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An Optical Model of Gravitation: Schrödinger-Like behavior of ElectroMagnetic Stationary Waves

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Abstract
In this work, we extend the gravitational-optical analogy by incorporating an effective refractive index $n_{eff}(r) = 1 + \sigma\frac{R_s}{r}$ into Maxwell's equations. This modification leads to a generalized Helmholtz equation describing the propagation of electromagnetic waves in a spherically symmetric inhomogeneous medium. Reformulating the problem in radial coordinates, we derive a wave equation for $u(r)$ which exhibits a formal analogy with the radial Schrödinger equation. However, the physical interpretation of the wave function $\psi(r)$ differs – it represents the amplitude of a stationary electromagnetic wave, not a probability amplitude. We analyze both $\ell = 0$ and $\ell \neq 0$ cases using WKB-method, identifying a dimensionless gravitational coupling constant $\eta = \sigma k R_s$, analogous in role to the fine-structure constant in the Coulomb problem. Our results reveal gravitationally induced phase shifts and asymptotic modulations of the wave function, with possible observational consequences near compact astrophysical objects such as neutron stars and black holes. This work lays the groundwork for a covariant formulation and opens new perspectives on gravitational wave analogies in optical media.

Keywords
Gravitational optics; Effective refractive index; EM stationary waves; Gravitational phase shift; Helmholtz equation; Radial wave equation; Schrödinger-like analogy; Variable light speed; Metric-optical equivalence; Wave propagation in curved spacetime; WKB-method.

Introduction

Understanding the influence of gravitation on the propagation of light remains a cornerstone of relativistic physics [1], [2], [9]. General Relativity (GR) explains the phenomenon through the curvature of spacetime induced by mass-energy, leading to measurable effects such as gravitational lensing [2], [15], redshift, and Shapiro delay, [6], [7], [8].
In contrast to this geometrical interpretation, the present work develops an alternative but complementary approach based on a gravitationally-induced optical index, where spacetime curvature is effectively modelled by a spatially refractive index, [3], [4].
This fifth publication extends the theoretical foundation of this model by studying the behavior of stationary electromagnetic (EM) waves in a medium where the local speed of light varies with the gravitational potential. Starting from a modified Helmholtz equation, derived from Maxwell's equations with a position-dependent speed of light $c(r) = c/n_{eff}(r)$, the radial behavior of the EM field is analyzed using analogies with quantum mechanics – particularly the radial Schrödinger equation. Though the physical interpretation differs, this formal analogy reveals new insights how gravity could modulate the phase and amplitude of EM fields.
This framework provides a novel interpretation of classical relativistic effects through the lens of wave optics, using an effective refractive index of the form: $n_{eff}(r) = 1 + \sigma\frac{R_s}{r}$; [17].

The solution to the resulting wave equations, their asymptotic behavior, and potential observables are rigorously analyzed.
A specific attention is given to the two cases of the number of angular modes or the multipolar index: $\ell = 0$ and $\ell \neq 0$, as well as a WKB-resolution for weak gravitational fields.
This approach opens new possibilities of connecting gravitational influence on light to laboratory-scale refractive effects, thereby suggesting potential pathways for experimental investigation beyond standard relativistic frameworks.

1- Foundations of the optical-gravitational analogy from Maxwell's equations

1-1 Definition of the effective refractive index

General relativity describes gravity as a curvature of spacetime [1], [2], [9]. However, in the regime of weak gravitational fields and high-frequency electromagnetic (EM) waves, an alternative approach can offer complementary insight: gravity may be viewed as inducing an optical medium [3], [4], [15], [16], [17]. In this framework, the presence of mass modifies the velocity of light as if space were endowed with a spatially varying refractive index [11].
The idea leads to a gravitationally induced optical model where the EM wave is studied within an inhomogeneous dielectric medium instead of vacuum. This approach preserves the essence of relativistic effects while using the mathematical tools of classical wave propagation.
In order to establish a sound foundation for our optical-gravitational analogy, we begin with the source-free Maxwell's equations in vacuum. Then, we introduce a modification through an effective position-dependant speed of light $c(r)$. This variation is interpreted as a gravitational influence modelled via a radially varying dielectric permittivity. In a previous publication [17], we calculated the local speed of light as:

$$c(r) = \frac{c}{n_{eff}(r)}; \ with \ n_{eff}(r) = 1 + \sigma \frac{R_s}{r} \quad (1)$$

Where R_s is the Schwarzschild radius of the massive body [5], that creates a weak gravitational field. If we remind the expression of the constant light celerity in the vacuum, we can write:

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \quad (2)$$

Where ε_0; μ_0 are respectively the permittivity, the magnetic permeability in the vacuum. If we replace equation (2) in the local light speed expression (1), it leads to:

$$c(r) = \frac{1}{\sqrt{n_{eff}^2(r)\varepsilon_0 . \mu_0}} \quad (3)$$

If we suppose that the magnetic permeability μ_0 is kept constant, corresponding to non-magnetic vacuum, then the expression (3) reveals a possible definition of a radially-dependant permittivity as:

$$\varepsilon(r) = n_{eff}^2(r)\varepsilon_0 \rightarrow \boxed{n_{eff}(r) = \frac{c}{c(r)} = \sqrt{\frac{\varepsilon(r)}{\varepsilon_0}}} \quad (4)$$

So, the effective refractive index $n_{eff}(r)$ finds its origin from the relative dielectric permittivity in the optical medium. This quantity is introduced for taking into account the gravitational field. This establishes a direct link between gravitational influence and optical response, encoded in $n_{eff}(r)$, without requiring full relativistic spacetime curvature.
The local light speed becomes, after some transformation of equation (3), with equation (4):

$$\boxed{c(r) = \frac{1}{\sqrt{\varepsilon(r)\mu_0}}} \quad \text{and} \quad \boxed{n_{eff}(r) = c\sqrt{\varepsilon(r)\mu_0}} \tag{5}$$

1-2 Maxwell's equations in an inhomogeneous medium

We recall the standard Maxwell's equations in a non-magnetic, source free, isotropic medium where $\varepsilon(r)$ is the radially dependent permittivity introduced previously to account for the gravitational field:

$$\begin{cases} \vec{\nabla}.\vec{\mathrm{E}} = 0 \\ \vec{\nabla}.\vec{\mathrm{B}} = 0 \end{cases} \qquad \begin{cases} \vec{\nabla} \times \vec{\mathrm{E}} = -\frac{\partial \vec{\mathrm{B}}}{\partial t} \\ \vec{\nabla} \times \vec{\mathrm{H}} = \frac{\partial \vec{\mathrm{D}}}{\partial t} \end{cases} \tag{6}$$

With the constitutive equations:

$$\vec{\mathrm{D}} = \varepsilon(r)\vec{\mathrm{E}}; \qquad \vec{\mathrm{B}} = \mu_0\vec{\mathrm{H}} \tag{7}$$

We replace these equations by the only vectors fields, electric $\vec{\mathrm{E}}$ and magnetic $\vec{\mathrm{H}}$, we get some slightly transformed equations (6) as:

$$\begin{cases} \vec{\nabla}.\vec{\mathrm{E}} = 0 \\ \vec{\nabla}.\vec{\mathrm{H}} = 0 \end{cases} \qquad \begin{cases} \vec{\nabla} \times \vec{\mathrm{E}} = -\mu_0\frac{\partial \vec{\mathrm{H}}}{\partial t} \\ \vec{\nabla} \times \vec{\mathrm{H}} = \varepsilon(r)\frac{\partial \vec{\mathrm{E}}}{\partial t} \end{cases} \tag{8}$$

First, let us take the curl of the Faraday's law:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{\mathrm{E}}\right) = -\mu_0\frac{\partial(\vec{\nabla}\times\vec{\mathrm{H}})}{\partial t} = -\varepsilon(r)\mu_0\frac{\partial^2\vec{\mathrm{E}}}{\partial t^2} \tag{9}$$

And, second, let us combine a result of the vectorial analysis with equation (8) as:

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{\mathrm{E}}\right) = \vec{\nabla}\left(\vec{\nabla}.\vec{\mathrm{E}}\right) - \nabla^2\vec{\mathrm{E}} = -\nabla^2\vec{\mathrm{E}} \tag{10}$$

So, by putting equation (9) in (10) and taking into account equation (8), it leads to a variant of the vectorial Helmholtz equation as:

$$\nabla^2\vec{\mathrm{E}} - \varepsilon(r)\mu_0\frac{\partial^2\vec{\mathrm{E}}}{\partial t^2} = 0 \tag{11}$$

Considering the definition (5) of the effective refractive index $n_{eff}(r)$ which introduces the light speed in vacuum c, we obtain the generalized wave equation for the electric field $\vec{\mathrm{E}}$ as:

$$\boxed{\nabla^2\vec{\mathrm{E}} - \frac{n_{eff}^2(r)}{c^2}\frac{\partial^2\vec{\mathrm{E}}}{\partial t^2} = 0} \tag{12}$$

Similarly with the magnetic field $\vec{H}$, we can demonstrate that the wave equation has the same expression, a well-known result:

$$\boxed{\nabla^2\vec{H} - \frac{n_{eff}^2(r)}{c^2}\frac{\partial^2\vec{H}}{\partial t^2} = 0} \quad (13)$$

1-3 Optical analogy and radial Helmholtz equation

We consider a monochromatic electromagnetic wave $\Psi(r,\theta,\varphi,t)$ propagating in a spherically symmetric gravitational field, described by an effective refractive index $n_{eff}(r)$ and assume the following decomposition, stationary, time-harmonic mode, written as follows:

$$\boxed{\Psi(r,\theta,\varphi,t) = \psi(r)Y_{lm}(\theta,\varphi)e^{-i\omega t}} \quad (14)$$

The first term of the Helmholtz equation (12) is the Laplacian operator of the function $\Psi(r,\theta,\varphi,t)$, the calculations (given in Annex 1) lead to:

$$\nabla^2\Psi = \left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right)\right]Y_{lm}e^{-i\omega t} + \left[\frac{1}{r^2}\left(\frac{1}{sin}\frac{\partial}{\partial\theta}\left(sin\theta.\frac{\partial Y_{lm}}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2 Y_{lm}}{\partial\varphi^2}\right)\right]\psi e^{-i\omega t} \quad (15)$$

From the last parenthesis appears the Laplacian angular operator of Cassini such as:

$$\nabla_\Omega^2 Y_{lm}(\theta,\varphi) = \left\{\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta.\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_{lm}(\theta,\varphi) \quad (16)$$

In Annex 1, the following equation (17) can be obtained where ℓ, the number of angular modes or the multipolar index, only refers to the angular part and to the corresponding symmetry properties of the EM wave as:

$$\nabla^2\Psi = \left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) - \frac{l(l+1)}{r^2}\psi\right]Y_{lm}(\theta,\varphi)e^{-i\omega t} \quad (17)$$

Then, by considering the second time derivative (see Annex 2) in the Helmholtz equation (12), the vacuum wave number $\boxed{k = \frac{\omega}{c}}$. appears and, after simplification by the term $Y_{lm}(\theta,\varphi)e^{-i\omega t}$, we find the radial Helmholtz equation:

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + \left[k^2 n_{eff}^2(r) - \frac{l(l+1)}{r^2}\right]\psi = 0} \quad (18)$$

Considering the following substitution: $\psi(r) = \frac{u(r)}{r}$, and after some calculations reported in Annex 3, it leads to the radial Helmholtz equation which reduces to a Schrödinger-like equation:

$$\boxed{\frac{d^2u}{dr^2} + \left[k^2 n_{eff}^2(r) - \frac{\ell(\ell+1)}{r^2}\right]u(r) = 0} \quad (19)$$

This section 1 justifies rigorously why the refractive index can encode gravitational influence in an equivalent optical medium.

The proposed substitution of $\psi(r)$ is consistent with spherical wave decomposition. The assumption of radial-only dependence is entirely valid in the context of spherical symmetry and matches with the Schwarzschild-like nature of the effective medium.

1-4 A radial Helmholtz equation analogous to a radial Schrödinger-like equation

The link between the variable dielectric permittivity $\varepsilon(r)$ and the refractive index $n_{eff}(r)$, defined by equation (5), anchors the formal analogy in classical electromagnetism.
Now, with the definition (1) of $n_{eff}(r)$, up to the first order in R_s/r in weak gravitational field, we can obtain the following expression for $n_{eff}^2(r)$:

$$n_{eff}(r) = 1 + \sigma\frac{R_s}{r} \rightarrow n_{eff}^2(r) = \left(1 + \sigma\frac{R_s}{r}\right)^2 = 1 + \frac{2\sigma R_s}{r} + \mathcal{O}\left(\left(\frac{R_s}{r}\right)^2\right) \quad (20)$$

By considering the two main terms of the development, the radial equation (19) becomes:

$$\frac{d^2u}{dr^2} + \left[k^2 + \frac{2\sigma k^2 R_s}{r} - \frac{\ell(\ell+1)}{r^2}\right]u(r) = 0 \quad (21)$$

Last, the structure of the equation (21) paves the way for exploring analogies with quantum mechanics, especially hydrogen atoms. The equation (21) resembles the radial Schrödinger equation for the Coulomb potential in quantum mechanics as:

$$\boxed{\frac{d^2u}{dr^2} + \left[\frac{2\mu E}{\hbar^2} - \frac{2\mu Ze^2}{\hbar^2 r} - \frac{\ell(\ell+1)}{r^2}\right]u(r) = 0} \quad (22)$$

By analogy, we can define a gravitational coupling parameter η, dimensionless as follows:

$$\boxed{\eta = \sigma k R_s} \quad (23)$$

Then, by incorporating this parameter η in the radial equation (21), it leads to the differential equation to solve in our problem of a wave propagation of stationary spherical EM:

$$\boxed{\frac{d^2u}{dr^2} + \left[k^2 + \frac{2\eta k}{r} - \frac{\ell(\ell+1)}{r^2}\right]u(r) = 0} \quad (24)$$

We must be very careful with the formal analogy between both equations (22) and (24) since the involved quantities belong to two different physical domains as explained after:

Domain	Quantum mechanics	Optical Gravitational Model
Wave function $\psi(r)$	Probability amplitude of a particle	Electric field amplitude of the EM wave
Potential $V(r)$	Coulomb interaction $-\frac{2\mu Ze^2}{\hbar^2 r}$	Effective gravitational refractive effect $+2\sigma k^2 R_s/r$
Coupling constant, dimensionless	$\frac{\mu Ze^2}{\hbar^2 k}$	$\eta = \sigma k R_s$
Constant k	Related to energy: $E = \hbar^2 k^2/2\mu$	Vacuum wave number: $k = \omega/c$

The gravitational term is positive, meaning that it enhances the phase velocity locally, unlike the attractive Coulomb potential.

2- Transformation of the radial Helmholtz equation and its resolution

2-1 Looking for a dimensionless variable $\boldsymbol{\rho}$

It is possible to transform the radial Helmholtz equation (24) by dividing all terms by the constant k^2, it then leads to:

$$\frac{1}{k^2}\frac{d^2u}{dr^2} + \left[1 + \frac{2\eta}{kr} - \frac{\ell(\ell+1)}{(kr)^2}\right]u(r) = 0 \tag{25}$$

Naturally, a dimensionless variable visible, noted $\boxed{\rho = kr}$, can transform the equation (25) in (26) where the expression between brackets is dimensionless and the new function to solve in this differential equation is now: $u(\rho)$

$$\boxed{\frac{d^2u}{d\rho^2} + \left[1 + \frac{2\eta}{\rho} - \frac{\ell(\ell+1)}{(\rho)^2}\right]u(\rho) = 0} \tag{26}$$

This expression is very interesting since it is analogous to a Schrödinger-like equation with an analogous "effective potential" defined as follows:

$$V_{eff}(\rho) = -\frac{2\eta}{\rho} + \frac{\ell(\ell+1)}{(\rho)^2} \tag{27}$$

Considering the dimensionless radial Helmholtz equation (26) in the presence of an effective gravitational potential, we must look at the two expressions in the brackets, compared to 1.
In a <u>weak gravitational field</u> where this model applies, characterized by the ratio $\frac{R_S}{r} \ll 1$, considering the definition of η, the gravitational coupling parameter (23), the <u>potential term</u> $\frac{2\eta}{\rho}$ can be expressed as follows where one simplification occurs by the vacuum wave number k:

$$\boxed{\frac{2\eta}{\rho} = 2.\frac{\sigma k R_S}{kr} = 2\sigma.\frac{R_S}{r} \ll 1} \tag{28}$$

This term remains small (despite the possible high value of η), due to the $1/r$ decay at large distances or high frequencies.
Similarly, the <u>centrifugal term</u> $\frac{\ell(\ell+1)}{(\rho)^2}$ vanishes at large distances or high frequencies $\rho \gg 1$.

2-2 General resolution by the Wentzel-Kramers-Brillouin (WKB) method

The Wentzel-Kramers-Brillouin method is explained hereafter; this method of resolution assumes that the solution $u(\rho)$ can be written as follows:

$$\boxed{u(\rho) = A(\rho)e^{i\Phi(\rho)}} \tag{29}$$

With two functions $A(\rho)$ and $\Phi(\rho)$ to be determined.
We can now compute the first and second ρ-derivatives before replacing in the radial Helmholtz equation (26). In order to simplify the notations, we state that the quantity between brackets is a known function $Q(\rho)$ defined as follows:

$$\boxed{Q(\rho) = 1 + \frac{2\eta}{\rho} - \frac{\ell(\ell+1)}{(\rho)^2}} \qquad (30)$$

The first derivative gives:

$$\frac{du}{d\rho} = (A' + iA\Phi')e^{i\Phi} \qquad (31)$$

And the second derivative leads to the following expression, with much terms:

$$\frac{d^2u}{d\rho^2} = (A'' + 2iA'\Phi' - A(\Phi')^2 + iA\Phi'')e^{i\Phi} \qquad (32)$$

If we insert the second derivative in the equation (26), we can collect terms as:

$$[A'' + 2iA'\Phi' - A(\Phi')^2 + iA\Phi'' + QA]e^{i\Phi} = 0 \qquad (33)$$

Regrouping the real and the imaginary parts, and dividing by $e^{i\Phi}$, we have:

$$[A'' - A(\Phi')^2 + QA] + i[2A'\Phi' + A\Phi''] = 0 \qquad (34)$$

Let first, consider the imaginary part of the complex equation (34), it will lead to the determination of the quantity $A(\rho)$ from what it is called the WKB conservation since:

$$2A'\Phi' + A\Phi'' = 0 \quad or \quad \frac{d}{d\rho}(A^2\Phi') = 0 \quad or \quad A^2\Phi' = cte \qquad (35)$$

Let second, consider the real part, according to the WKB approximation where $A'' \ll A(\Phi')^2$, valid when Φ' varies slowly with ρ and the potential is "smooth"; this yields to:

$$A(\Phi')^2 = QA \qquad (36)$$

Simplifying by A never null, we can calculate the first derivative: Φ' of the phase since Q is a known function. In paragraph 2-1, we saw that both terms, in condition of weak gravitational field, gravitational and angular momentum terms, are small compared to 1, leading to the positivity of the Q function.

$$(\Phi')^2 = Q = 1 + \frac{2\eta}{\rho} - \frac{\ell(\ell+1)}{(\rho)^2} \rightarrow \Phi'(\rho) = \sqrt{Q(\rho)} = \sqrt{1 + \frac{2\eta}{\rho} - \frac{\ell(\ell+1)}{(\rho)^2}} \qquad (37)$$

In that case, we can calculate the total phase shift Φ by the general expression:

$$\boxed{\Phi(\rho) = \int \sqrt{1 + \frac{2\eta}{s} - \frac{\ell(\ell+1)}{(s)^2}}\, ds} \qquad (38)$$

This expression reveals that the total phase shift accumulates along ρ. Since Φ' is chosen positive, we can calculate the amplitude function $A(\rho)$, such as:

$$A^2\Phi' = cte = a^2 \quad \rightarrow \quad \boxed{A(\rho) = \frac{a}{\sqrt{\Phi'(\rho)}}} \qquad (39)$$

As we are in condition of weak gravitational field, we can determine the asymptotic solution by integrating equation (38) when the square root expression is approximated.

2-3 Exact solution $u(\rho)$ by the Wentzel-Kramers-Brillouin method

If we remind equation (37), the derivative of the phase shift is given by the square root of the Q known function as:

$$\Phi'(\rho) = \sqrt{Q(\rho)} = \sqrt{1 + \frac{2\eta}{\rho} - \frac{\ell(\ell+1)}{(\rho)^2}} \qquad (40)$$

Then, the phase shift is given by the integration of this expression:

$$\Phi(\rho) = \int \sqrt{1 + \frac{2\eta}{s} - \frac{\ell(\ell+1)}{(s)^2}}\, ds \qquad (41)$$

The two functions, which allows $u(\rho)$ to be calculated, can be assembled to give the exact real solution of the radial Helmholtz equation (26), if we introduce a constant which depends on ℓ (if non null) as follows:

$$\boxed{u(\rho) = \frac{a}{\sqrt{\Phi'(\rho)}} sin\left[\int \sqrt{1 + \frac{2\eta}{s} - \frac{\ell(\ell+1)}{(s)^2}}\, ds + \delta_l\right] = \frac{a}{[Q(\rho)]^{1/4}} sin\left[\int \sqrt{Q(s)} ds + \delta_l\right]} \qquad (42)$$

Despite the fact that we produce the exact solution, the physical interpretation of equation (42) is not easy at all. In order to fulfil this objective, we must develop, at the first order, the square root since we are in condition of weak gravitational field.

2-4 Asymptotic solution $u(\rho)$ and physical interpretation

We analyse the phase shift $\Phi(\rho)$ when $\rho \gg 1$, by first expanding the square root, in the expression of: $\Phi'(\rho)$, using a binomial approximation:

$$\Phi'(\rho) = \sqrt{Q(\rho)} = \sqrt{1 + \frac{2\eta}{\rho} - \frac{\ell(\ell+1)}{(\rho)^2}} \approx 1 + \frac{\eta}{\rho} - \frac{\ell(\ell+1)+\eta^2}{2(\rho)^2} \qquad (43)$$

By putting into equation (41), easy integrations can be performed as:

$$\Phi(\rho) \approx \int \left[1 + \frac{\eta}{s} - \frac{\ell(\ell+1)+\eta^2}{2(s)^2}\right] ds = \rho + \eta . ln(\rho) + \frac{\ell(\ell+1)+\eta^2}{2\rho} + cte \qquad (44)$$

So, the WKB asymptotic solution can be rewritten as follows:

$$\boxed{u(\rho) \approx \frac{a}{\sqrt{\Phi'(\rho)}} sin(\rho + \eta . ln\rho + \delta_l) = \frac{a}{[Q(\rho)]^{1/4}} sin(\rho + \eta . ln\rho + \delta_l)} \qquad (45)$$

Where δ_l is a constant phase term which only depends on boundary conditions.

Physical interpretation of the phase $\Phi(\rho)$.
The leading term ρ corresponds to the flat phase propagation of the EM wave.

The logarithmic term $\eta. ln\rho$ represents the phase delay due to gravitational slowing down of light in our model where $\eta = \sigma R_s k$ encodes the strength of the gravitational field.

The last term decay in $1/\rho$ contains contributions from both the angular momentum barrier and the second-order gravitational effects.

The asymptotic solution (45) confirms that:
- in the high-frequency and/or large ρ limit, the phase shift accumulates logarithmically due to the gravity-induced index variation;
- the WKB method remains valid even for large η;
- in case of $\ell \neq 0$, a sinusoidal form is well justified with an additive phase shift.

Both cases $\ell \neq 0$ and $\ell = 0$ are treated according to the same WKB method.

2-5 Asymptotic solution $u(r)$ and wave amplitude $\psi(r)$.
In our problem, we consider stationary electromagnetic waves propagating in a gravitational optical medium, we have:

$$\psi(r) = \frac{u(r)}{r} \tag{46}$$

Where, for example, $\psi(r)$ is the amplitude of the electric field, measured in V/m. The unit of $u(r)$ is given by the equation: $[u(r)] = [\psi(r)][r] = \left(\frac{V}{m}\right).m = V$. We have used a dimensionless variable $\rho = kr$, however this does not change the physics of our problem, then:

$$u(r) = u(\rho) \tag{47}$$

According to equation (45), the asymptotic solution of our problem leads to the calculation of the amplitude $\psi(r)$ of the electric field as follows:

$$\begin{cases} \psi(r) = \frac{1}{r}.\frac{a}{[Q(kr)]^{1/4}} sin(kr + \eta.\ln(kr) + \delta_l) \\ Q(kr) = 1 + \frac{2\eta}{kr} - \frac{\ell(\ell+1)+\eta^2}{(kr)^2};\ \eta = \sigma k R_s \end{cases} \tag{48}$$

The gravitational effects given by the η constant, which depends on the Schwarzschild radius of the astrophysical object, influence both the amplitude and the phase shift of the electromagnetic wave.

3- Validity of the solution and numerical estimates for astrophysical objects

3-1 Limitations of the optical gravitational model and consequences
Our model is applicable in conditions of weak gravitational field or is valid for the exterior of the Schwarzschild radial variable r, it has the following consequences considering the definition of the gravitational coupling constant η:

$\frac{R_s}{r} \ll 1 \;\; or \;\; r \gg R_s \rightarrow kr = \rho \gg kR_s = \frac{\eta}{\sigma} = \rho_{min}$ (49)

So, it means that an inferior limitation ρ_{min} is naturally introduced in the analysis of all functions of the dimensionless variable ρ. As we mainly found the value 1 for the σ-parameter in other publications; the validity of our approach leads to:

$$\boxed{\rho \gg \rho_{min} = \eta \quad \rightarrow \quad \frac{\eta}{\rho} \ll 1} \tag{50}$$

In order to apply the WKB-method, the ratio $\frac{\eta}{\rho}$ must be less than 1 in the expression of the asymptotic development of $\Phi'(\rho)$ at the first order, it is exactly the case:

$$\Phi'(\rho) = \sqrt{1 + \frac{2\eta}{\rho} - \frac{\ell(\ell+1)+\eta^2}{(\rho)^2}} = 1 + \frac{\eta}{\rho} + \mathcal{O}\left(\left(\frac{\eta}{\rho}\right)^2\right) \tag{51}$$

One strong condition for applying the WKB-method is that the function $\Phi'(\rho)$ evolutes slowly with the variable ρ. It is the case in our problem for the analysis of the propagation of EM waves in our gravitational-optical medium.
As $\boxed{\rho_{min} = kR_s}$, it means that this value depends both of the Schwarzschild radius R_s of the astrophysical object (its mass) and of the frequency (or wave-length) of the considered EM wave. It also means that the gravitational effects are more pronounced for high frequency EM waves (high k value) and/or massive objects (high R_s).
In the next paragraph, we will estimate numerically the phase shift due to gravitational effects only and encoded by the logarithmic term: $\eta.\ln(kr)$.

3-2 Numerical estimates for 3 different astrophysical objects

Let us now compute numerical estimates of the asymptotic phase shift accumulated due to gravity in our optical model for various astrophysical objects. We will base our estimates on the WKB-solution deduced from equation (48), given by the following equation, limited at the first order:

$$\boxed{\Phi(r) \approx kr + \eta.\ln(kr) \qquad with \quad \eta = \sigma k R_s = \sigma R_s.\frac{2\pi}{\lambda}} \tag{52}$$

For a given astrophysical massive object, the Schwarzschild radius R_s is defined as follows:

$$R_s = \frac{2GM}{c^2} \tag{53}$$

The σ-parameter is equal to 1 in our optical model and λ is the wavelength of the EM waves chosen as: $\lambda = 500\ nm = 5.00\ 10^{-7}\ m$ (visible light). The other quantities are well known constants in fundamental physics as:

- the gravitational constant: $G = 6.674\ 10^{-11}\ m^3kg^{-1}s^{-2}$
- the celerity of light in the vacuum: $c \approx 3.00\ 10^8\ ms^{-1}$.

In order to differentiate the numerical estimates of the gravitational phase shift: $\eta.\ln(kr)$, we will study 3 different astrophysical objects as the Earth, the Sun and a neutron star.
All the results are presented in the following table.

Object	M (kg)	Rs (m)	η	$\eta.\ln(kr)$
Earth	$5.97\ 10^{24}$	$8.87\ 10^{-3}$	$1.11\ 10^{5}$	$4.60\ 10^{6}$
Sun	$1.99\ 10^{30}$	$2.95\ 10^{3}$	$3.70\ 10^{10}$	$1.54\ 10^{12}$
Neutron star	$3.98\ 10^{30}$	$5.90\ 10^{3}$	$7.40\ 10^{10}$	$3.08\ 10^{12}$

These are huge phase shifts, showing that our model predicts a significant effect of gravitational index modulation on wave propagation – even in weak fields (like the Earth).
The logarithmic phase delay becomes very large due to the multiplicative factor η.
Interpretation
This phase shift affects interference patterns, resonance conditions and propagation delays. In principle, it could be measurable in high-precision optical or radio-interferometry, especially around compact stars. The WKB approximation justifies using the phase shift formula:

$$\boxed{\Phi(r) \approx kr + \eta.\ln(kr)} \tag{54}$$

The perturbative method, in quantum mechanics, breaks down for the analysis of EM wave propagation in our optical medium, due to the large value of η.

4- How could we observe or measure the gravitational phase shift $\boldsymbol{\delta(r) = \eta.ln(kr)}$?

We would like to explore a few realistic approaches to detect or make this gravitational phase shift experimentally relevant.

4-1 Interferometric methods

In principle, gravitationally induced phase shifts could be detected by comparing the phase of an EM wave travelling through two different paths:

- One passing near a massive object (e.g. Sun),
- Another path far away, in "free space".

This is analogous to gravitational lensing however we do not measure just the deflection but phase shift as:

$$\boxed{\Delta\phi = \eta.ln\left(\frac{kr_{near}}{kr_{far}}\right) = \eta.ln\left(\frac{r_{near}}{r_{far}}\right)} \tag{55}$$

Challenges: it requires very high-precision phase control at astronomical distances; however, it seems hard to differentiate this from the other propagation effects (scattering, etc).

4-2 Phase shifts in Pulsar Timing or Quasar Light

Pulsars emit periodic EM signals. If such a signal passes near a massive object, it will accumulate a gravitational phase shift according to our model.
If one could track changes in the arrival phase as the path changes (e.g. during a gravitational conjunction), this might reflect the cumulative $\delta(r)$.
What is needed? A precise timing, high-frequency radio waves (smaller λ, larger k, enhancing the gravitational coupling constant: η).

4-3 Scattering/Interference patterns in strong gravitational fields

This method considers how interference fringes change a wave propagates near a strong gravitational field. The optical model, presented in this paper, predicts a modulation of the amplitude due to the term of the asymptotic WKB-solution:

$$\boxed{\psi(r) \sim \frac{1}{r}\sin\left(kr + \delta(r)\right)} \tag{56}$$

If $\delta(r)$ varies significantly with r, the fringe pattern changes; this might be visible in:

- high resolution radio-telescopes observing sources lensed by neutron stars or black holes;
- carefully analysing frequency-dependant patterns.

4-4 Laboratory analogue models

Our model is based on an optical analogy, so why not <u>engineer a material</u> with:

$$n_{eff}(r) = 1 + \frac{R_s}{r} \tag{57}$$

Or approximate it using gradient-index (GRIN) optics? Then, it could be possible to measure the phase shift of light across different paths within the medium. It could be done on Earth, it could offer a direct visualization and could connect with experimental analogue gravity platforms.

4-5 Summary of Observational Strategies

The following table gathers all the possible strategies to validate our model.

Method	Feasibility	Required precision	Comments
Interferometry near massive body	Low	Extremely high	Analogous to gravitational Aharonov-Bohm effect
Pulsar timing (phase delay)	Moderate	Nanosecond-scale	Promises if signal passes close to massive object
Interference/fringe pattern shifts	Moderate	High	Needs very accurate modelling
Optical analogue material lab test	High	Lab-scale precision	Best near-term option for controlled confirmation

Conclusion

In this study, we have developed a theoretical framework for modelling the propagation of stationary electromagnetic waves in a gravitationally modified optical medium, characterized by an effective refractive index: $n_{eff}(r) = 1 + \sigma \frac{R_s}{r}$.

By recasting Maxwell's equations into a generalized Helmholtz equation in spherical symmetry, we have revealed a deep formal analogy with the radial Schrödinger equation, while emphasizing the distinct physical interpretation of the wave amplitude $\psi(r)$. Through analytical techniques, mainly the WKB approximation (unlike the perturbation theory in quantum mechanics), we explored both the $\ell = 0$ and $\ell \neq 0$ regimes and demonstrated how gravitational effects induce non-negligible phase shifts and amplitude modulations.

Our analysis introduces a gravitational coupling constant: $\eta = \sigma k R_s$, whose potentially large magnitude in astrophysical contexts challenges standard perturbative approaches and reinforces the relevance of semiclassical methods. The resulting predictions—such as logarithmic phase shifts and gravitational lensing-like modulations—may be of observational significance near compact objects like neutron stars or black holes.

This work sets the stage for further investigations, including the formulation of a fully covariant model of wave propagation in curved spacetime analogues, and opens intriguing prospects for modelling gravitational wave phenomena through optical analogies.

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Annex 1: Laplacian operator of the function: $\boldsymbol{\Psi(r,\theta,\varphi,t)=\psi(r)Y_{lm}(\theta,\varphi)e^{-i\omega t}}$
In spherical coordinates (r,θ,φ), the Laplacian operator leads to the following expression:

$$\nabla^2\Psi=\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right)+\frac{1}{r^2}\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta.\frac{\partial\Psi}{\partial\theta}\right)+\frac{1}{r^2}\frac{1}{sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2} \qquad \text{(a)}$$

Considering the decomposition in functions which only depends on the radial coordinate for the radial amplitude $\psi(r)$, and on the angular coordinates (θ,φ) for the function $Y_{lm}(\theta,\varphi)$, the Laplacian operator can be slightly modified in the following expression:

$$\nabla^2\Psi=\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right)\right]Y_{lm}(\theta,\varphi)e^{-i\omega t}+\left[\frac{1}{r^2}\left(\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta.\frac{\partial Y_{lm}}{\partial\theta}\right)+\frac{1}{sin^2\theta}\frac{\partial^2Y_{lm}}{\partial\varphi^2}\right)\right]\psi(r)e^{-i\omega t} \qquad \text{(b)}$$

After extraction of the variable r in the angular bracket expression, the Cassini angular operator emerges as:

$$\nabla^2_\Omega Y_{lm}(\theta,\varphi)=\left\{\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta.\frac{\partial}{\partial\theta}\right)+\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_{lm}(\theta,\varphi) \qquad \text{(c)}$$

Then the Laplacian operator applied to the Ψ function leads to:

$$\nabla^2\Psi=\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right)\right]Y_{lm}e^{-i\omega t}+\left\{\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta.\frac{\partial}{\partial\theta}\right)+\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_{lm}(\theta,\varphi)\frac{\psi(r)}{r^2}e^{-i\omega t} \qquad \text{(d)}$$

The spherical harmonics $Y_{lm}(\theta,\varphi)$ are the proper functions of the angular Cassini operator, so, it leads to the number of angular modes or the multipolar index ℓ which appears, defined as:

$$\left\{\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta.\frac{\partial}{\partial\theta}\right)+\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_{lm}(\theta,\varphi)=-\ell(\ell+1)Y_{lm}(\theta,\varphi) \qquad \text{(e)}$$

Then, the simplified expression for the Laplacian operator is:

$$\nabla^2\Psi=\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right)\right]Y_{lm}e^{-i\omega t}-\ell(\ell+1)Y_{lm}\frac{\psi(r)}{r^2}e^{-i\omega t} \qquad \text{(f)}$$

If the product $Y_{lm}e^{-i\omega t}$ is factorized, the Laplacian operator leads to:

$$\boxed{\nabla^2\Psi=\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right)-\ell(\ell+1)\frac{\psi}{r^2}\right]Y_{lm}e^{-i\omega t}} \qquad \text{(g)}$$

This is exactly the equation (17) in the text.

Annex 2: obtention of the radial Helmholtz equation for the electric field amplitude
We will begin with the vectorial field equation, identical for the electric or magnetic field:

$$\boxed{\nabla^2\vec{E}-\frac{n^2_{eff}(r)}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=0} \qquad \text{(h)}$$

If we replace the vector $\vec{E}$ by its complex expression $\Psi(r,\theta,\varphi,t)$, then we have:

$$\nabla^2\Psi(r,\theta,\varphi,t) - \frac{n_{eff}^2(r)}{c^2}\frac{\partial^2\Psi(r,\theta,\varphi,t)}{\partial t^2} = 0 \quad \text{(i)}$$

For a time-harmonic stationary wave propagation, we have written the following time-harmonic mode decomposition:

$$\Psi(r,\theta,\varphi,t) = \psi(r)Y_{lm}(\theta,\varphi)e^{-i\omega t} \quad \text{(j)}$$

The first term (Laplacian operator) of equation (i) does not change with time and was treated in Annex 1; however, it is not the case for the second term of equation (i) since we must take the second time-derivative of this complex number as follows in two steps:

$$\frac{\partial}{\partial t}[\psi(r)Y_{lm}(\theta,\varphi)e^{-i\omega t}] = \psi(r)Y_{lm}(\theta,\varphi)\frac{\partial}{\partial t}[e^{-i\omega t}] = \psi(r)Y_{lm}(\theta,\varphi)(-i\omega)e^{-i\omega t} = (-i\omega)\Psi(r,\theta,\varphi,t) \quad \text{(k)}$$

And, for the second time-derivative:

$$\frac{\partial^2}{\partial t^2}\Psi(r,\theta,\varphi,t) = \frac{\partial}{\partial t}[(-i\omega)\Psi(r,\theta,\varphi,t)] = (-i\omega)^2\Psi(r,\theta,\varphi,t) = -\omega^2\Psi(r,\theta,\varphi,t) \quad \text{(l)}$$

Replacing in equation (b), it leads to:

$$\nabla^2\Psi(r,\theta,\varphi,t) + \omega^2\frac{n_{eff}^2(r)}{c^2}\Psi(r,\theta,\varphi,t) = 0 \quad \text{(m)}$$

The Laplacian operator leads to the following expression, already given in the text by equation (18), and entirely demonstrated in Annex 1:

$$\nabla^2\Psi(r,\theta,\varphi,t) = \left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) - \frac{l(l+1)}{r^2}\psi\right]Y_{lm}(\theta,\varphi)e^{-i\omega t} \quad \text{(n)}$$

Considering this previous equation in the equation (m), it leads to the radial Helmholtz equation in the brackets to be solved in our problem:

$$\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) - \frac{l(l+1)}{r^2}\psi + \omega^2\frac{n_{eff}^2(r)}{c^2}\psi\right]Y_{lm}(\theta,\varphi)e^{-i\omega t} = 0 \quad \text{(o)}$$

After simplification by the term $Y_{lm}(\theta,\varphi)e^{-i\omega t}$, the radial Helmholtz equation is found as follows, by putting the radial amplitude in factor for the second and third terms in brackets:

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + \left[k^2n_{eff}^2(r) - \frac{l(l+1)}{r^2}\right]\psi = 0} \quad \text{(p)}$$

The only radial amplitude function $\psi(r)$, a scalar field, is the unknown in our problem.

Annex 3: obtention of a Schrödinger-like equation for the electric field amplitude

If we start from the previous equation (p):

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) + \left[k^2n_{eff}^2(r) - \frac{\ell(\ell+1)}{r^2}\right]\psi(r) = 0 \quad \text{(q)}$$

As our problem is formally analogous to a Coulomb's problem, we operate the following substitution for sake of simplicity as:

$$\psi(r) = \frac{u(r)}{r} \rightarrow \frac{d\psi}{dr} = \frac{u'.r-u}{r^2} \quad \text{(r)}$$

Then, we calculate:

$$\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = \frac{d}{dr}(u'.r-u) = u''.r+u'-u'=u''.r \quad \text{(s)}$$

By replacing in the radial Helmholtz equation, the first is written as follows:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = \frac{1}{r^2}\left(r\frac{d^2u}{dr^2}\right) = \frac{1}{r}\frac{d^2u}{dr^2} \quad \text{(t)}$$

After simplification by $1/r$, the equation to analyse takes the following form as:

$$\boxed{\frac{d^2u}{dr^2} + \left[k^2n_{eff}^2(r) - \frac{\ell(\ell+1)}{r^2}\right]u(r) = 0} \quad \text{(u)}$$

This equation is a Schrödinger-like equation in the Coulomb's problem in quantum mechanics; however, the analogy is purely formal and must be taken with care in our problem.