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The Axiom of Infinity is Inconsistent

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Abstract.-This article contains the shortest proof I have been able to develop of the inconsistency of the actual infinity and the Axiom of Infinity. The proof is based on the dual numerable and densely ordered nature of the rational interval $(0, 1)$, and is a consequence of assuming that there exist all rational numbers greater than zero and less than 1, without there being a first rational number greater than zero (a similar proof could be based on the assumed existence of all rational number less than 1 without there being a last rational number less than 1).

1 Introduction

Although with little hope, I have included an abridged version of the following argument in other publications. And always for the same reason: to convince of the inconsistency of the actual infinity. I have decided to publish it here independently, in case any reader wants to waste five minutes reading it, and can help to spread it if he is convinced. Anyway, it seems to me very difficult to respond to more than 120 years of absolutely hegemonic and dominant mathematical infinitism. But we have to try, because it is not a trivial matter: the inconsistency of the actual infinity changes everything, not only in mathematics, but also in a good part of physical theories, especially those committed to the infinitist spacetime continuum.

2 On the inconsistency of the Axiom of Infinity

Before starting to develop the argument included in this section on the inconsistency of the Axiom of Infinity it is convenient to recall the few technicalities included in it. We say that a set A is densely ordered if between its elements there exists a binary relation $<$ such that this relation is:

1. Irreflexive: $\forall a \in A$: not $a < a$.
2. Asymetric: $\forall a, b \in A$: If $a < b$ then not $b < a$.
3. Transitive: $\forall a, b, c \in A$: If $a < b$ and $b < c$, then $a < c$.
4. Connected: $\forall a, b \in A$: If $a \neq b$ then either $a < b$ or $b < a$.
5. Dense: $\forall a, b \in A$: $\exists c$: $a < c < b$.

An example of densely ordered set is the open rational interval $(0, 1)$ in its natural order of precedence. Recall that the infinity of a set is the actual infinity (not the potential infinity) if the set is a complete totality: any element that could be in the set, is in the set. Or in other words, in an actual infinite set it is not possible to add to the set any element that is not already in the set.

The complete argument for the inconsistency of the actual infinity, and therefore of the Axiom of Infinity, includes all the formal elements that follow:

Definition 1 (of Complete Totality) *A complete totality is a set defined by comprehension in which every element that satisfies the corresponding membership definition of the set is in the set.*

In consequence, to a complete totality of a certain type of elements, it is not possible to add new elements of that type because it already contains *all of them*.

Definition 2 (of the types of sets) *A set is finite if it has a definite and finite number of elements. A set is infinite if it has a definite and transfinite number of elements of a certain type that exist as a complete totality. A set of elements of a certain type is potentially infinite if it cannot contain all the elements of that type, because new elements of that type that are not in the set can always be added to it.*

Definition 3 (of the types of infinities) *The actual infinity is the infinity of the infinite sets. The potential infinity is the infinity of the potentially infinite sets.*

Definition 4 (of consistent set) *A set is inconsistent if a contradiction can be deduced from its elements or from a part of them.*

Definition 5 (of infinite set) *A set is infinite if it can be put into one-to-one correspondence with one of its proper subsets.*

As the reader will have recognized, this is the well-known Dedekind's definition of infinite set [1, p. 115]. But giving a definition of infinite set does not justify its existence, so we need an axiom that formally legitimizes that existence: the Axiom of Infinity, which can be expressed in different more or less abstract ways, but all of them compatible with the following ordinary language expression :

Axiom 1 (of Infinity) *There exists at least one infinite set.*

Where an infinite set is one that satisfies Dedekind's definition of an infinite set (Definition 5).

Definition 6 (of denumerable set) *A set is denumerable if its cardinal is the smallest infinite cardinal $\aleph_0$ of the infinite set of all natural numbers. An infinite set is non-denumerable if its cardinal is greater than the smallest infinite cardinal $\aleph_0$.*

Cardinals greater than $\aleph_0$ are, for example, $2^{\aleph_0}$ or $\aleph_1$. Now it is Immediate to Prove the Following Results:

Theorem 1 (of the Axiom of Infinity) *The infinity subsumed in the Axiom of Infinity can only be the actual infinity.*

Proof.- Since potentially infinite sets do not exist as complete totalities, only two proper subsets with the same number of elements of the same potentially infinite set could be put into one-to-one correspondence, and then we would have a one-to-one correspondence between two proper subsets of a potentially infinite set, instead of a one-to-one correspondence between a set and one of its proper subsets, as required by the definition of an infinite set (Definition 5). Therefore, the potential infinity cannot be the infinity of an infinite set. Only the actual infinity can be the infinity of the infinite set whose existence is established by the Axiom of Infinity. □

Theorem 2 (of Denumerable Sets) *It is always possible to define a one-to-one correspondence between any two denumerable sets.*

Proof.- Let A and B be any two denumerable sets. They have the same number of elements, exactly $\aleph_0$ elements (Definition 6). Therefore, their respective elements can be put into one-to-one correspondence, i.e. each of the different elements of A can be paired with a different and exclusive element of B, so that all elements of A and B result exclusively paired. □

Theorem 3 (of Non-Denumerable Sets) *Every non-denumerable set has denumerable proper subsets.*

Proof.- Let X be any non-denumerable set. Since its cardinal is greater than $\aleph_0$ (Definition 6), X contains proper subsets with only $\aleph_0$ elements, all of which are denumerable proper subsets of X (Definition 6). □

Theorem 4 (of Indexation) *The elements of a denumerable set can be reordered with the same order as the elements of any other denumerable set.*

Proof.-Let $A = \{a, b, c, \dots\}$ and $B = \{\alpha, \beta, \dots\}$ be any two denumerable sets. There exists at least one bijection f between the elements of A and B (Theorem 2). Consequently, f pairs each element k of A with a unique and exclusive element, say δ, of B, which can be used to exclusively index that element k of A, so that element k can be rewritten as $a_d elta$. Consequently, the elements of the set A can be reordered and rewritten to define the set $A' = \{a_\alpha, a_\beta, a_\gamma, \dots\}$ which has exactly the same elements as A, and ordered in the same way as the elements of B. □.

The infinity of infinite sets is the actual infinity, not the potential infinity (Theorem 1 of the Axiom of Infinity). This implies the existence of certain infinite sets that are also complete totalities (Definition 1). For example the set of ALL natural numbers in their natural order of precedence. It is not possible, then, to add new natural numbers to the set of natural numbers because it already contains them all. And the same is true of many other numerical or non-numerical sets. For many authors, the existence of these ordered and complete totalities without a last element that completes them (or without a first element that initiates them) is a proven conclusion independent of the Axiom of Infinity. It is not. It is an existence assumed and legitimized by the Axiom of Infinity. Their existence is, therefore, as debatable as the Axiom of Infinity itself. So it is as legitimate to argue about that axiom as it is to argue about the existence of those complete totalities. This fully justifies the following:

Theorem 5 (of the Denumerable Infinity) *The denumerable sets are inconsistent.*

Proof.- Let A be any denumerable set. The set A allows us to define the set A' with the same elements as A but reordered as the set $\mathbb{N}$ of natural numbers in their natural order of precedence: $A' = \{a_1, a_2, a_3, \}$ (Theorem 4). The open interval of rational numbers $(0, 1)$ is densely ordered in the natural order of precedence (represented by the symbol $<$) defined by the natural values of the rational numbers. It is also a denumerable set, so there exists a bijection f between A' and $(0, 1)$ (Theorem 2). Consequently, $(0, 1)$ can be reordered and rewritten as the set $\mathbb{Q}_{01} = \{q_{a_1}, q_{a_2}, q_{a_3}, \dots\}$, where $q_{a_i} = f(a_i), \forall a_i \in A'$, and the successive elements $q_{a_1}, q_{a_2}, q_{a_3}, \dots$ of $\mathbb{Q}_{01}$ are ordered by the successive natural numbers in their natural order of precedence, and not by their respective values as rational numbers. Let x now be a rational variable defined initially as q_{a_1}. And let the value of x be $<$-compared (i.e., compared according to the values of the rational numbers) with the successive elements of the set $\mathbb{Q}_{01}$, with x being redefined as the compared element q_{a_i} if, and only if, $q_{a_i} < x$.

For short, let us call comparison* this $<$-comparison and redefinition of x if, and only if, the value of the compared element is smaller than the current value of x. It is immediate to prove that for each natural number v it is possible to perform the first v comparisons* of x with the first v successive elements of $\mathbb{Q}_{01}$. Indeed, if it were not possible, there would be at least one natural number $n \leq v$ such that x could not be compared* with q_{a_n}, which is impossible because q_{a_n} is a rational number of $\mathbb{Q}_{01}$ that can be compared* with the current value of x, which is also a rational number. Once all possible comparisons* of x with the successive elements $q_{a_1}, q_{a_2}, q_{a_3}, \dots$ of $\mathbb{Q}_{01}$ have been made, the current value of x, whatever it may be, could only be the smallest rational number of that set. Indeed, if once performed all possible comparisons* of x with the successive elements of $\mathbb{Q}_{01}$ the current value of x were not the smallest rational number of $\mathbb{Q}_{01}$, there would be at least one element q_{a_n} in $\mathbb{Q}_{01}$ such that $q_{a_n} < x$. But that is impossible because n is a natural number; the first n comparisons* have been carried out; and therefore x was compared* with q_{a_n} and redefined as q_{a_n}; and in all subsequent comparisons*, x could only be redefined with values smaller than q_{a_n}. Therefore, it is impossible for $q_{a_n} < x$. But, on the other hand, it is also immediate to prove that once all possible comparisons* of x with the successive elements of $\mathbb{Q}_{01}$ have been made, the current value of x is not the smallest rational number of that set: every element of the infinite set $\{x/2, x/3, x/4 \dots\}$ is an element of $\mathbb{Q}_{01}$ smaller than x. This contradiction proves that the set A', defined exclusively with the elements of A, is inconsistent. Therefore A' and A are inconsistent (Definition 4). And A being any denumerable set, it must be concluded that all denumerable sets are inconsistent. □

It is immediate to turn the proof of the above theorem into a supertask: it suffices to suppose that the comparison* of x with each successive q_{ai} is performed at each successive instant t_i of a strictly increasing and convergent sequence $\langle t_i \rangle$ of instants within the finite time interval (t_a, t_b), whose limit is t_b. The instant t_b is the first instant after all instants of $\langle t_i \rangle$, and therefore the first instant after having performed all possible comparisons* of x with the successive elements of Q_{01}. At the instant t_b the rational variable x will still be a rational variable with a certain value, whatever it is; and not, for example, an elephant (in which case anything could be proved). The problem is that the value of x at the instant t_b is and is not the least rational of Q_{01}.

From the previous theorem, we can immediately deduce, among many others, the following results:

Corollary 1 (of the Infinite Sets) *All infinite sets are inconsistent.*

Proof.-Let X be any infinite set. If X is denumerable, then it is inconsistent. (Theorem 5). If X is non-denumerable, then it has denumerable proper subsets (Theorem 3 of Non-Denumerable Sets) and is also inconsistent (Definition 4). Therefore, all infinite sets are inconsistent. □

Corollary 2 (of the Inconsistent Axiom of Infinity) *The axiom of infinity is inconsistent.*

Proof.-This is an immediate consequence of Corollary 1. □

Theorem 6 (of the Actual Infinity) *The actual infinity is inconsistent.*

Proof.-The actual infinity is the infinity subsumed in the Axiom of Infinity (Theorem 1). That axiom only establishes the existence of at least one infinite set, and therefore of a set whose only declared property is that of being actual infinite (Axiom 1). But the Axiom of infinity is inconsistent (Corollary 2). Therefore, the existence of a set whose only declared property is that of being actual infinite is inconsistent; which is only possible if the actual infinity (Definition 3) is inconsistent. □

Corollary 3 (of Infinite Divisibility) *The actual infinite divisibility of any formal or physical object is inconsistent.*

Proof.- From the actual infinite divisibility of any formal or physical object can only result an inconsistent infinite set of parts (Corollary 1). So that actual infinite divisibility is inconsistent. □

Theorem 7 (of the Inconsistent Continuum) *The spacetime continuum is inconsistent.*

Proof.- Being $\mathbb{R}$ the set of all real numbers, the spacetime continuum is, by definition, the Cartesian product $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ of all quaternions of real numbers (x, y, z, t). And since $\mathbb{R}$ is an infinite set (Definition 5), it is inconsistent (Corollary 1). Therefore, the spacetime continuum $\mathbb{R}^4$, of which $\mathbb{R}$ is a part, is also inconsistent (Definition 4). □

Bibliographical Reference

[1] Richard Dedekind. *Qué son y para qué sirven los números (Was sind Und was sollen die Zahlen (1888)).* Alianza, Madrid, 1998. Definición de conjunto infinito p. 115.