping of the value is the identity function [Rosen 2012a] in transforming the value to itself. The TPQ function maps the sign of the value of the real number to non-negative, zero 0, or positive.

Thus two transformations are done by the TPQ function, one is the identity function for the magnitude of the value, and the other is the sign of the value which is transformed into to non-negative form.

3.1 Two Elements in Analysis

The two transformations of a real number value mapped by the TPQ function require a real number to be analyzed for those two transformations.

3.1.1 Elements of Real Numbers

For analysis, two elements in a real number are:

- Sign - the sign of a real number as *c* is positive, c=+1, negative, c=-1, or zero, c=0.
- Magnitude - the value of *m* or the magnitude of a real number.

These two elements are a product together, the sign multiplied by a value or magnitude as a real number. The contrast is that the sign is mutually exclusive of the value or magnitude.

3.1.1 Number as Product of Sign and Value

The sign of a number is an integer constant *c* that can be four forms of 0, +1, -1, or ±1. These forms are then multiplied times the real value magnitude m. Thus a real number is $c \cdot m$ as: $0 \cdot m$, $+1 \cdot m$, $-1 \cdot m$, $\pm 1 \cdot m$.

For example, the real number *e* or Euler’s constant sign is +1 the magnitude is +2.71828 and the form is $+1 \cdot 2.71828$.

Using this form of a number as the product of the constant sign times the value or magnitude is used in the analysis of each power function, and the composition of the functions together.

3.1.3 Composition of Power Function

The composition of the three functions with the real number of the form $c \cdot m$ is the transformation from the domain to the range, for each function, and the composite of the functions.

3.1.3.1 Inner Power Function

For the inner function, the power function x^2 of the transformation is, using the form of a real number:

$$F(x^2) : \{0, +1, -1, \pm 1\} \cdot m^2 \rightarrow \{0, +1\} \cdot m^2 = \left\{0, +1 \cdot m^2\right\} \text{ where } m \in \mathrm{R}$$

For any sign 0, +1, -1, ±1 of a real number, the result is a non-negative real number 0, or a positive real number.

The magnitude is squared from m^1 to m^2.

3.1.3.2 Middle Power Function

$$F(x^{1/4}) : \{0, +1\} \cdot m^2 \rightarrow \{0, \pm 1\} \cdot m^{1/2} = \left\{0, \pm 1 \cdot m^{1/2}\right\} \text{ where } m \in \mathrm{R}$$

For the 0 or +1 sign of a real number, the result is zero, or positive-negative. The magnitude is the square root of the magnitude m as m^2 to $m^{1/2}$.

3.1.3.3 Outer Power Function

$$F(x^2) : \{0, \pm 1\} \cdot m^{1/2} \rightarrow \{0, +1\} \cdot m^{1/1} = \left\{0, +1 \cdot m^2\right\} \text{ where } m \in \mathrm{R}$$

For the 0 or ±1 sign of a real number, the result is non-negative, zero, or positive. The magnitude is the square root of the square or the original magnitude of the real number m.

3.2 Case of Real Number Zero

For the case of the real number zero 0, the magnitude m = 0, and the sign is zero 0, or c = 0.

Applying the power function for each step in the TPQ function:

1. $F(x^{2/1})$: $(c=0 \cdot m=0) \equiv F(0^{2/1}) \rightarrow \{ c=0 \cdot m=0 \equiv 0 \}$
2. $F(x^{1/4})$: $(c=0 \cdot m=0) \equiv F(0^{1/4}) \rightarrow \{ c=0 \cdot m=0 \equiv 0 \}$
3. $F(x^{2/1})$: $(c=0 \cdot m=0) \equiv F(0^{2/1}) \rightarrow \{ c=0 \cdot m=0 \equiv 0 \}$

At each step of the TPQ function, the sign c = 0 remains unchanged as integer zero 0, and the magnitude m = 0 remains unchanged.

Thus the absolute value calculated using the TPQ function is:

1. abs $(c=0 \cdot m=0) \equiv$ TPQ$(c=0 \cdot m=0) \rightarrow \{ c=0 \cdot m=0 \equiv 0 \}$
2. abs(0) $\equiv$ TPQ(0) $\equiv 0$

The sign c = 0 is transformed into zero 0, for each step of the TPQ function, and the final sign is zero 0.

The value or magnitude m = 0 is transformed into zero 0, for each step of the TPQ function, and the final magnitude m = 0 for a value that is zero 0.

Thus the integer value zero 0, or real value zero 0.0, is correct in that the absolute value is zero 0.

3.3 Case of Non-Zero Real Numbers

For the case of non-zero real numbers, the two cases for the number, the magnitude is m = 1.0 with the sign c = { -1, +1 } for each case. The magnitude 1.0 is immutable, but the sign c is transformed, for -1 into +1, and for +1 into +1.

3.3.1 Positive Real Number

For the case of the real number non-zero m > 0, the magnitude m > 0, and the sign is the real number one 1.0, or c = 1.0. For the real number m, the value is 1.0 or m = 1.0.

Applying the power function for each step in the TPQ function:

1. $F(x^{2/1})$: $(c=+1 \cdot m=1.0) \equiv F(+1.0^{2/1}) \rightarrow \{ c=+1 \cdot m=1.0 \equiv +1.0 \}$
2. $F(x^{1/4})$: $(c=+1 \cdot m=1.0) \equiv F(+1.0^{1/4}) \rightarrow \{ c=+1 \cdot m=1.0 \equiv +1.0 \}$
3. $F(x^{2/1})$: $(c=+1 \cdot m=1.0) \equiv F(+1.0^{2/1}) \rightarrow \{ c=+1 \cdot m=1.0 \equiv +1.0 \}$

At each step of the TPQ function, the sign c = +1 remains unchanged as integer one +1, and the magnitude m = 1.0 remains unchanged.

Thus the absolute value calculated using the TPQ function is:

1. abs (c=+1 · m=1.0) ≡ TPQ(c=+1 · m=1.0) → { c=+1 · m=1.0 ≡ +1.0 }
2. abs(+1) ≡ TPQ(+1) ≡ +1

The sign c = +1 is transformed into a positive integer one +1, for each step of the TPQ function, and the final sign is positive one +1.

The value or magnitude m = 1.0 is transformed into a real number 1.0, for each step of the TPQ function, and the final magnitude m = 1.0 for a value that is one 1.0.

Thus the integer value one +1, or real value one +1.0, is correct in that the absolute value is positive one +1.

3.3.2 Negative Real Number

For the case of the real number non-zero m < 0, the magnitude is m < 0, and the sign is the real number -1.0, or c = -1.0. For the real number m, the value is -1.0 or m = -1.0.

Applying the power function for each step in the TPQ function:

1. F($x^{2/1}$): (c=-1 · m=1.0) ≡ F($-1.0^{2/1}$) → { c=+1 · m=1.0 ≡ +1.0 }
2. F($x^{1/4}$): (c=+1 · m=1.0) ≡ F($+1.0^{1/4}$) → { c=+1 · m=1.0 ≡ +1.0 }
3. F($x^{2/1}$): (c=+1 · m=1.0) ≡ F($+1.0^{2/1}$) → { c=+1 · m=1.0 ≡ +1.0 }

At each step of the TPQ function, the sign c = -1 is transformed as integer one +1, and the magnitude m = 1.0 remains unchanged.

Thus the absolute value calculated using the TPQ function is:

1. abs (c=-1 · m=0) ≡ TPQ(c=-1 · m=1.0) → { c=+1 · m=1.0 ≡ +1.0 }
2. abs(-1) ≡ TPQ(-1) ≡ +1

The sign c = -1 is transformed into a positive integer one +1, for each step of the TPQ function, and the final sign is positive one +1.

The value or magnitude m = -1.0 is transformed into a positive one, +1.0, so is unchanged, for each step of the TPQ function, and the final magnitude m = 1.0 for a value that is one 1.0.

Thus the integer value one -1, or real value one -1.0, is correct in that the absolute value is positive one +1.

3.4 Examples in Practice

The three examples of the TPQ function are from the Law of Trichotomy [Burn 2015] as stated:

> *"If a is a number, then either a = 0, or a is positive or -a is positive, and only one of these is true. When -a is positive, a is said to be negative. This property is called the trichotomy law because of the three possibilities."*

Thus, there are three illustrative examples in practice, which are for zero, positive, and negative real numbers.

3.4.1 Zero Real Number

The real number 0.0 has the sign $c = 0$, and the magnitude or value $m = 0.0$. The form of the real number zero 0.0 is the product $0 \cdot 0.0$.

As illustrated before, the absolute value for integer value 0 or real value 0.0 is zero 0.

3.4.2 Positive Real Number

The positive odd prime, real number, positive seven +7.0 has the sign $c = +1$, and the magnitude or value $m = 7.0$. The form of the real number seven 7.0 is the product of $+7 \equiv +1 \cdot 7.0$.

The sign $c = +1$ is transformed by the three functions in the TPQ function:

$$\mathrm{TPQ}(+1) \rightarrow (+1)^2 \rightarrow (+1)^{1/4} \rightarrow (+1)^2 \equiv +1 \text{ or } \mathrm{TPQ}(+1) = +1$$

The magnitude of the real number $m = 7.0$ is transformed by the three functions in the TPQ function:

$$\mathrm{TPQ}(7.0) \rightarrow (7.0)^2 \rightarrow (49.0)^{1/4} \rightarrow (49.0^{1/4})^2 \equiv (49.0)^2 = 7 \text{ or } \mathrm{TPQ}(7) = 7$$

Together positive real number +7.0 with the sign and magnitude is:

$$\mathrm{TPQ}(+7.0) \rightarrow (+7.0)^2 \rightarrow (+49.0)^{1/4} \rightarrow (+49.0^{1/4})^2 \equiv (+49.0)1/^{2} = +7 \text{ or } \mathrm{TPQ}(+7) = +7$$

3.4.3 Negative Real Number

The negative even number, real number negative four -4.0, as the sign c = -1, and the magnitude or value m = 4.0. The form of the real number negative four -4.0 is the product as $-4 \equiv -1 \cdot 4.0$.

The sign c = -1 is transformed by the three functions in the TPQ function:

$$\text{TPQ}(-1) \rightarrow (-1)^2 \rightarrow (+1)^{1/4} \rightarrow (+1)^2 \equiv +1 \text{ or } \text{TPQ}(-1) = +1$$

The magnitude of the real number m = 4.0 is transformed by the three functions in the TPQ function:

$$\text{TPQ}(4.0) \rightarrow (4.0)^2 \rightarrow (16.0)^{1/4} \rightarrow (16.0^{1/4})^2 \equiv (4.0)^{1/2} = 4.0 \text{ or } \text{TPQ}(4.0) = 4$$

Together positive real number +7.0 with the sign and magnitude is:

$$\text{TPQ}(-4.0) \rightarrow (-4.0)^2 \rightarrow (+16.0)^{1/4} \rightarrow (+16.0^{1/4})^2 \equiv (+16.0)^{1/2} = +4.0 \text{ or } \text{TPQ}(-4.0) = +4.0$$

3.5 Table Illustrating TPQ with Power Functions

Table 3.5 of integers and the power function with different powers, and the composite of the TPQ function is given to illustrate the definition of the TPQ function.

X	X^2	X^1/4	X^2	TPQ
-16	+256	+4	+16	+16
-4	+16	+2	+4	+4
-1	+1	+1	+1	1
0	0	0	0	0
+1	+1	+1	+1	1
+4	+16	+2	+4	+4
+16	+256	+4	+16	+16

Table of Integers with Power Functions for TPQ Function

The table contains a set X of integers where X ⊆ Z for the integer value of x, where x ∈ X, where the set X = { -16, -4, -1, 0, +1, +4, +16 }. These integers, except for zero, are powers of two to keep each power function for each step in the table Table 3.5 of the TPQ function simple.

The value of x is then shown across each row for each step in the power function, and then the overall result of the TPQ function is given. Each step is the power function with the real values of +2, +1/4, and +2 that result in the final value of the TPQ function. The TPQ function is the mapping from the column of the integer value for x, and then to the column of the TPQ value. The mapping is x ∈ Z, TPQ: x → { 0, +1 } or a zero or positive integer value.

4. Derivative of Absolute Value Functions

The derivative of the absolute value function demonstrates the mathematical nature of the absolute value function in the traditional mathematical definition and the TPQ mathematical function.

4.1 Derivative of the Traditional Absolute Value Function

The derivative of the traditional absolute value function $|x|$ is:

$$\frac{d}{dx}|x| = \frac{x}{|x|}$$

The problem with the derivative of the traditional absolute function $|x|$ is that $x = 0$the slope of the line is $\frac{0}{0}$ indeterminate, or the answer of “no answer” is the result. This is the inflection point in the graph of the traditional absolute value function.

4.2 Derivative of If-Then-Else Absolute Value Function

The derivative of the If-Then-Else definition of the absolute value function is the derivative of each sub-function in the piecewise function.

The two sub-functions are:

1. $f_{ge0}(x) = +1 \cdot x$ — the case for greater than or equal to zero
2. $f_{lt0}(x) = -1 \cdot x$ — the case for less than zero

The derivatives of the two sub-functions are:

$$\frac{d}{dx} f_{ge0}(x) = \frac{d}{dx} + 1 \cdot x = + 1$$

$$\frac{d}{dx} f_{lt0}(x) = \frac{d}{dx} - 1 \cdot x = - 1$$

Both piecewise functions have the same derivative, the constant function that is the real number one ±1.0.

4.3 Derivative of Two Sub-Functions of the TPQ Function

The derivative of the TPQ function is the derivative of the piecewise functions that form the function through functional composition.

4.3.1 Derivative of Power Function Parameter Real Number 2.0

The derivative for the power function at real number value 2.0, or simply the derivative with respect to real variable x for the function x^2, is:

$$\frac{d}{dx} x^2 = 2 \cdot x$$

The derivative is the product of 2 times the real number x.

4.3.2 Derivative of Power Function Parameter Real Number 1/4

The derivative for the power function at real number value 1/4 or 0.25, or simply the derivative concerning the real variable x for the function, $x^{\frac{1}{4}}$ is:

$$\frac{d}{dx} x^{\frac{1}{4}} = \frac{1}{4 \cdot x^{\frac{3}{4}}}$$

The derivative is the quotient of real value 1.0 divided by real value 4.0 multiplied by the real variable x for the function $x^{\frac{3}{4}}$.

4.3.3 Derivative of Composite of Power Functions

The composite of the two functions is the real variable x to the power of the real number 1/2 or 0.5, or more simply, the square root of the real variable x is then:

$$x^{\frac{1}{2}}$$

Thus, the derivative is the functional composition of both functions $x^{\frac{1}{2}}$:

$$\frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2 \cdot x^{\frac{1}{2}}}$$

The derivative [Stewa 2008c] is the composite, which is the quotient of the real value 1.0 divided by the square root of the real variable x, multiplied by the real value 2.0.

4.4 Derivative of TPQ Function

The overall derivative of the TPQ function is the derivative of the composition of the three forms of the power function.

Starting with the TPQ function equation:

$$y = \left(x^2\right)^{\frac{1}{4}^{2}}$$

Determine the derivative of the TPQ function that is:

$$\frac{d}{dx}\left(y = \left(x^2\right)^{\frac{1}{4}^{2}}\right)$$

The derivative is calculated [Symbo 2025] in a four-step process:

Step 1. Simplify the equation:

$$= \frac{d}{dx}\left(\left(x^{2}\right)^{\frac{1}{2}}\right)$$

Step 2. Apply the Chain Rule:

$$= \frac{1}{2\left(x^{2}\right)^{\frac{1}{2}}}\frac{d}{dx}\left(x^{2}\right)$$

Step 3. Simplify the Result

$$= \frac{1}{2\left(x^{2}\right)^{\frac{1}{2}}} \cdot 2x$$

Step 4: Further Simplification:

$$= \frac{2x}{2\left(x^{1}\right)} = \frac{2x}{2x} = \frac{1}{1} = 1$$

The result is the derivative:

$$\frac{d}{dx}\left(y = \left(x^{2}\right)^{{\frac{1}{4}}^{2}}\right) = 1$$

Thus, the derivative of the TPQ function f(x), or f'(x) = 1. The derivative of the TPQ function is:

$$\frac{d}{dx}x^{2{\frac{1}{4}}^{2}} = 1$$

Simplifying the function composition, the derivative is:

$$\frac{d}{dx}x^1 = 1$$

The derivative of the TPQ function is the constant function one or a real number value +1.0.

4.5 Limit Approaching Zero from Left and Right

The TPQ function is continuous and differentiable at a real number value of zero 0.0.

The proof is that the derivative of the TPQ function converges to the same real number value from both the left and right. Stewart [2008b] gives this formal definition of a function f is continuous. Thus, the limit converges at the real number value, and therefore, the TPQ function is differentiable at zero.

4.5.1 TPQ Function and Derivative

The derivative of the TPQ function f(x) with the definition is:

$$f(x) \rightarrow \{0, +1\} \text{ where } x \in \mathbf{R}$$

Where x is a real number value, then the derivative f'(x) is:

$$f'(x) = 1.$$

For the TPQ function to be differentiable at real value zero 0, both limits approaching real value zero 0 from the left and right must converge to the same real value at zero 0.

4.5.2 Limit Approaching Zero from Left

The limit approaching zero from the left, where x < 0, is:

$$\lim_{x \to 0^-} f'(x) = 1 \text{ where } x \in \mathbf{R}$$

The limit approaching zero from the left is the real number +1.0.

4.5.3 Limit Approaching Zero from Right

The limit approaching zero from the left, where x > 0, is:

$$\lim_{x\to 0^+} f'(x) = 1 \text{ where } x \in \mathbf{R}$$

The limit approaching zero from the right is the real number +1.0.

4.5.4 Limit Approaching Zero from Left and Right Converges

The limit approach of zero from both the left and the right for the derivative of the TPQ function converges to +1.0. Stewart [Stewa 2008a] gives the formal theorem and states: "...a two-sided limit exists if and only if both of the one-sided limits exist and are equal."

Thus, the overall limit approaches the real number value +1.0 for the derivative TPQ function. Hence, the limit is equivalent, and thus the TPQ function at the real number value 0.0 is continuous.

5. Conclusion

The absolute value function as defined and taught, is a mathematically flawed definition of a real value mathematical function that is over two centuries old. The absolute value function has remained unchanged and dormant in terms of mathematical consideration and analysis since the mid-nineteenth century.

5.1 Properties of Triple Power Quad Function

The TPQ function defines the absolute value using the composition of the power function, with different parameters on the variable containing the real value. This triple composition of the power function defines the TPQ function.

The properties of the TPQ function are:

1. Continuous for the entire real number line or set of real numbers.
2. Differentiable at all real number values on the real number line.
3. The derivative of the TPQ function is the real number value 1.0.

5.2 Contradistinction of TPQ and Absolute Value Function

The two primary distinctions in the definition of the absolute value are between the traditional definition of the absolute value and the TPQ function.

These two contradistinctions are:

1. Continuous and differentiable at all real values on the number line including the real value 0. The derivative of the TPQ function is a simple real number value of 1.0 for all real values along the real number line.
2. The TPQ function uses a composition of the power function with different parameters whereas the traditional definition of the absolute value function is by if-then-else or with the product of the sign of a number function of the real value with the real value.

An obvious contradistinction is that the TPQ function has a constant value of real number value 1 for the derivative whereas the absolute value has an inconsistent derivative.

5.3 Consider Both the Sign and Magnitude of a Real Number

For the mathematical definition of the absolute value, either the traditional definition of the product of the sign of a real value multiplied by the real value, where x is the real number value. In the TPQ function, both the sign of a real value and the magnitude of the real value are significant. In the formal mathematical non-logical definition, the emphasis is on the sign of the real number value.

The TPQ function focuses on both the sign of a real number value and the magnitude of the real number value. The composition of the three power functions for the sign is always a positive sign of a real number value, and the magnitude of the real number value is the identity function.

5.4 Summary

The absolute value function is a fundamental, foundational function in existing mathematics. Yet the absolute value definition, properties, and functions used and known are from the nineteenth century. This definition has known mathematical flaws and defects, but has remained unresolved and unconsidered. The absolute value function as it exists in the domain of mathematical knowledge and thought is “old knowledge” that is foundational but never reconsidered.

The TPQ function gives a consistent and mathematically elegant definition of the absolute value function. The same mathematical result is achieved, but the TPQ function has none of the inconsistencies and flaws of the traditional absolute value function. By reconsidering the “old knowledge” of mathematics in the form of the absolute value function, a better mathematical definition is formulated for a better formal mathematical foundation and simpler, elegant result.

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