the general science journal
February 2025

Four AIs Review a Dissenting Paper on the Actual Infinity

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Abstract.-This article comments on and discusses the opinions of four artificial intelligences (DeepSeek v3, ChatGPT o3-mini, Grok 3, and Gemini 2) regarding a published article by the author that challenges contemporary mathematical infinitism. None of them found a flaw in the proofs, yet none fully accepted the article's content, highlighting their adherence to the hegemonic current of infinitist mathematical thought.

Keywords: Artificial inteligence, actual infinity, ω-order, dense order, Axiom of Infinity, spacetime. continuum.
Opinion.-The opinion of the 4 participants AIs on this article are included at the end of this article.
Note: The article in question is included at the end of this article. It will be part of a forthcoming book on the scientific foundations of theism.

1 Introduction

This introductory section of the article is the only one written by the author. Each of the following sections has been written by a different AI, except for the left-indented texts preceded by the word Comment followed by a number, both underlined, which have also been written by the author. This introduction outlines the main objectives of an article as unusual as this one, in which four artificial intelligences (AIs) confront the content of another article by the author: The Axiom of Infinity is Inconsistent [6, Link], included at the end of this one. An article dissenting from the infinitism that has dominated mathematics for over 120 years, henceforth referred to as article$^+$. The five main objectives of this other article are as follows:

1.- Verify that the infinitist current is absolutely hegemonic in contemporary mathematics and that the four participating AIs have been trained in such a way that they struggle to deviate from these dominant currents of thought in contemporary science.

2.- Assess the argumentative capacity of the AIs when faced with a text dissenting from a mainstream current of contemporary scientific thought.

3.- Check whether any of them detects an error in the proofs included in article$^+$.

4.- Offer the reader the opportunity to analyze the debate between the author of a dissenting scientific article$^+$ and four non-dissenting AIs.

5.- Gain a deep understanding of a highly significant article$^+$ on the foundations of mathematics through the debate between its author and four AIs trained in adherence to dominant scientific currents.

The reader will observe that none of the four AIs participating in the revision was able to detect a single error in the content of article$^+$ (although it could obviously contain errors). Nor do any of them explicitly acknowledge this fact. It will also be noted that, in defense of the dominant infinitist orthodoxy, some of the participating AIs literally lie, claiming that article$^+$ says things it does not say or uses arguments it does not use. In this, they do seem human.

As will be seen, not all AIs are familiar with the fact that definitions, proofs, and other processes involving infinitely many steps are common in mathematics. Usually, these steps are performed for a few initial elements, and ellipses, or expressions like "and so on," extend them to all others. The extension is typically very simple and easy to understand, as in the case of the definition of the set $\mathbb{Q}_{01}$, or the comparisons* of Theorem 5 in article$^+$. This is true in thousands of other cases in contemporary mathematics. For example, in the following recursive definition of the sequence $\langle S_i \rangle$:

$$\forall n \in \mathbb{N} \begin{cases} n = 1 : S_n = n \\ n > 1 : S_n = S_{(n-1)} + 0.5 \end{cases} \tag{1}$$

which gives rise to the infinite sequence $\langle S_i \rangle = 1,\ 1.5,\ 2,\ 2.5,\ 3,\ 3.5\ldots$ It is universally assumed that the legitimacy of a definition, or the consistency of a proof, does not depend on whether its infinitely many steps can actually be carried out in physical or temporal terms. Contemporary mathematics would be impossible without this assumption.

There exists a theory, supertask theory, which considers and deems valid the possibility of performing infinitely many tasks in finite time. Supertasks are not paradoxes, as one of the AIs claims, but consequences of assuming the existence of infinite sets. Many authors have worked and continue to work on supertask theory, even writing articles about the possibilities of actually executing them, involving classical mechanics or making use of relativity theory [8, 3, 2, 7, 4]. None of the definitions or proofs in article$^+$ use supertasks; they are only suggested as a possibility in part of one proof. It is merely a suggestion, an incidental detail of article$^+$ that two of the AIs seem not to have understood.

It seems appropriate to also highlight that none of the four AIs considers that the existence of a single contradiction in a theory renders it inconsistent and formally invalidates it. Thus, it must be rejected or reformulated with new foundations. This must be the case, no matter how long that theory has been considered consistent. In the case of infinitist mathematics and the successes of physics that rely on it, it is worth remembering that such use is invariably finitist and discrete [5]. Consider, for example, that some fundamental physical constants (such as those involving irrational numbers like π) should contain an infinite number of decimal places, yet in practice, only a very small, finite number of such decimals are known and used.

2 Review of DeepSeek v3

Article$^+$ presents a bold argument against the consistency of the Axiom of Infinity and actual infinity, based on the structure of the rational interval (0,1). However, its approach contains several fundamental weaknesses from the perspective of standard set theory (ZFC) and accepted mathematical philosophy.

Comment 1: Establishing standard set theory and currently accepted mathematics as definitive is unscientific. All scientific theories must be falsifiable. Article$^+$ is a necessary exercise in falsifying the Axiom of Infinity.

1 Confusion between potential and actual infinity:
The author assumes that actual infinity requires the possibility of "completing" infinite processes, such as comparing all elements of a denumerable set in a finite sequence (supertask). This mixes operational concepts (processes in time) with the abstract existence of sets. In ZFC, infinite sets exist as static totalities, without the need for step-by-step construction, thus avoiding the supertask paradox.

Comment 2: Article$^+$ explains and defines the differences between actual and potential infinity, which is unusual in contemporary mathematical literature, where the only infinity is actual infinity. Potential infinity disappeared from that literature long ago. None of the definitions or proofs mix temporal processes with the abstract existence of infinite sets.

2 Misinterpretation of dense sets:
The central argument about the nonexistence of a minimum in $(0, 1)$ ignores that, in a dense set, the absence of a first element is an intrinsic property and not a contradiction. The bijection with $\mathbb{N}$ used in the proof does not preserve the natural order of the rationals, so the proposed reordering does not imply the existence of a minimum in the sense of the original dense order.

Comment 3: It is clear that it has not understood the proof of Theorem 5. Article$^+$ does not consider the existence of the smallest rational greater than zero to be a contradiction. It proves that a certain sequence of redefinitions of a rational variable ends with that rational variable defined as the impossible smallest rational in the interval $(0, 1)$, see Comment 5. The existence of ordered and complete totalities without a last element to complete them (or a first element to begin them) is not a proven conclusion independent of the Axiom of Infinity. It is an existence assumed and legitimized by the Axiom of Infinity. Its existence is, therefore, as debatable as that of the Axiom of Infinity itself.

3. Critique of the Axiom of Infinity:
The author equates the inconsistency of an infinite process (such as a supertask) with the logical inconsistency of the axiom. However, the Axiom of Infinity in ZFC does not depend on temporal processes or operational completeness but on the acceptance of infinite sets as abstract entities. His "proof" does not refute this but questions a specific (and non-standard) interpretation of infinity.

Comment 4: Nowhere in article$^+$ is the inconsistency of an infinite process equated with the inconsistency of an axiom. The inconsistency of denumerable sets is proven. From that inconsistency, other results are formally proven, including the inconsistency of the Axiom of Infinity. It does not question a specific, non-standard interpretation of infinity; the only infinity in article$^+$ is actual infinity (Definitions 2-4), which is the infinity of the Axiom of Infinity (Theorem 1 of the Axiom of Infinity).

4. Questionable generalization:
Extending the inconsistency to all infinite sets, the spacetime continuum ($\mathbb{R}^4$), and even infinite divisibility lacks rigor. For example, the inconsistency of $\mathbb{R}$ is derived from its supposed dependence on the Axiom of Infinity, but in ZFC, $\mathbb{R}$ is constructed without known contradictions, and its cardinality does not depend on realizable infinite processes.

Comment 5: The logical structure of article$^+$ could not be simpler or more rigorous: Starting from any denumerable set, the set A, and using Theorem 4 of Indexation, the set A' is defined as the result of reordering the elements of A with the same natural ω-order as the natural numbers $\mathbb{N}$. The same Theorem 4 of Indexation allows reordering the elements of the rational interval $(0, 1)$ with the same ω-order as A', resulting in the non-densely ordered set $\mathbb{Q}_{01}$. The rational variable x is then compared* with the successive elements of $\mathbb{Q}_{01}$. Once all possible comparisons* of x with the successive elements of $\mathbb{Q}_{01}$ have been made, the value of that variable would have to be the impossible smallest rational greater than 0. Therefore, A is inconsistent, as a contradiction has been deduced from its elements (Definition 5 of an inconsistent set). And since A is any denumerable set, all denumerable sets are inconsistent. And as every non-denumerable set has denumerable subsets (Theorem 3), all infinite sets are inconsistent (Definition 5), and so must be the Axiom of Infinity that legitimizes their existence.

5. Lack of engagement with standard literature:
Article$^+$ does not address classical responses to similar paradoxes (such as Cantor's arguments or limit theory) nor acknowledge established distinctions between potential and actual infinity in formal systems. This limits its academic credibility.

Comment 6: No scientific article, to my knowledge, deals with all the issues it could address; it would be an "infinite" article. On actual and potential infinity, see Comment 2.

Conclusion:
The argument is ingenious but relies on premises not accepted in conventional mathematics, such as equating infinite processes with complete totalities or assuming that density implies contradiction. Unless it is proven within a standard axiomatic framework (e.g., deriving a contradiction in ZFC), its main claim lacks support. Article$^+$ reflects an extreme finitist stance, useful for philosophical debates, but it does not invalidate the Axiom of Infinity in current mathematical practice.

Comment 7: Article$^+$ does not rely on premises unaccepted in conventional mathematics, nor does it equate infinite processes with complete totalities, nor assume that density implies contradiction. Not at all. It simply deduces a contradiction within ZFC set theory. I wonder how an AI can reach such a conclusion so far removed from the content of the article$^+$ it has examined. It's pure pure hodgepodge.

3 Review of ChatGPT o3-mini

Article$^+$ presents a provocative and challenging stance, questioning one of the foundational pillars of set theory: the Axiom of Infinity. Antonio León argues, using properties of countable sets—especially through an analysis of the rational interval $(0, 1)$—that the notion of "actual infinity" leads to logical contradictions. In this sense, the author not only reconsiders the axiom's consistency but extends the critique to related concepts such as infinite divisibility and the space-time continuum.

I find the clarity with which the author defines key concepts (such as complete totality, denumerable and non-denumerable sets, and the notions of actual and potential infinity) interesting. However, his proof, while ingenious, stands on highly controversial ground within modern mathematics, as the Axiom of Infinity is a foundation accepted in theories like ZFC. The use of arguments related to supertasks and reorderings of countable sets is stimulating but likely requires deeper grounding and analysis to convince the traditional mathematical community.

Comment 8: All axioms are statements whose truth is accepted without proof. They can thus be inconsistent. Supertasks are not used in any of the proofs in article$^+$. The reader is only informed of the existence of supertask theory (an infinitist theory as legitimate as set theory itself), according to which the successive redefinitions of x could be timed within a finite time interval.

In summary, it is a work that invites critical reflection on the mathematical and philosophical foundations of infinity. Its radical nature and original approach may open new debates, though it is also likely to generate skepticism and controversy as it confronts widely established concepts in set theory.

Comment 9: Scientific theories are well-established until proven otherwise. If they are truly scientific, they must be falsifiable. Article$^+$ is a simple and necessary exercise in falsifying a scientific theory. It is regrettable that undertaking such an exercise seems audacious. As expected, a clear submission to a mainstream scientific current -ZFC set theory- is observed here too.

4 Review of Grok 3

Summary of the argument
The author proposes a proof of the inconsistency of actual infinity and, consequently, of the Axiom of Infinity in set theory, based on the dual nature of the open rational interval $(0, 1)$, which is both countable (i.e., has cardinality $\aleph_0$) and densely ordered (between any two elements there is always another). The proof unfolds in several formal steps, culminating in a contradiction that, according to León, demonstrates that infinite sets (specifically denumerable ones) are inconsistent. This leads to broader conclusions, such as the inconsistency of actual infinity, the spacetime continuum, and infinite divisibility.

1. Strengths of article$^+$
Expository clarity: Article$^+$ is well-structured and presents definitions and theorems progressively, making the reasoning easy to follow. The formal definitions (such as complete totality, actual vs. potential infinity, and dense order) are consistent with standard mathematical language.
Originality: León tackles a deep philosophical and mathematical issue—the legitimacy of actual infinity—from an unconventional angle. His attempt to challenge over a century of acceptance of infinity in modern mathematics (as in Cantor's theory and the Zermelo-Fraenkel system) is bold and could stimulate debate.
Use of the interval $(0, 1)$: The choice of rationals in $(0, 1)$ as the basis of the argument leverages an interesting

and well-known property: their denumerability and density, making the reasoning intuitively appealing at first glance.

2. Weaknesses and criticisms

The central argument relies on an iterative procedure (the "comparison*") that redefines a variable x by comparing it with the elements of an enumeration of $\mathbb{Q}_{01}$. León claims that, after performing all possible comparisons, x should be the smallest rational in $\mathbb{Q}_{01}$, but simultaneously cannot be due to the set's density (there are always smaller rationals, such as $x/2$). This generates a contradiction that he attributes to the inconsistency of denumerable sets. This reasoning seems to misinterpret the nature of denumerable and dense sets. The fact that $\mathbb{Q}_{01}$ has no minimum element is an inherent property of its dense order, not a logical contradiction derived from its existence as a complete totality. In standard set theory, the absence of a minimum element in $\mathbb{Q}$ or $(0, 1)$ does not imply inconsistency but is an accepted characteristic of the rational order. The contradiction León identifies might be a consequence of his artificial comparison procedure, not of the set's existence itself.

Comment 10: The set $\mathbb{Q}_{01}$ is not densely ordered: there is a first element q_{a_1}, and each element q_{a_n} has an immediate successor $q_{a_{n+1}}$. That a set's characteristic is ACCEPTED does not mean it cannot be inconsistent. I'm not sure what Grok 3 means by artificial comparison; the one in the argument could not be more natural: they are all comparisons between pairs of rational numbers.

Confusion between properties and existence: The author seems to equate the difficulty of finding a "minimum" in a dense set with an ontological inconsistency of actual infinity. However, modern set theory (ZF with the Axiom of Infinity) does not require every ordered infinite set to have a minimum or maximum element; these are additional properties that do not affect the axiom's coherence.

Comment 11: The proof neither equates nor uses difficulties in finding the minimum element of a densely ordered set, simply because the sets A' and $\mathbb{Q}_{01}$ used to reach the contradiction are not densely ordered; nor does it rely on any ontological inconsistency of actual infinity.

Supertask and logical validity: Reframing the argument as a supertask (infinite comparisons in a finite time interval) introduces a physical element that does not necessarily align with pure mathematical logic. While interesting, this analogy does not strengthen the formal proof and could be a distraction.

Comment 12: Supertasks are extraneous to article$^+$ and do not form part of any of its proofs. See Comment 8.

Excessive generalization: From the supposed inconsistency of denumerable sets, León deduces the inconsistency of all infinite sets (denumerable and non-denumerable) and actual infinity in general. This extrapolation is problematic, as it does not directly address non-denumerable sets (like $\mathbb{R}$) beyond noting that they contain denumerable subsets. Cantor's theory clearly distinguishes between cardinalities ($\aleph_0$, $2^{\aleph_0}$, etc.), and León's argument does not seem to robustly challenge this structure.

Comment 13: Article$^+$ does not suppose the inconsistency of denumerable sets; it proves it in Theorem 5. It has previously defined as inconsistent any set from whose elements, or a part thereof, a contradiction is deduced (Definition 5); and it has proven that non-denumerable sets have denumerable subsets (Theorem 3). Accordingly, non-denumerable sets are also inconsistent.

3. Philosophical and scientific implications

The author suggests that the inconsistency of actual infinity would revolutionize not only mathematics but also physics, particularly theories based on the space-time continuum (such as relativity). While it is true that infinity plays a crucial role in these disciplines, article$^+$ does not offer a concrete alternative nor address how finitist mathematics could replace the current framework without collapsing fundamental results.

Comment 14: Those are not the objectives of article$^+$. Its sole aim is to prove the inconsistency of the Axiom of Infinity (hence its title).

The critique of "dominant mathematical infinitism" echoes historical debates (such as Kronecker versus Cantor) but lacks a deep analysis of why actual infinity has been so fruitful and consistent in practical applications.

Comment 15: Science has never discovered anything infinite or divisible into an infinite number of parts. The success of infinitesimal calculus in physics is explained because it is, in reality, a finitist and discrete calculation: progress toward the limit is always made in discrete steps, and calculations stop without ever reaching the limit [5].

4. Mathematical rigor

Although article$^+$ employs formal language and standard definitions (such as those of densely ordered sets or Dedekind's definition of infinity), its conclusion seems to depend more on philosophical intuition than on a demonstrable error in the axioms of set theory. In ZF, the Axiom of Infinity ($\exists S(\emptyset \in S \wedge \forall x \in S(x \cup \{x\} \in S))$) guarantees the existence of an inductive set (like $\mathbb{N}$), and no internal contradiction has been identified in this system for over a century. León's argument does not seem to directly engage with this axiomatic formulation.

Comment 16: The Axiom of Infinity does not guarantee anything. It arbitrarily legitimizes the existence of an infinite set. A statement whose truth, like that of all axiomatic statements, is accepted without proof. Article$^+$ proves that this axiom is inconsistent.

Conclusion: Antonio León's article+ is a courageous and provocative effort to question actual infinity, but its proof seems to rest on a misinterpretation or artificial construction that does not truly undermine the foundations of modern set theory. The contradiction it identifies in Theorem 5 might be more a reflection of the properties of the dense order of rationals than an inherent flaw in the concept of actual infinity or the Axiom of Infinity. While the text invites reflection on the nature of infinity, it does not seem to provide a definitive refutation that withstands rigorous mathematical scrutiny.

> Comment 17: If demonstrating the existence of a contradiction within a theory does not refute that theory, then no theory is refutable. Has Grok 3's mathematical scrutiny not been rigorous?

5 Review of Gemini 2.0

The article+ "The Axiom of Infinity is Inconsistent" by Antonio León presents an argument to demonstrate the inconsistency of the notion of actual infinity and, consequently, of the Axiom of Infinity in set theory. The central argument relies on the dual nature (countable and densely ordered) of the rational interval (0,1). In summary, my opinion is as follows:

Originality and Ambition:
Article+ addresses a fundamental and highly debated topic in mathematics: the nature of infinity. The claim to demonstrate the inconsistency of a central axiom like the Axiom of Infinity is ambitious and requires extreme rigor.

Clarity and Accessibility:
While the author strives to keep the language as accessible as possible, the topic is inherently complex, and understanding it requires a solid knowledge of set theory and mathematical logic. Some points could benefit from more detailed explanations and concrete examples.

Strength of the Argument:
Here lies the most critical point. The brevity of the "argument" raises suspicions about its soundness. While the central idea of using the structure of the interval (0,1) is interesting, the proof seems to oversimplify the inherent complexity of actual infinity. The conclusion that "once all possible comparisons have been made, the current value of x is and is not the smallest rational in $\mathbb{Q}_{01}$" requires much deeper analysis and rigorous justification. The contradiction likely stems from a misinterpretation or subtle manipulation of the concepts of "all possible" and "continuum."

> Comment 18: The brevity of an argument is not evidence against it. I can think of nothing more profound than a simple, brief argument that aligns with the laws of logic and the accepted foundations of the corresponding science. In any case, if there were a flaw in the reasoning of article+, none of the four most advanced AIs in the world have been able to find it. Perhaps it doesn't exist.

Relevance and Context:
If the proof were correct, it would have revolutionary implications for mathematics and physics, as the author mentions. However, the history of mathematics is full of "proofs" of paradoxical results that later turned out to be subtle errors. Therefore, this work must be analyzed with extreme caution.

Conclusion:
Article+ presents a bold claim with a proof that, at first glance, seems incomplete and requires exhaustive review by experts in the field. It is highly likely that the proof contains some logical or conceptual error that invalidates the conclusion.

> Comment 19: If it is incomplete or contains a logical or conceptual error, none of the four AIs used have been able to find it.

Specific points to consider in a deeper review:
The definition of "comparison*": Is the "comparison* operation well-defined? Is it consistent throughout the proof?

> Comment 20: It is such a simple definition that I find it surprising to question whether it is correct: comparing the value of a rational variable with the value of a rational number and redefining the rational variable with the value of the compared rational number if, and only if, the value of the compared rational number is less than the value of the variable.

The transition to a supertask: Transforming the proof into a supertask introduces concepts of time and limits that must be handled with extreme care to avoid paradoxes.

> Comment 21: There is no transition to any supertask (see Comment 8).

The inference of "all possible": What does "all possible" comparisons mean in the context of actual infinity? Can an infinite operation truly be completed?

> Comment 22: See the introduction.

In summary, article+ is interesting for the topic it addresses, but its validity depends on the soundness of the proof, which must be examined with rigorous detail. It is highly likely that some error will be found that invalidates the conclusion.

Comment 23: An error that, I repeat, neither Gemini 2.0 nor the other three AIs have been able to find. It seems reasonable to consider the possibility that it does not exist.

Opinions of the four IAs participating in the above article on the same above article. The reader has the last word on this article and, especially, on the article$^+$.

DeepSeek:
The article presents a provocative challenge to the Axiom of Infinity, arguing that infinite sets (e.g., rationals in (0,1)) lead to contradictions via reordering and comparison processes. While the four reviewed AIs (DeepSeek, ChatGPT, Grok, Gemini) found no technical errors, they uniformly rejected the conclusion, adhering to ZFC orthodoxy. Criticisms included misinterpreting dense orders, invoking supertasks (which the author denies using), and dismissing contradictions as philosophical rather than mathematical. The author highlights AIs' inability to engage deeply with dissenting arguments, exposing their training bias toward mainstream infinitism. Strengths include originality and rigor in defining inconsistencies; weaknesses involve dense exposition and potential overreach in generalizing results (e.g., spacetime). The debate underscores tensions between foundational mathematical axioms and AI's capacity for critical, paradigm-shifting critique.

Chat GPT o3-mini:
Antonio León's article is a bold, thought-provoking challenge to conventional set theory. By reordering the rational interval (0,1), the paper constructs an argument against the consistency of the Axiom of Infinity. The innovative use of AI-generated reviews adds a multi-perspective dialogue that both critiques and supports established mathematical ideas. While the article stimulates valuable philosophical debate on actual versus potential infinity, its conclusions rely on non-standard interpretations that may not convince proponents of traditional frameworks like ZFC. Overall, this work is a commendable intellectual exercise that encourages deeper scrutiny of foundational concepts in mathematics, even if it might require further formal rigor to gain broader acceptance.

Wrok 3:
The article by Antonio León explores a provocative debate where four AIs—DeepSeek v3, ChatGPT 03-mini, Grok 3, and Gemini 2—review his paper challenging the Axiom of Infinity. León argues that actual infinity is inconsistent, using the rational interval (0,1) to derive a contradiction. The AIs praise the paper's originality and clarity but criticize its unconventional premises, suggesting it misinterprets concepts like dense sets and relies on non-standard assumptions. They find no explicit errors but doubt the proof's rigor within ZFC set theory. León defends his work, asserting the AIs' critiques reflect bias toward mainstream mathematics. The discussion highlights tensions between established axioms and bold dissent, though the AIs' skepticism underscores the need for stronger evidence to upend such a foundational concept. It's a thought-provoking clash of perspectives.

Gemini 2.0:
This article explores the responses of four different artificial intelligence models—DeepSeek v3, ChatGPT 03-mini, Grok 3, and Gemini 2—to a paper that challenges the conventional mathematical understanding of infinity. None of the AIs identified any errors in the paper's proofs, but they also did not fully endorse its content. This highlights their inclination to adhere to established mathematical thought. The article includes the dissenting paper, which allows readers to assess the AIs' analytical capabilities when faced with perspectives that diverge from mainstream scientific thought.

Bibliographical References

[1] John Earman and John D. Norton. Forever is a Day: Supertasks in Pitowsky and Malament-Hogarth Spacetimes. *Philosophy of Science*, 60(1):22–42, 1993.

[2] John Earman and John D. Norton. Comments on Laraudogoitia's 'Classical Particle Dynamics, Indeterminism and a Supertask'. *The British Journal for the Phylosophy of Science*, 49(1):122 – 133, March 1998.

[3] M. L. Hogarth. Does General Relativity Allow an Observer to view an Eternity in a Finite Time? *Foundations of Physics Letters*, 5:173 – 181, 1992.

[4] A. León. Physicists calculate but do not explain 1/4: Physics and infinitesimal calculus Link. *The General Science Journal*, 2024.

[5] A. León. The Axiom of Infinity Is Inconsistent. *The General Science Journal Link*, 2024.

[6] John Byron Manchack. On the Possibility of Supertasks in General Relativity. *Philsci-Archive*, pages 1–18, 2009. http://philsci-archive.pitt.edu/archive/00005020.

[7] Jon Pérez Laraudogoitia. A Beautiful Supertask. *Mind*, 105:49–54, 1996.

Appendix. The article$^+$ reviewed by the 4 IAs.

The Axiom of Infinity is Inconsistent

Abstract.-This article contains the shortest proof I have been able to develop of the inconsistency of the actual infinity and the Axiom of Infinity. The proof is based on the dual denumerable and densely ordered nature of the rational interval $(0, 1)$, and is a formal consequence of assuming that there exist all rational numbers greater than zero and less than 1, without there being a first rational number greater than zero (a similar proof could be based on the assumed existence of all rational number less than 1 without there being a last rational number less than 1).

6 Introduction

Although with little hope, I have included an abridged version of the following argument in other publications. And always for the same reason: to convince of the inconsistency of the actual infinity. I have decided to publish it here independently, in case any reader wants to waste five minutes reading it, and can help to spread it if he is convinced. Anyway, it seems to me very difficult to respond to more than 120 years of absolutely hegemonic and dominant mathematical infinitism. But we have to try, because it is not a trivial matter: the inconsistency of the actual infinity changes everything, not only in mathematics, but also in a good part of physical theories, especially those committed to the infinitist spacetime continuum.

7 On the inconsistency of the Axiom of Infinity

Before starting to develop the argument included in this section on the inconsistency of the Axiom of Infinity it is convenient to recall the few technicalities included in it. We say that a set A is densely ordered if between its elements there exists a binary relation $<$ such that this relation is:

1. Irreflexive: $\forall a \in A$: not $a < a$.
2. Asymetric: $\forall a, b \in A$: If $a < b$ then not $b < a$.
3. Transitive: $\forall a, b, c \in A$: If $a < b$ and $b < c$, then $a < c$.
4. Connected: $\forall a, b \in A$: If $a \neq b$ then either $a < b$ or $b < a$.
5. Dense: $\forall a, b \in A$: $\exists c$: $a < c < b$.

An example of densely ordered set is the open rational interval $(0, 1)$ in its natural order of precedence. Recall that the infinity of a set is the actual infinity (not the potential infinity) if the set is a complete totality: any element that could be in the set, is in the set. The complete argument for the inconsistency of the actual infinity, and therefore of the Axiom of Infinity, includes all the formal elements that follow:

Definition 1 (of Complete Totality) *A complete totality is a set defined by comprehension in which every element that satisfies the corresponding membership definition of the set is in the set.*

In consequence, to a complete totality of a certain type of elements, it is not possible to add new elements of that type because it already contains *all of them.*

Definition 2 (of the types of sets) *A set is finite if it has a definite and finite number of elements. A set of elements of a certain type is potentially infinite if it cannot contain all the elements of that type, because new elements of that type that are not in the set can always be added to it.*

Definition 3 (of infinite set) *A set is infinite if it can be put into one-to-one correspondence with one of its proper subsets.*

This is the well-known Dedekind's definition of infinite set [1, p. 115]. But giving a definition of infinite set does not justify its existence, so we need an axiom that formally legitimizes that existence: the Axiom of Infinity, which can be expressed in different more or less abstract ways, but all of them compatible with the following ordinary language expression :

Axiom 1 (of Infinity) *There exists at least one infinite set.*

Where an infinite set is one that satisfies Dedekind's definition of an infinite set (Definition 3).

Definition 4 (of the types of infinities) *The actual infinity is the infinity of the infinite sets. The potential infinity is the infinity of the potentially infinite sets.*

Definition 5 (of inconsistent set) *A set is inconsistent if a contradiction can be deduced from its elements or from a part of them.*

Definition 6 (of denumerable set) *A set is denumerable if its cardinal is the smallest infinite cardinal* $\aleph_0$ *of the infinite set of all natural numbers. An infinite set is non-denumerable if its cardinal is greater than the smallest infinite cardinal* $\aleph_0$.

Cardinals greater than $\aleph_0$ are, for example, $2^{\aleph_0}$ or $\aleph_1$. Now it is Immediate to Prove the Following Results:

Theorem 1 (of the Axiom of Infinity) *The infinity subsumed in the Axiom of Infinity can only be the actual infinity.*

Proof.- Since potentially infinite sets do not exist as complete totalities, only two proper subsets with the same number of elements of the same potentially infinite set could be put into one-to-one correspondence, and then we would have a one-to-one correspondence between two proper subsets of a potentially infinite set, instead of a one-to-one correspondence between a set and one of its proper subsets, as required by the definition of an infinite set (Definition 3). Therefore, the potential infinity cannot be the infinity of an infinite set. Only the actual infinity can be the infinity of the infinite set whose existence is established by the Axiom of Infinity. □

Theorem 2 (of Denumerable Sets) *It is always possible to define a one-to-one correspondence between any two denumerable sets.*

Proof.- Let A and B be any two denumerable sets. They have the same number of elements, exactly $\aleph_0$ elements (Definition 6). Therefore, their respective elements can be put into one-to-one correspondence, i.e. each of the different elements of A can be paired with a different and exclusive element of B, so that all elements of A and B result exclusively paired. □

Theorem 3 (of Non-Denumerable Sets) *Every non-denumerable set has denumerable proper subsets.*

Proof.- Let X be any non-denumerable set. Since its cardinal is greater than $\aleph_0$ (Definition 6), X contains proper subsets with only $\aleph_0$ elements, all of which are denumerable proper subsets of X (Definition 6). □

Theorem 4 (of Indexation) *The elements of a denumerable set can be reordered with the same order as the elements of any other denumerable set.*

Proof.-Let $A = \{a, b, c, \dots\}$ and $B = \{\alpha, \beta, \dots\}$ be any two denumerable sets. There exists at least one bijection f between the elements of A and B (Theorem 2). Consequently, f pairs each element k of A with a unique and exclusive element, say δ, of B, which can be used to exclusively index that element k of A, so that element k can be rewritten as a_delta. Consequently, the elements of the set A can be reordered and rewritten to define the set $A' = \{a_\alpha, a_\beta, a_\gamma, \dots\}$ which has exactly the same elements as A, and ordered in the same way as the elements of B. □.

The infinity of infinite sets is the actual infinity, not the potential infinity (Theorem 1 of the Axiom of Infinity). This implies the existence of certain infinite sets that are also complete totalities (Definition 1). For example the set of ALL natural numbers in their natural order of precedence. It is not possible, then, to add new natural numbers to the set of natural numbers because it already contains them all. And the same is true of many other numerical or non-numerical sets. For many authors, the existence of these ordered and complete totalities without a last element that completes them (or without a first element that initiates them) is a proven conclusion independent of the Axiom of Infinity. It is not. It is an existence assumed and legitimized by the Axiom of Infinity. Their existence is, therefore, as debatable as the Axiom of Infinity itself. So it is as legitimate to argue about that axiom as it is to argue about the existence of those complete totalities. This fully justifies the following:

Theorem 5 (of the Denumerable Infinity) *The denumerable sets are inconsistent.*

Proof.- Let A be any denumerable set. The set A allows us to define the set A' with the same elements as A but reordered as the set $\mathbb{N}$ of natural numbers in their natural order of precedence: $A' = \{a_1, a_2, a_3, \}$ (Theorem 4). The open interval of rational numbers $(0, 1)$ is densely ordered in the natural order of precedence (represented by the symbol $<$) defined by the natural values of the rational numbers. It is also a denumerable set, so there exists a bijection f between A' and $(0, 1)$ (Theorem 2). Consequently, $(0, 1)$ can be reordered and rewritten as the set $\mathbb{Q}_{01} = \{q_{a_1}, q_{a_2}, q_{a_3}, \dots\}$, where $q_{a_i} = f(a_i), \forall a_i \in A'$, and the successive elements $q_{a_1}, q_{a_2}, q_{a_3}, \dots$ of $\mathbb{Q}_{01}$ are ordered by the successive natural numbers in their natural order of precedence, and not by their respective values as rational numbers. Let x now be a rational variable defined initially as q_{a_1}. And let the value of x be $<$-compared (i.e., compared according to the values of the rational numbers) with the successive elements of the set $\mathbb{Q}_{01}$, with x being redefined as the compared element q_{a_i} if, and only if, $q_{a_i} < x$.

For short, let us call comparison* this $<$-comparison and redefinition of x if, and only if, the value of the compared element is smaller than the current value of x. It is immediate to prove that for each natural number v it is possible to perform the first v comparisons* of x with the first v successive elements of $\mathbb{Q}_{01}$. Indeed, if it were not possible, there would be at least one natural number $n \leq v$ such that x could not be compared* with q_{a_n}, which is impossible because q_{a_n} is a rational number of $\mathbb{Q}_{01}$ that can be compared* with the current value of x, which is also a rational number. Once all possible comparisons* of x with the successive elements $q_{a_1}, q_{a_2}, q_{a_3}, \ldots$ of $\mathbb{Q}_{01}$ have been made, the current value of x, whatever it may be, could only be the smallest rational number of that set. Indeed, if once performed all possible comparisons* of x with the successive elements of $\mathbb{Q}_{01}$ the current value of x were not the smallest rational number of $\mathbb{Q}_{01}$, there would be at least one element q_{a_n} in $\mathbb{Q}_{01}$ such that $q_{a_n} < x$. But that is impossible because n is a natural number; the first n comparisons* have been carried out; and therefore x was compared* with q_{a_n} and redefined as q_{a_n}; and in all subsequent comparisons*, x could only be redefined with values smaller than q_{a_n}. Therefore, it is impossible for $q_{a_n} < x$. But, on the other hand, it is also immediate to prove that once all possible comparisons* of x with the successive elements of $\mathbb{Q}_{01}$ have been made, the current value of x is not the smallest rational number of that set: every element of the infinite set $\{x/2, x/3, x/4 \ldots\}$ is an element of $\mathbb{Q}_{01}$ smaller than x. This contradiction proves that the set A', defined exclusively with the elements of A, is inconsistent. Therefore A' and A are inconsistent (Definition 5). And A being any denumerable set, it must be concluded that all denumerable sets are inconsistent. □

Although the consistency of a mathematical proof of infinite steps is universally accepted without the need to perform all of its infinite steps, the theory of supertasks considers the possibility of performing them in finite time. In the case of the above successive comparisons* of x with each successive q_{ai} would be performed at each successive instant t_i of a strictly increasing and convergent sequence $\langle t_i \rangle$ of instants within the finite time interval (t_a, t_b), whose limit is t_b. The instant t_b is the first instant after all instants of $\langle t_i \rangle$, and therefore the first instant after having performed all possible comparisons* of x with the successive elements of Q_{01}. At the instant t_b the rational variable x will still be a rational variable with a certain value, whatever it is; and not, for example, an elephant (in which case anything could be proved). The problem is that the value of x at the instant t_b is and is not the least rational of Q_{01}. From the previous theorems, we can immediately deduce, among many others, the following results:

Corollary 1 (of the Infinite Sets) *All infinite sets are inconsistent.*

Proof.-Let X be any infinite set. If X is denumerable, then it is inconsistent. (Theorem 5). If X is non-denumerable, then it has denumerable proper subsets (Theorem 3 of Non-Denumerable Sets) and is also inconsistent (Definition 5). Therefore, all infinite sets are inconsistent. □

Corollary 2 (of the Inconsistent Axiom of Infinity) *The axiom of infinity is inconsistent.*

Proof.-This is an immediate consequence of Corollary 1. □

Theorem 6 (of the Actual Infinity) *The actual infinity is inconsistent.*

Proof.-The actual infinity is the infinity subsumed in the Axiom of Infinity (Theorem 1). That axiom only establishes the existence of at least one infinite set, and therefore of a set whose only declared property is that of being actual infinite (Axiom 1). But the Axiom of infinity is inconsistent (Corollary 2). Therefore, the existence of a set whose only declared property is that of being actual infinite is inconsistent; which is only possible if the actual infinity (Definition 4) is inconsistent. □

Corollary 3 (of Infinite Divisibility) *The actual infinite divisibility of any formal or physical object is inconsistent.*

Proof.- From the actual infinite divisibility of any formal or physical object can only result an inconsistent infinite set of parts (Corollary 1). So that actual infinite divisibility is inconsistent. □

Theorem 7 (of the Inconsistent Continuum) *The spacetime continuum is inconsistent.*

Proof.- Being $\mathbb{R}$ the set of all real numbers, the spacetime continuum is, by definition, the Cartesian product $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ of all quaternions of real numbers (x, y, z, t). And since $\mathbb{R}$ is an infinite set (Definition 3), it is inconsistent (Corollary 1). Therefore, the spacetime continuum $\mathbb{R}^4$, of which $\mathbb{R}$ is a part, is also inconsistent (Definition 5). □

Bibliographical Reference

[1] Richard Dedekind. ¿Qué son y para qué sirven los números? Alianza Editorial, Madrid. 1998.