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Impossible Integrals.
(Combinational approximating integral hybrids – approximating non-elementary integrals with elementary integrals)

Abstract: This paper attempts to show that non-elementary integrals can be expressed in an elementary way, by producing curves that closely approximate the true solution.
Although there are a great many integrals that are considered impossible, in that they cannot be expressed in an elementary way, here I show that they can be approximated quite closely. I present a technique that allows for the generation of functions that closely approximate the solution to so-called non elementary integrals. I will only provide a small number of examples in this paper, although the technique can be applied to any integral, to give it an approximate elementary solution. It is my hope that from this work, a universal set of combinational approximating integral hybrids will be generated, and universally agreed upon, that may be used in conventional mathematics in place of non elementary solutions. In this paper, I shall evaluate the integrals ∫Sin(x)/(x) dx and ∫Cos(x)/(x) dx.

Deriving ∫Sin(x)/(x) dx

The non-elementary integral ∫Sin(x)/(x) dx is considered impossible to express in an elementary way, but here, I show that it actually can be done, and generate a function that closely approximates the actual solution. My work is only a rough approximation, and in time, once this integral evaluation technique is understood, others will produce an answer that more closely approximates the true solution.

The steps are as follows: derive components and properties of the integral as expected by conventional calculus, then combine these components and properties in a way that approximates the exact solution. Let us derive our first property or component of the integral ∫Sin(x)/(x) dx:

From conventional calculus, we know that:
$d/dx\ (Sin(x)/x) = Cos(x)/x – Sin(x)/x^2$
Proof: The product rule: Y' = UV' + VU'
Let Y= UV, where U = 1/x, and V= Sinx
Therefore $U' = -1/x^2$, V' = Cos(x)
Hence: $d/dx\ (Sin(x)/x) = Cos(x)/x – Sin(x)/x^2$

Now, lets multiply both sides by –x:
-x d/dx (Sin(x)/x) = -Cos(x) + Sin(x)/x
Re- arranging gives:
-x d/dx (Sin(x)/x) +Cos(x) = Sin(x)/x
Since $\int -x = -x^2/2$:
$d/dx\ (-x^2/2) * d/dx\ (Sin(x)/x) + Cos(x) = Sin(x)/x$
Integrating both sides gives us:

$\int[d/dx\ (-x^2/2) * d/dx\ (Sin(x)/x) + Cos(x)] = \int Sin(x)/x\ dx$

Using a calculus rule I derived myself: **∫[d/dx(A) . d/dx(B)] dx = ∫[d/dx(A . B)] dx** This becomes:
$\int\{d/dx\ [(-x^2/2) * (Sin(x)/x)] + Cos(x)\}dx = \int Sin(x)/x\ dx$
Multiplying the derivative terms in the derivate bracket on the left hand side gives us:
$\int\{d/dx\ [(-x^2 Sin(x))/2x] + Cos(x)\}\ dx = \int Sin(x)/x\ dx$
Simplifying the derivative gives us:
$\int\{d/dx\ [(-xSin(x))/2] + Cos(x)\}\ dx = \int Sin(x)/x\ dx$

From conventional calculus, we know that:
d/dx [(-xSin(x))/2] = -xCos(x)/2 – Sin(x)/2
Proof: The product rule: Y' = UV' + VU'
Let Y= UV, where U = -x/2, and V= Sin(x)
Therefore U'=-1/2, V' = Cos(x)
d/dx [(-xSin(x))/2] = -xCos(x)/2 – Sin(x)/2

Inserting this into our equation gives us:
$\int\{-xCos(x)/2 - Sin(x)/2 + Cos(x)\}\ dx = \int Sin(x)/x\ dx$
Since $\int Cos(x)\ dx = Sin(x)$:
$\int\{-xCos(x)/2 - Sin(x)/2\}dx + Sin(x) = \int Sin(x)/x\ dx$
Now, from conventional calculus, we know that:
$\int\{-xCos(x)/2 - Sin(x)/2\}dx = (-xSin(x))/2 - (Cos(x))/2 + (Cos(x))/2$
Inserting this into our equation, gives us:
$(-xSin(x))/2 - (Cos(x))/2 + (Cos(x))/2 + Sin(x) = \int Sinx/x\ dx$
Simplifying this expression gives us:
$(-xSin(x))/2 + Sin(x) = \int Sin(x)/x\ dx$……….(Component – Property 1)

Here, we have derived our first component - property of the ∫Sin(x)/x dx integral. This component alone is not the solution, but shall be part of a larger function to approximate the exact solution of ∫Sin(x)/x dx.

Now, Lets find another component – property:

Lets look at the equation: $Sin^2(x) + Cos^2(x) = 1$.
Dividing both sides by x^2 gives us:
$Sin^2(x)/x^2 + Cos^2(x)/x^2 = 1/x^2$
Re-arranging in terms of $Sin^2(x)/x^2$ gives us:
$Sin^2(x)/x^2 = 1/x^2 - Cos^2(x)/x^2$
Taking the positive root of both sides yields:

$|Sin(x)/x| = [(1 - Cos^2(x))/x^2]^{1/2}$
Integrating both sides yields:

$\int|\,\text{Sin}(x)/x\,| = \int[(1 - \text{Cos}^2(x))/x^2]^{1/2}$

From previously, it was shown that according to conventional calculus, ∫Sin(x)/(x) dx = Sin(x)-(xSinx)/2. Thus in this case, we may say:

$\int|\,\text{Sin}(x)/x\,| = \int \text{Sin}(x)/(x)\,dx = \text{Sin}(x)-(x\text{Sin}x)/2$……….(Component – Property 2)

Thus, here we see an interesting component - property. We see that ∫Sin(x)/(x) dx only takes on absolute values. In other words, when constructing a function as a solution to this integral, we shall only look at positive x values, as you cannot take the square root of a negative number.

Since an integral is equal to the area under a graph, we can evaluate how the exact solution will appear as a graph. Please refer to figure 1.

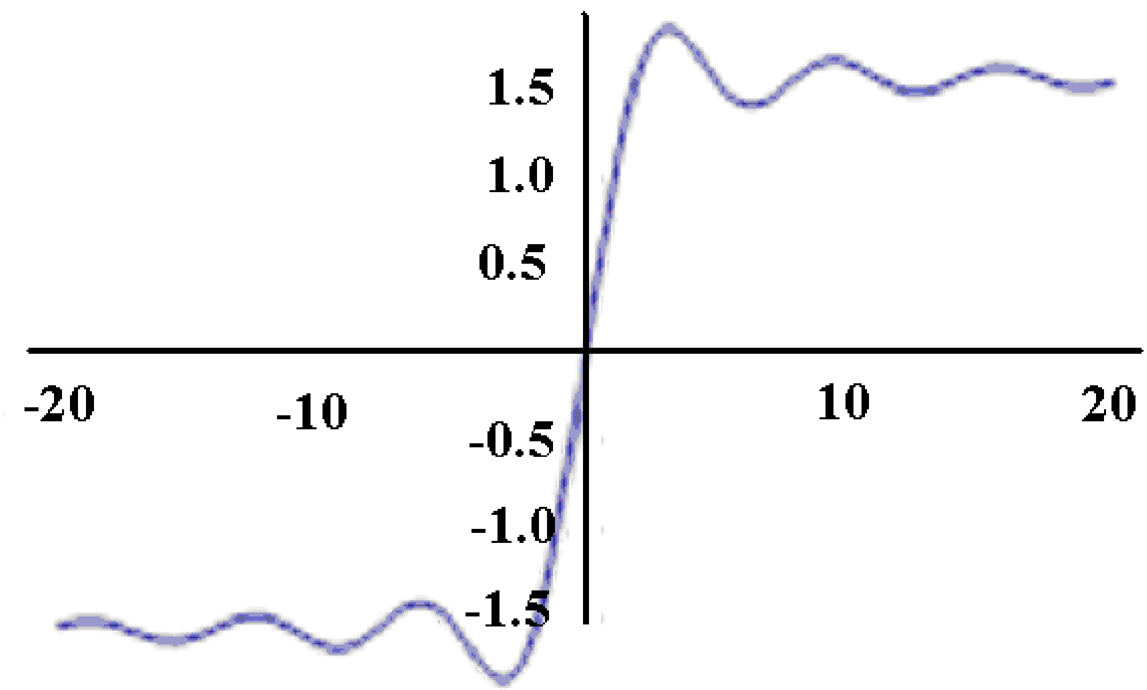

Figure 1: The graph of ∫Sin(x)/x dx, from Wolfram. Link: http://reference.wolfram.com/mathematica/ref/SinIntegral.html

Here, we see the graph we must approximate. From component – property 2 identified earlier, we realize that an elementary solution to this problem must ignore negative x values. In other words, to calculate negative x values for our function, we must use positive x values, then simply multiply the final answer by -1.

Notice figure 1 rises up to about 1.5, then levels out. This shall be component – property 3. We can put this into our function by using a function that levels out after its early or first x- values. The function 1/(e^x) behaves in a similar way, so let the function 1/(e^x) be component – property 3. We could use another function in place of this one, as other functions behave this way too. For example, we could use 1/(h^x), where h is a constant, but for the case of this paper, I shall use 1/(e^x), where e^x is the exponential function, where it's inverse is the natural logarithm to base e. Note: e^(1) = 2.71828182846.

Also notice how after reaching it's first peak, the peaks in the graph reduce as the x values increase. The graph of d/dx (Sin(x)/x) = Cos(x)/x – Sin(x)/x^2 behaves this way also, because part of the function was divided by x^2. Therefore, dividing by x^2 shall be component - property 4.

Since component - property 3 describes the behavior of the overall function we want, we will make this our base function, and unifying this component property with component properties 1, 2 and 4, we may arrive at the theoretical combinational approximating integral hybrid expression:

$$\int \mathrm{Sin}(x)/x\ dx \sim \frac{D}{e^{\left(\frac{\mathrm{Sin}(x \pm A)-(0.5)(x \pm B)\mathrm{Sin}(x \pm C)}{x^2}\right)}}$$

Where A, B, C, and D are constants to be adjusted to closely approximate the graph of ∫Sinx/x dx on the positive x axis only. For calculations involving negative x-values, we must use positive x values, then simply multiply the final answer by -1. Also note that this theoretical combinational approximating integral hybrid is just one theoretical form that may be generated.

Now comes the fun part- we must adjust the values of A, B, C and D to make our function approximate the graph in figure 1, while still keeping true to all the component - property rules and functions. Using a graphing program, I was able to find a series of values we can use to do this- although a number of different series values are possible. These values make our function approximate the exact answer to the function ∫Sinx/x dx, for all positive x values, where we must multiply by -1 to obtain the result for negative x-values.
Doing this, our function becomes:

$$\int Sin(x)/x\, dx \sim \frac{1.5}{e^{\left(\frac{Sin(x-1.955)-(0.5)(x+3.142)Sin(x-1.955)}{x^2}\right)}}$$

Where x is always positive. For negative x values, simply multiply the result of the final function by -1.

If you wanted to enter this into a graphing program, you would type:
(1.5/(exp((sin(x-1.955)-((0.5)(x+3.142)sin(x-1.955)))/(x^2))))
This function closely approximates all the positive x-values shown in figure 1, although I will leave the perfecting of this technique to evaluate ∫Sin(x)/x dx to others. In time, a combinational approximating integral hybrid shall be generated for ∫Sin(x)/x dx, that will be universally agreed upon and taken to be equal to the exact answer- within the limits of all the component – properties.

Deriving ∫Cos(x)/(x) dx

The steps are as follows: derive components and properties of the integral as expected by conventional calculus, then combine these components and properties in a way that approximates the exact solution. Let us derive our first property or component of the integral ∫Cos(x)/(x) dx:

From conventional calculus, we know that:
d/dx (Cos(x)/x) = -Sin(x)/x – Cos(x)/x^2
Proof: The product rule: Y' = UV' + VU'
Let Y= UV, where U = 1/x, and V= Cos(x)
Therefore U'=-1/x^2, V' = -Sin(x)
Hence: d/dx (Cos(x)/x) = -Sin(x)/x – Cos(x)/x^2

Now, lets multiply both sides by –x:
-x d/dx (Cos(x)/x) = Sin(x) + Cos(x)/x
Re- arranging gives:
-x d/dx (Cos(x)/x) -Sin(x) = Cos(x)/x
Since ∫-x = -x^2/2:
d/dx (-x^2/2) *d/dx (Cos(x)/x) -Sin(x) = Cos(x)/x
Integrating both sides gives us:
∫[d/dx (-x^2/2) *d/dx (Cos(x)/x) - Sin(x)] = ∫Cos(x)/x dx

Using a calculus rule I derived myself: **∫[d/dx(A) . d/dx(B)] dx = ∫[d/dx(A . B)] dx** This becomes:

$\int\{d/dx\ [(-x^2/2) * (\mathrm{Cos}(x)/x)] - \mathrm{Sin}(x)\}dx = \int \mathrm{Cos}(x)/x\ dx$

Multiplying the derivative terms in the derivate bracket on the left hand side gives us:

$\int\{d/dx\ [(-x^2\mathrm{Cos}(x))/2x] - \mathrm{Sin}(x)\}\ dx = \int \mathrm{Cos}(x)/x\ dx$

Simplifying the derivative gives us:

$\int\{d/dx\ [(-x\mathrm{Cos}(x))/2] - \mathrm{Sin}(x)\}\ dx = \int \mathrm{Cos}(x)/x\ dx$

From conventional calculus, we know that:

$d/dx\ [(-x\mathrm{Cos}(x))/2] = x\mathrm{Sin}(x)/2 – \mathrm{Cos}(x)/2$

Proof: The product rule: Y’ = UV’ + VU’

Let Y= UV, where U = -x/2, and V= Cos(x)

Therefore U’=-1/2, V’ = -Sin(x)

$d/dx\ [(-x\mathrm{Cos}(x))/2] = x\mathrm{Sin}(x)/2 – \mathrm{Cos}(x)/2$

Inserting this into our equation gives us:

$\int\{x\mathrm{Sin}(x)/2 – \mathrm{Cos}(x)/2 - \mathrm{Sin}(x)\}\ dx = \int \mathrm{Cos}(x)/x\ dx$

Since $\int -\mathrm{Sin}(x)\ dx = \mathrm{Cos}(x)$:

$\int\{x\mathrm{Sin}(x)/2 – \mathrm{Cos}(x)/2\}dx + \mathrm{Cos}(x) = \int \mathrm{Cos}(x)/x\ dx$

Now, from conventional calculus, we know that:

$\int\{x\mathrm{Sin}(x)/2 – \mathrm{Cos}(x)/2\}dx = (-x\mathrm{Cos}(x))/2 – (\mathrm{Sin}(x))/2 + (\mathrm{Sin}(x))/2$

Inserting this into our equation, gives us:

$(-x\mathrm{Cos}(x))/2 – (\mathrm{Sin}(x))/2 + (\mathrm{Sin}(x))/2 + \mathrm{Cos}(x) = \int \mathrm{Cos}x/x\ dx$

Simplifying this expression gives us:

$(-x\mathrm{Cos}(x))/2 + \mathrm{Cos}(x) = \int \mathrm{Cos}(x)/x\ dx$……….(Component – Property 1)

Here, we have derived our first component - property of the ∫Cos(x)/x dx integral. This component alone is not the solution, but shall be part of a larger function to approximate the exact solution of ∫Cos(x)/x dx.

Now, Lets find another component – property:

Lets look at the equation: $\mathrm{Sin}^2(x) + \mathrm{Cos}^2(x) = 1$.

Dividing both sides by x^2 gives us:

$\mathrm{Sin}^2(x)/x^2 + \mathrm{Cos}^2(x)/x^2 = 1/x^2$

Re-arranging in terms of $\mathrm{Cos}^2(x)/x^2$ gives us:

$\mathrm{Cos}^2(x)/x^2 = 1/x^2 - \mathrm{Sin}^2(x)/x^2$

Taking the positive root of both sides yields:

$|\mathrm{Cos}(x)/x| = [(1 - \mathrm{Sin}^2(x))/x^2]^{1/2}$

Integrating both sides yields:

$\int |\mathrm{Cos}(x)/x| = \int[(1 - \mathrm{Sin}^2(x))/x^2]^{1/2}$

From previously, it was shown that according to conventional calculus, ∫Cos(x)/(x) dx = Cos(x)-xCos(x)/2. Thus in this case, we may say:

∫│Cos(x)/x│ = ∫Cos(x)/(x) dx = Cos(x)-(xCosx)/2……….(Component – Property 2)

Thus, here we see an interesting component - property. We see that ∫Cos(x)/(x) dx only takes on absolute values. In other words, when constructing a function as a solution to this integral, we shall only look at positive x values, as you cannot take the square root of a negative number.

Since an integral is equal to the area under a graph, we can evaluate how the exact solution will appear as a graph. Please refer to figure 2.

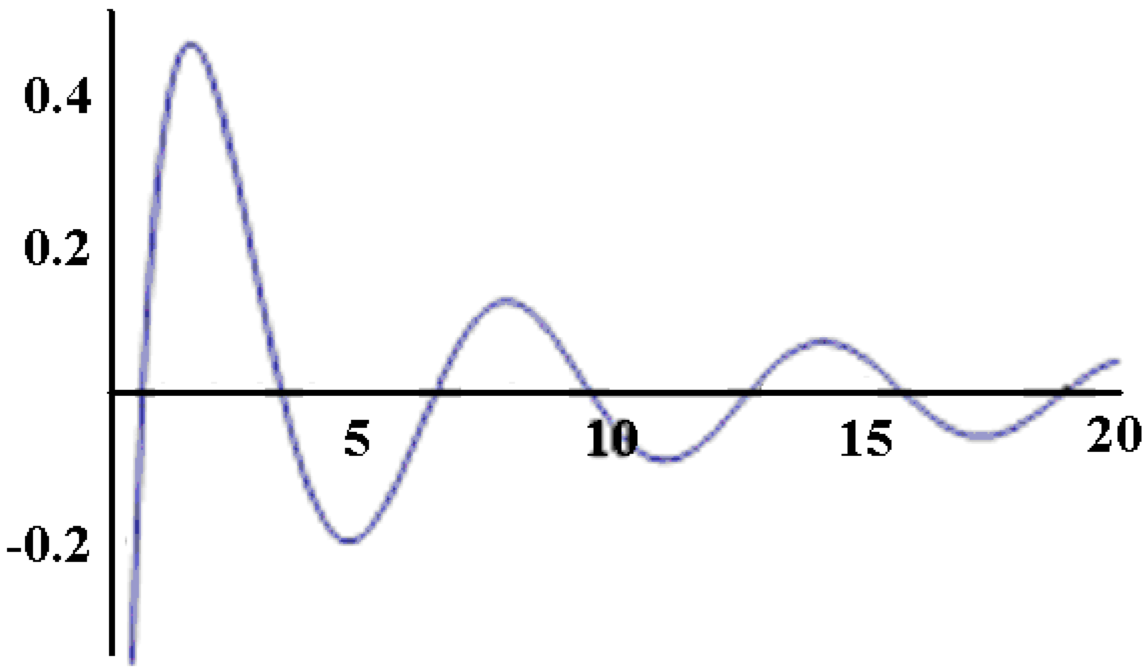

Figure 2: The graph of ∫Cos(x)/x dx, from Wolfram. Link:
http://reference.wolfram.com/mathematica/ref/CosIntegral.html

Here, we see the graph we must approximate. From component – property 2 identified earlier, we realize that an elementary solution to this problem must ignore negative x values. In other words, to calculate negative x values for our function, we must use positive x values, then simply multiply the final answer by -1.

From Figure 2, we see that our graph oscillates about the x- axis, as a normal Cos function would. We therefore do not need to shift our graph up or down, but can leave it as it is. Our base function in this case will therefore be equal to 1. Let this be component – property 3.

Also notice how after reaching it's first peak, the peaks in the graph reduce as the x values increase. The graph of d/dx (Cos(x)/x) = -Sin(x)/x – Cos(x)/x^2 behaves this way also, because part of the function was divided by x^2. Therefore, dividing by x^2 shall be component - property 4.

Since component - property 3 describes the behavior of the overall function we want, we will make this our base function, and unifying this component property with component properties 1, 2 and 4, we may arrive at the theoretical combinational approximating integral hybrid expression:

$$\int \mathrm{Cos}(x)/(x)\,dx \sim D\left(\frac{\mathrm{Cos}(x \pm A)-(B)(0.5x)\mathrm{Cos}(x \pm C)}{x^2}\right)$$

Where A, B, C, and D are constants to be adjusted to closely approximate the graph of ∫Sinx/x dx on the positive x axis only. For calculations involving negative x-values, we must use positive x values, then simply multiply the final answer by -1. Also note that this theoretical combinational approximating integral hybrid is just one theoretical form that may be generated.

Now comes the fun part- we must adjust the values of A, B, C and D to make our function approximate the graph in figure 2, while still keeping true to all the component - property rules and functions. Using a graphing program, I was able to find a series of values we can use to do this- although a number of different series values are possible. These values make our function approximate the exact answer to the function ∫Cosx/x dx, for all positive x values, where we must multiply by -1 to obtain the result for negative x-values.
Doing this, our function becomes:

$$\int \mathrm{Cos}(x)/(x)\,dx \sim 0.95\left(\frac{\mathrm{Cos}(x-4.72)-(0.89)(3.142)(0.5x)\mathrm{Cos}(x-4.72)}{x^2}\right)$$

Note: in this expression, I have set the value of B to (0.89 * 3.142).
If you wanted to enter this into a graphing program, you would type:
((0.95)((cos(x-4.72)-((0.89)(3.142)(0.5x)cos(x-4.72)))/(x^2)))
This function closely approximates all the positive x-values shown in figure 2, although I will leave the perfecting of this technique to evaluate ∫Cos(x)/x dx to others. In time, a combinational approximating integral hybrid shall be generated for ∫Cos(x)/x dx, that will

be universally agreed upon and taken to be equal to the exact answer- within the limits of all the component – properties.

__Update:__ some even better approximations I have found, ***__valid for all values of x__***, are:

$$\int \frac{\text{Sin}(x)}{x} \simeq [\text{sgn}(x)]\left[\frac{-\text{Cos}\left(|x| - \frac{\pi}{7}\right)}{|x| + \frac{\pi}{5.5}} + \frac{\pi}{1.999}\right] \text{,For all x values} \quad \text{..........equation (1}$$

$$\int \frac{\text{Cos}(x)}{x} \simeq \left[\frac{\text{Sin}\left(|x| - 0.2\right)}{|x| + \frac{\pi}{5}}\right] \text{,For all x values} \quad \text{..........equation (2}$$

To enter equation 1 into a graphing program, you would type:
(sgn(x))*((-cos(|x|-(π/7)))/(|x|+(π/5.5))+(π/1.999))

To enter equation 2 into a graphing program, you would type:
(sin(|x|-0.2))/(|x|+(π/5))

Note: sgn is known as the **signum or sign function, thus sgn(x)= +1 or -1**

To compare these approximations (equations 1 and 2) with the actual graphs, I used the online wolfram graphing program, at:
http://www.wolframalpha.com/widget/widgetPopup.jsp?p=v&id=1f7f038e713da9929b0f9a3ce48f97cd&title=Math+Help+Boards:+Graph+Plotter&theme=blue
See figures 3 and 4 below:

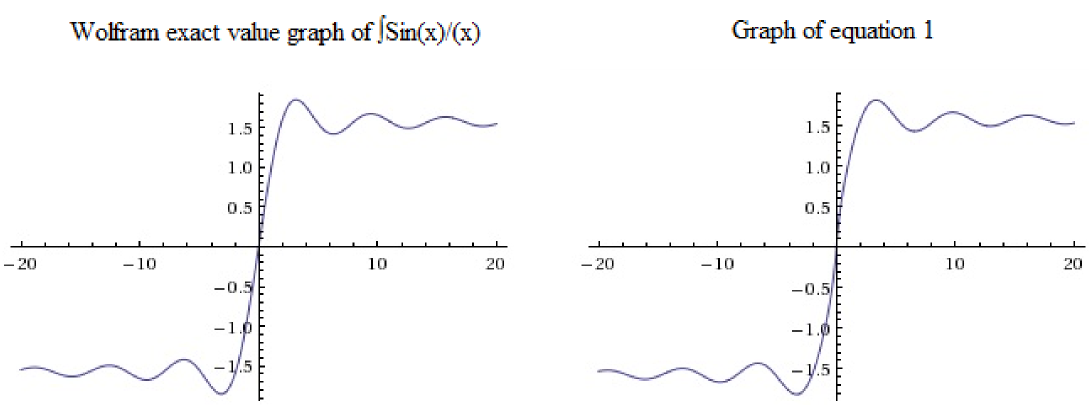

Figure 3: A comparison of the exact value graph of ∫Sin(x)/x, with equation 1:

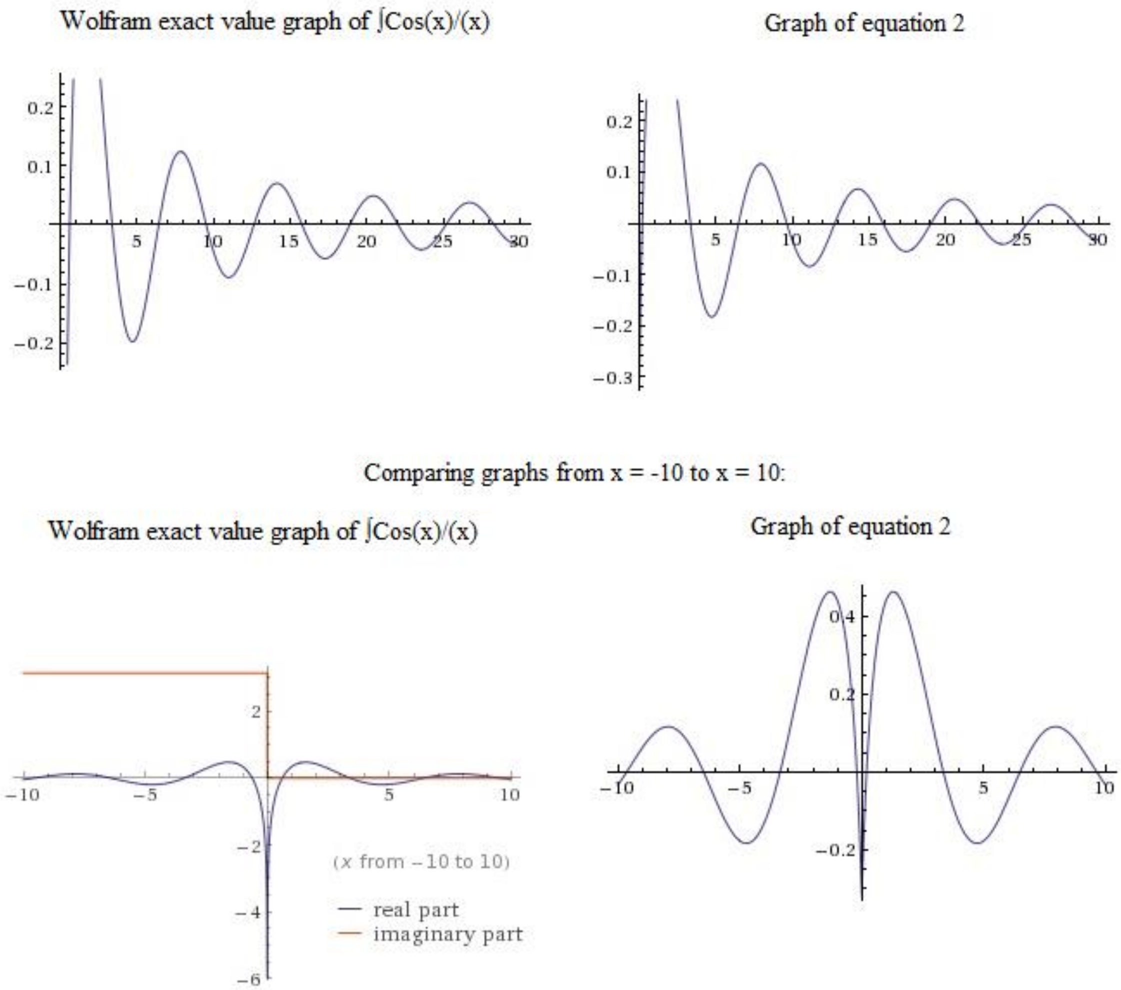

Figure 4: A comparison of the exact value graphs of ∫Cos(x)/x, with equation 2:

As can be seen, equations 1 and 2 almost exactly match the exact values.

The integral of Tan(x)/(x) is a tough one. I have not been able to find any reference material on it to compare a graph with. Still, since the integral of a graph is equal to the area under the graph, I can make a prediction as to what the integral would look like. I predict that:

$$\int \frac{\mathrm{Tan}(x)}{x} \approx \frac{-\,\mathrm{Ln}\left(\left|\mathrm{Cos}(x-0.0005)\right|\right)}{x+0.0005}$$

for all x values

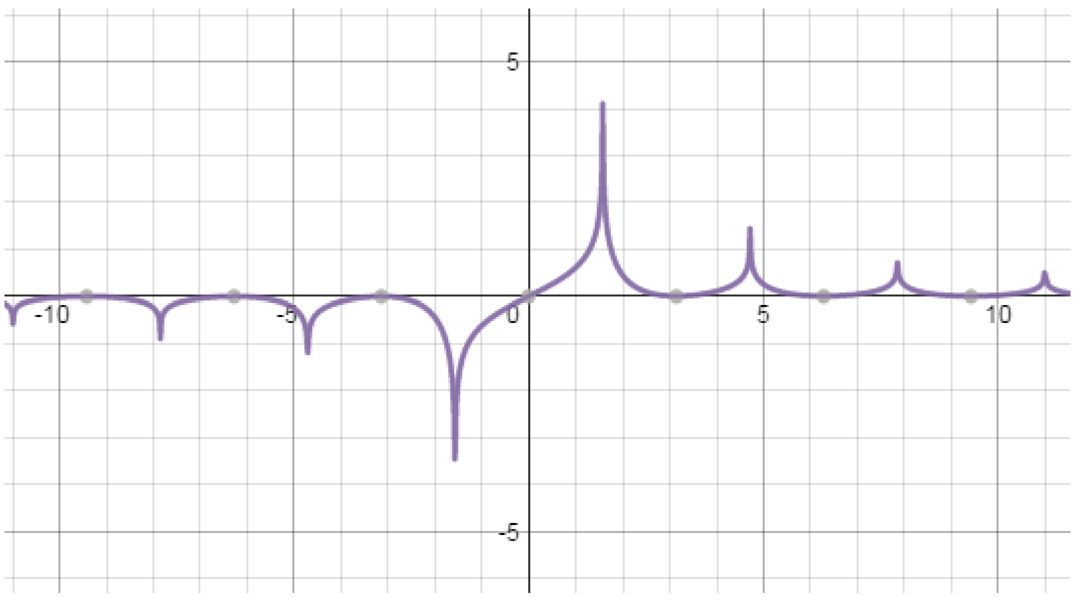

Figure 5: The graph of (–ln(abs(cos(x-0.0005))))/(x+0.0005), from the Desmos online graphing calculator, at: https://www.desmos.com/calculator

If we combine the graph of Tan(x)/(x) with (–ln(abs(cos(x-0.0005))))/(x+0.0005), we have figure 6 below:

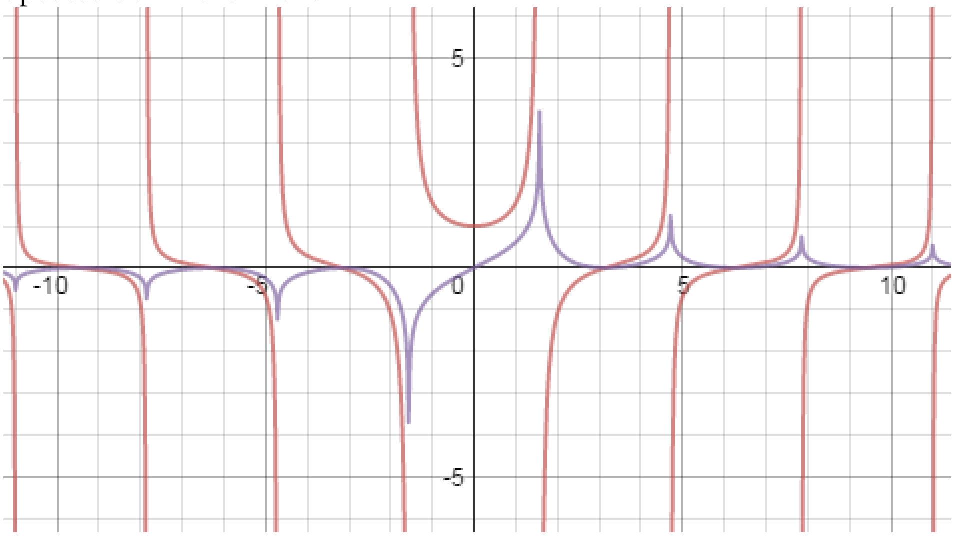

Figure 6: The graph of (–Ln(abs(cos(x-0.0005))))/(x+0.0005), (purple graph) and Tan(x)/x (red graph) from the Desmos online graphing calculator, at: https://www.desmos.com/calculator . Starting at zero, as the red graph curves up to infinity, the area below it increases, but then the red graph curves up to zero from negative infinity, the area decreases back to its original value. Hence, I predict that the purple graph may approximate the integral of the red graph. I need more data to verify this.

References:

Desmos online integrator (2013). Link: https://www.desmos.com/calculator
Link last accessed 30th March, 2015.

Wolfram online graphing program, link:
http://www.wolframalpha.com/widget/widgetPopup.jsp?p=v&id=1f7f038e713da9929b0f9a3ce48f97cd&title=Math+Help+Boards:+Graph+Plotter&theme=blue
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Wolfram online integrator (2013). Link: http://integrals.wolfram.com/
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