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Towards A Discrete Cosmology 3/17

Finite versus infinite

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Abstract.-Once made the necessary distinction between the actual infinity and the potential infinity, this paper 3 of the series proves that the infinity of the Axiom of Infinity can only be the actual infinity, and that ω-ordered collections are inconsistent, which in turn implies the inconsistency of the Axiom of Infinity itself, and the inconsistency of any infinite set when considered as a complete totality. It is also shown here (and the results will be much used in subsequent discussions) that every set is either discrete or can be ordered discretely (where being discrete means having a first and a last element and that every element (except the first) has an immediate predecessor, and an immediate successor (except the last). Obviously, the infinitist mathematics of modern physics (rarely put to the test) will be seriously affected by the inconsistency of the actual infinity (fortunately experimental physics can only be finitist and discrete). The consequences of this conclusion will be deduced in this and the following articles of the series. Here, one of such physical consequences will be demonstrated: in a consistent reality only a finite number of universes (if more than one) could exist, each with a finite number of physical objects.

Keywords: actual infinity, potential infinity, Dedekind definition, axiom of infinity, Hilbert's machine, inconsistency of ω-order, inconsistency of the actual infinity, inconsistency of actual infinite sets, theorem of the finite universe.

1. The actual and the potential infinity

(This section includes published texts by the author [37, p. 32-35])

In common parlance, the word infinite is used to refer to the quality of being immense, gigantic, unlimited, etc. C. F. Gauss (*Princeps Mathematicorum* [73, p. 1188]) said that infinity is a way of speaking (C.F. Gauss, Letter to astronomer H.C. Schumacher, 12 July 1831 [26, Vol. II, p. 268]):

> I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a *façon de parler* [a way of speaking], the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction.

The consideration of an infinite magnitude (or an infinite sequence, for instance of numbers) as something completed is what we call *actual infinity* since Aristotle, who introduced the distinction between the potential infinity and the actual infinity [5, 6, Books III, VIII]. It is remarkable the fact that in the above quotation, Gauss implicitly includes the distinction between both infinities (see below in this section). I have the impression that most physicists think, like Gauss, in terms of the potential infinity, without worrying about the fact that they are building physics with the mathematics of the actual infinity.

As will be seen below, it is possible to give a precise definition of the concept infinity, albeit based on a primitive concept: the concept of set. However, the concept of set could be defined in operational, non-Platonic terms [37, p. 360]:

Definition 1 (of Set) *A set is the theoretical object that results from a mental grouping of different arbitrary objects previously defined.*

This definition has the advantage of avoiding the entanglements caused by self-reference, simply by requiring that the elements to be grouped be previously defined, which seems quite reasonable if we intend to know what we are grouping. It is convenient, on the other hand, to consider that some sets exist as complete totalities, i.e. as sets satisfying the following:

Definition 2 (of Complete Totality) *A complete totality is a set defined by comprehension in which every element that satisfies the corresponding membership definition of the set is in the set.*

In consequence, to a complete totality of a certain type of elements, it is not possible to add new elements of that type because it already contains *all of them*.

But returning to the concept of infinity, and apart from Gauss's opinion, the word "infinite" has also a precise meaning based on the primitive concept of set:

Definition 3 (of Infinite Set) *A set is said infinite if it can be put into a one to one correspondence with one of its proper subsets.*

which is the well known Dedekind's definition of infinite set [19, p. 115], an important element of the foundations of modern infinitist mathematics, which began its development at the end of the 19th century. As is well known, the controversial history of the (philosophical and) mathematical infinity has its roots in the pre-Socratic times, although here we are not interested in the details of that history (there is an abundant and excellent literature on the history of infinity, for instance: [74, 41, 63, 9, 60, 16, 38, 45, 47, 35, 36, 1, 48, 46, 15, 69, 7, 59]).

Now I will try to explain the distinction between the two infinities, the actual and the potential. The set of the natural numbers and *supertask theory* are two suitable instruments to evidence such a distinction. The set of the natural numbers needs no presentation. With respect to supertask theory it must be recalled that it is an infinitist theory based, as set theory, on the Axiom of Infinity (introduced in Section 2). It originated as a consequence of a seminal discussion about the possibility, or impossibility, of performing an infinite number of actions (tasks) in a finite time interval. [66, 10, 65, 70, 8].

Although the main objective of supertask theory was, and continue to be, the discussion on the actual infinity, its physical implications (including special relativity) have also been discussed in the last years [52, 53, 57, 62, 29, 31, 30, 53, 54, 55, 24, 56, 49, 3, 4, 58, 71, 33, 22, 23, 49, 21, 64]. In short:

> A supertask consists in performing an infinite sequence of actions $\langle a_i \rangle$ within a finite time interval $[t_a, t_b)$, each action a_i being performed at the precise instant t_i of a strictly increasing and convergent sequence of instants $\langle t_i \rangle$ within $[t_a, t_b)$, being t_b the mathematical limit of $\langle t_i \rangle$.

where the elements of $\langle a_i \rangle$ and $\langle t_i \rangle$ are ordered in the same way as the set of the natural numbers in their natural order of precedence: ω-order: 1, 2, 3,.... Notice in this ordering the set exist as a complete totality (Definition 2) and each element n has an immediate successor $n+1$ (Peano's Axiom of the Successor, [51, p. 1]), where immediate successor is defined according to:

Definition 4 (of Immediate Successor) *All elements of an ordered set A succeeding (preceding) a given element n of A are successors (predecessors) of n in the considered order of A. An element n of an ordered set A is said the immediate successor (predecessor) of another element m of A if n succeeds (precedes) m in the considered ordering of A and no other element of A exists between m and n in that ordering.*

We are now in the appropriate position to analyze the difference between the actual and the potential infinity, Indeed, consider the list L_n of the natural numbers in their natural order of precedence:

$$L_n = 1, 2, 3, \ldots \tag{1}$$

The list L_n can be considered in two different ways:

a) As a complete totality, i.e. as a list in which every element that could be in the list, is in the list (actual infinity).

b) As an unlimited and uncompletable totality (potential infinity).

According to the Hypothesis of the Actual Infinity, the list L_n of the natural numbers in their natural order of precedence 1, 2, 3,... exists as a *complete totality*, i.e as a totality that contains, all at once, all natural numbers. The ellipsis (...) in $1, 2, 3, \ldots$ stands for *all* natural numbers. For all. The word "actual" in *actual infinity* means, therefore, that all elements of an infinite collection as L_n, exist all at once (in the *act*), as a complete totality. In consequence, the list L_n of the natural numbers in their natural order of precedence is considered as a complete totality despite the fact that no last number completes the list. To assume the Hypothesis of the Actual Infinity means, therefore, to assume that it is possible to complete the incompletable, as Aristotle would surely say [6, p. 291]. Or that the incompletable can exist as complete.

To emphasize this sense of completeness, let us consider the task of counting the successive elements of L_n, i.e. the successive natural numbers 1, 2, 3,... in their natural order of precedence. In agreement with the Hypothesis of the Actual Infinity we could count *all* natural numbers in a finite time, for example in an hour, or in a millisecond. The task of counting all natural numbers in a finite time interval, even in less than a second, is an example of supertask:

- *Count each of the successive natural numbers 1, 2, 3... at each of the successive instants t_1, t_2, t_3... of a strictly increasing sequence of instants $\langle t_i \rangle$ within the finite real interval (t_a, t_b), being t_a and t_b any two instants such that $t_a < t_b$, and t_b the mathematical limit of the sequence $\langle t_i \rangle$. For instance, the classical sequence defined by:*

$$t_n = t_a + (t_b - t_a)\,\frac{2^n - 1}{2^n} \tag{2}$$

As we will now prove, at t_b all natural numbers would have been counted. All. In effect, let each natural number n of the list L_n be counted at the precise instant t_n of $\langle t_i \rangle$. Being t_b the limit of $\langle t_i \rangle$, t_b is the first instant after all instants of $\langle t_i \rangle$, and all those instants do exist as a complete totality according to the Hypothesis of the Actual Infinity. So, the one to one correspondence f between L_n and $\langle t_i \rangle$ defined by:

$$f(n) = t_n, \ \forall n \in L_n \tag{3}$$

proves that at t_b all natural numbers of the list L_n has been counted. All. The reader can easily imagine why ellipsis and correspondences between sets are the key instruments for demonstrations in infinitist mathematics. Note, on the other hand, that the fact of pairing the elements of two infinite sequences (in our case the one of natural numbers and the other of instants) does not prove both sequences exist as complete totalities. They could also be potentially infinite with the same number of elements, a possibility usually ignored in modern infinitist mathematics.

The alternative to the Hypothesis of the Actual Infinity is the Hypothesis of the Potential Infinity, which rejects the existence of *complete* infinite totalities, and then the possibility to count all natural numbers. From this perspective, the natural numbers result from the *endless* process of counting: it is always possible to count a number greater than any given number (Peano's Axiom of the Successor [51, p. 1]). But it is impossible to complete the process of counting all of them, simply because there is not a last natural number to complete the process. So, the complete list of all natural numbers makes no sense, simply because it is incompletable.

The word "potential" in *potential infinity* means, therefore, that the elements of an infinite collection do not exist all at once, but potentially, as possible. The potential infinity is *the unlimited*, as the list L_n of the natural numbers in their natural order of precedence, but only finite collections can be considered as complete totalities, as large as wished but always finite. Similarly, only finite natural numbers can be considered, as large as wished but always finite. For the potential infinite there is not a last natural number (it is always possible to consider a number greater than any previously considered number), but neither is there the complete collection of *all* natural numbers. Contrarily to the actual infinity, the potential infinity assumes the incompletable cannot be completed, cannot exist as complete, precisely because it is not completable.

In short, the actual infinite hypothesis states that the infinite collections are complete totalities, even if no last element completes the collection, as in the case of the ordered list of the natural numbers. On the contrary, the hypothesis of the potential infinite proposes that the infinite collections do not exist as complete totalities, the only complete totalities are the finite totalities, though they can be unlimited in the number of their possible elements. All of which can be summarized in the following definition:

Definition 5 (of Actual and Potential Infinity)*An ordered collection of elements is infinite if there is no last (first) element that completes (initiates) it. The collection is actually infinite if it is considered a complete totality, and potentially infinite if it is not considered a complete totality.*

where collection is by set, succession, sequence, list, etc. To be formally precise, the words *set*, *succession*, *sequence*, etc. should be replaced by the more general word *collection*. However, for the sake of brevity, it will not be necessary to do so. Therefore, in what follows all of them will be interchangeable with each other, unless otherwise specified.

The potential infinity (the 'improper' or 'non-genuine' infinity as Cantor called it [14, p. 70]) has never deserved the attention of contemporary mathematics. The infinity in Dedekind's Definition 3 of infinite set is the actual infinity (see next section). The infinitely many elements of an infinite set exist all at once, as a complete totality. Dedekind's Definition 3 is, therefore, based on the violation of the old Euclidean Axiom of the Whole and the Part (the whole is greater than the proper part) [25]. Set theory has been built on that violation.

The hegemony of the actual infinity in contemporary mathematics is absolute. As absolute as the submission of physics to infinitist mathematics. Some authors proceed as if the existence of complete infinite totalities had been formally demonstrated. Obviously, if that were the case we would not need the Axiom of Infinity to legitimize the existence of such infinite totalities. The Hypothesis of the Actual Infinity is just a hypothesis, not a proven fact. And physics should not be subject to infinitist mathematics. In fact, and in agreement with P. Dirac, it should not be subject to any kind of mathematics at all [20, p. VIII]:

> Mathematics is only a tool and one should learn to hold physical ideas in one's mind without reference to the mathematical form.

The three most important "proofs" of the existence of actual infinite totalities (by Bolzano, Dedekind and Cantor) are illustrative of what we could call *naive infinitism*. They also explain why modern infinitist mathematics had finally to establish the existence of actual infinite sets by an arbitrary law, i.e. by means of an arbitrary axiom (the Axiom of Infinity, which is introduced in the next section).

• Bolzano's proof goes as follow (taken from [46, p 112]):

> One truth is the proposition that Plato was Greek. Call this p_1. But then there is another truth p_2, namely the proposition that p_1 is true [But then there is another truth p_3, namely the proposition that p_2 is true]. And so *ad infinitum*. Thus the set of truths is infinite.

But the existence of an endless process (p_1 is true, then p_2 is true, then p_3 is true, then ...) does by no means prove the existence of a final result as a complete totality. At best it proves the existence of an endless (potentially infinite) process. But it does not prove the existence of an actual infinite totality.

• Dedekind's proof is similar (taken from [46, p 113]):

> Given some arbitrary thought s_1, there is a separate thought s_2, namely that s_1 can be object of thought [there is a separate thought s_3, namely that s_2 can be object of thought]. And so ad infinitum. Thus the set of thoughts is infinite.

The above comment on Bolzano proof also applies here. Dedekind gave another proof a little more detailed, albeit with the same formal defect, based on his definition of infinite set [19, p. 115].

• And finally, Cantor's proof: ([32, p 25], [46, p. 117]):

> Each potential infinite presupposes an actual infinity.

or ([13, p. 404] English translation [61, p. 3]):

> ... in truth the potential infinity has only a borrowed reality, insofar as a potentially infinite concept always points towards a logically prior actually infinite concept whose existence it depends on.

But this is an opinion, not a formal proof. It is now clear why the existence of an actual infinite set had to be finally established by law; that is, by means of an axiom.

Let us, finally, state a conventional use of the expressions actual infinite and potential infinity in this and the subsequent chapters of this book. From now on, and for sake of simplicity, the actual infinity will be referred to simply as infinity or actual infinity, while the potential infinity will always be referred to as potential infinity. Or put another way, the word "infinity" will always mean actual infinity, unless it is preceded by the word "potential", in which case it will obviously mean potential infinity. For the sames reasons of simplicity, the world "universe" will always denote the observable universe.

2. The infinity of the Axiom of Infinity

Nothing we have been able to observe and measure so far has been infinite. Nor has it been possible to divide anything into an infinite number of parts. On the other hand, and after more than twenty-seven centuries of arguments and discussions, it was not possible to prove (or disprove) the existence of the actual infinities. Infinitism had no choice but to accept that existence in axiomatic terms by means of the Axiom of Infinity. An axiom that simply states the existence of an infinite set:

Axiom 1 (of Infinity (ordinary language))

There exists an infinite set.

Or in abstract, symbolic, terms:

Axiom 2 (of Infinity (abstract form))

$$\exists A : \emptyset \in A \wedge \forall a \in A \ (a \cup \{a\} \in A \tag{4}$$

that reads: there exists a set A such that $\emptyset$ (the empty set) belongs to A and for every element a in A, the element $a \cup \{a\}$ also belongs to A. Although it is not explicitly declared the type of infinity involved in the set A, it can be easily proved that it is the actual infinity:

Theorem 1 (of the Actual Infinity) *The infinity in the Axiom of Infinity can only be the actual infinity.*

Proof.-Since potentially infinite sets do not exist as complete totalities, only two subsets with the same number of elements of the same potentially infinite set could be put into a one to one correspondence, and then Dedekind Definition 3 is not satisfied, because we would have a one to one correspondence between two proper subsets of a potentially infinite set, in the place of a one to one correspondence between a set and one of its proper subsets. In consequence, the infinity involved in the Axiom 1 of Infinity can only be the actual infinity. □

Obviously, an axiom is just an axiom, i.e. a statement that can be accepted or rejected. Some relevant authors as L.E.J. Brouwer, C. Hermite, S. Kleene, J. König, L. Kronecker, H. Poincaré, A. Robinson, L. Wittgenstein, or H. Weyl, among others, rejected the Axiom of Infinity, more or less explicitly. H. Poincaré went so far as to say that (quoted in [46, p. 121], [18, p. 1]):

> infinity is a perverse pathological illness that would one day be cured.

But the vast majority of contemporary mathematicians and physicists do not question the Axiom of Infinity. Indeed, in our days the criticism of the actual infinity is practically non-existent. And infinitism has become a current of thought absolutely hegemonic and quite intolerant of dissent, as if the existence of the actual infinite had been proven. And no, it has not been proven; it has been assumed. And one has the right and the duty to question that assumption, without being insulted and ostracized for it (as is currently the case).

3. A short proof of inconsistency

Over the last 30 years, and from different perspectives (set theory, supertask theory, transfinite cardinals, transfinite ordinals, transfinite arithmetics, geometry) I have developed more than forty formal proofs of the inconsistency of the Hypothesis of the Actual Infinity [37]. This section includes one of them, chosen for its brevity and simplicity: the next Theorem 5. First, however, it is necessary to consider the following formal elements:

Definition 6 (of the Types of Sets) *A set is finite if it has a definite and finite number of elements. A set is potentially infinite if it always contains a finite number of elements of a certain type and any finite numbers of new elements of that type can always be added to it, without the set ceasing to be potentially infinite and without it being necessary to change its name. Two sets are equipotent (have the same number of elements) if, and only if, there is a bijection between their respective elements.*

Definition 7 (of Inconsistent Set) *A set is inconsistent if a contradiction can be deduced from the number of its elements, or from the number of elements of at least one of its proper subsets.*

Corollary 1 (of Inconsistent Sets) *A set with the same number of elements as an inconsistent set, is also inconsistent.*

Proof.-It is an immediate consequence of Definition 7

Definition 8 (of Denumerable Set) *A set is denumerable if its cardinal is the smallest infinite cardinal $\aleph_0$ of the infinite set of all natural numbers. An infinite set is non-denumerable if its cardinal is greater than the smallest infinite cardinal $\aleph_0$.*

Definition 9 (of ω-Ordered Sets) *A set is ω-ordered if being denumerable, it has a first element, each element has an immediate successor and an immediate predecessor, except the first one which has no predecessor.*

Theorem 2 (of Denumerable Sets) *It is always possible to define a one-to-one correspondence between any two denumerable sets.*

Proof.-Let A and B be any two denumerable sets. Assume there is no one-to-one correspondence between their respective elements. In consequence, A and B would not have the same number of elements (Definition 6), which is not the case because, being both denumerable sets, they have exactly the same number of elements: just $\aleph_0$ elements (Definition 8). Therefore, there must be at least a one-to-one correspondence between the sets A and B, and then between any two denumerable sets. □

Theorem 3 (of non-Denumerable Sets) *Every non-denumerable set has denumerable proper subsets.*

Proof.- Let X be any non-denumerable set. Since its cardinal is greater than $\aleph_0$ (Definition 8), X contains proper subsets with only $\aleph_0$ elements, all of which are denumerable proper subsets of X (Definition 8). □

Theorem 4 (of Indexation) *The elements of a denumerable set can be reordered with the same order as the elements of any other denumerable set.*

Proof.-Let $A = \{a, b, c, \dots\}$ and $B = \{\alpha, \beta, \dots\}$ be any two denumerable sets. There exists at least one bijection f between the elements of A and B (Theorem 2). Consequently, f pairs each element k of A with a unique and exclusive element, say δ, of B, which can be used to exclusively index that element k of A, so that element k can be rewritten as a_δ. Consequently, the elements of the set A can be reordered and rewritten to define the set $A' = \{a_\alpha, a_\beta, a_\gamma, \dots\}$ which has exactly the same elements as A, and ordered in the same way as the elements of B. □.

The infinity of infinite sets is the actual infinity, not the potential infinity (Theorem 6 of the Axiom of Infinity). This implies the existence of certain infinite sets that are also complete totalities (Definition 2). For example the set $\mathbb{N}$ of ALL natural numbers in their natural order of precedence. It is not possible, then, to add new natural numbers to the set $\mathbb{N}$ of natural numbers because it already contains them all. And the same is true of many other numerical or non-numerical sets. For many authors, the existence of these ordered and complete totalities without a last element that completes them (or without a first element that initiates them) is a proven conclusion independent of the Axiom of Infinity. It is not. It is an existence assumed and legitimized by the Axiom of Infinity. Their existence is, therefore, as debatable as the Axiom of Infinity itself. So it is as legitimate to argue about that axiom as it is to argue about the existence of those complete totalities. This fully justifies the following:

Theorem 5 (of Denumerable Infinity) *All denumerable sets are inconsistent.*

Proof.- Let A be any denumerable set. The set A allows us to define the set A' with the same elements as A but reordered as the set $\mathbb{N}$ of natural numbers in their natural order of precedence: $A' = \{a_1, a_2, a_3, \}$ (Theorem 4). The open interval of rational numbers $(0, 1)$ is densely ordered in the natural order of precedence (represented by the symbol <) defined by the natural values of the rational numbers. It is also a denumerable set, so there exists a bijection f between A' and $(0, 1)$ (Theorem 2). Consequently, $(0, 1)$ can be reordered and rewritten as the set $\mathbb{Q}_{01} = \{q_{a_1}, q_{a_2}, q_{a_3}, \dots\}$, where $q_{a_i} = f(a_i), \forall a_i \in A'$, and the successive elements $q_{a_1}, q_{a_2}, q_{a_3}, \dots$ of $\mathbb{Q}_{01}$ are ordered by the successive natural numbers in their natural order of precedence, and not by their respective values as rational numbers. Let x now be a rational variable defined initially as q_{a_1}. And let the value of x be <-compared (i.e., compared according to the values of the rational numbers) with the successive elements of the set $\mathbb{Q}_{01}$, with x being redefined as the compared element q_{a_i} if, and only if, $q_{a_i} < x$.

For short, let us call comparison* this <-comparison and redefinition of x if, and only if, the value of the compared element is smaller than the current value of x. It is immediate to prove that for each natural number v it is possible to perform the first v comparisons* of x with the first v successive elements of $\mathbb{Q}_{01}$. Indeed, if it were not possible, there would be at least one natural number $n \leq v$ such that x could not be compared* with q_{a_n}, which is impossible because q_{a_n} is a rational number of $\mathbb{Q}_{01}$ that can be compared* with the current value of x, which is also a rational number. Once all possible comparisons* of x with the successive elements $q_{a_1}, q_{a_2}, q_{a_3}, \dots$ of $\mathbb{Q}_{01}$ have been made, the current value of x, whatever it may be, could only be the smallest rational number of that set. Indeed, if once performed all possible comparisons* of x with the successive elements of $\mathbb{Q}_{01}$ the current value of x were not the smallest rational number of $\mathbb{Q}_{01}$, there would be at least one element q_{a_n} in $\mathbb{Q}_{01}$ such that $q_{a_n} < x$. But that is impossible because n is a natural number; the first n comparisons* have been carried out; and therefore x was compared* with q_{a_n} and redefined as q_{a_n}; and in all subsequent comparisons*, x could only be redefined with values smaller than q_{a_n}. Therefore, it is impossible for $q_{a_n} < x$. But, on the other hand, it is also immediate to prove that once all possible comparisons* of x with the successive elements of $\mathbb{Q}_{01}$ have been made, the current value of x is not the smallest rational number of that set: every element of the infinite set $\{x/2, x/3, x/4 \dots\}$ is an element of $\mathbb{Q}_{01}$ smaller than x. This contradiction proves that the set A', defined exclusively with the elements of A, is inconsistent. Therefore A' and A are inconsistent (Definition 7). And A being any denumerable set, it must be concluded that all denumerable sets are inconsistent. □

Although the consistency of a mathematical proof of infinite steps is universally accepted without the need to perform all of its infinite steps, the theory of supertasks considers the possibility of performing them in finite time. In the case of the above successive comparisons* of x with each successive q_{ai} would be performed at each successive instant t_i of a strictly increasing and convergent sequence $\langle t_i \rangle$ of instants within the finite time interval (t_a, t_b), whose limit is t_b. The instant t_b is the first instant after all instants of $\langle t_i \rangle$, and therefore the first instant after having performed all possible comparisons* of x with the successive elements of Q_{01}. At the instant t_b the rational variable x will still be a rational variable with a certain value, whatever it is; and not, for example, an elephant (in which case anything could be proved). The problem is that the value of x at the instant t_b is and is not the least rational of Q_{01}. From the previous theorems, we can immediately deduce, among many others, the following results:

Corollary 2 (of Inconsistent ω-Order) *ω-ordered sets are inconsistent.*

Proof.-Since ω-ordered sets are also denumerable sets (Definition 9), they are inconsistent (Theorem 5). □

4. The axiom of infinity is inconsistent

The above Theorem 5 proves the inconsistency of any denumerable set. It is then immediate to prove the following results:

Theorem 6 (of the Axiom of Infinity) *The Axiom of Infinity is inconsistent.*

Proof.-Let us write the set A defined in Axiom 2:

$$\exists A : (\emptyset \in A \wedge \forall a \in A \ (a \cup \{a\} \in A)) \tag{5}$$

as:

$$A = \{a, s_1(a), s_2(a), s_3(a), \dots\} \tag{6}$$

where:

$$s_1(a) = a \cup \{a\} \tag{7}$$

$$s_2(a) = s_1(a) \cup \{s_1(a)\} \tag{8}$$

$$s_3(a) = s_2(a) \cup \{s_2(a)\}\} \tag{9}$$

$$s_4(a) = s_3(a) \cup \{s_3(a)\}\} \tag{10}$$

$$s_5(a) = s_4(a) \cup \{s_4(a)\}\} \tag{11}$$

$$\dots$$

Consider now the set $\mathbb{N}$ of the natural numbers, which is denumerable, and the set A defined by (6), which is the set whose existence claims the Axiom of Infinity. The one to one correspondence f between the denumerable set $\mathbb{N}$ and A defined according to:

$$f(n) = s_n(a), \ \forall n \in \mathbb{N} \tag{12}$$

proves that A is also an inconsistent set (Theorem 5 and Corollary 1). □

And from Theorems 1 and 6 it immediately follows the next three corollaries:

Corollary 3 (of the Inconsistent Infinity) *The actual infinity is inconsistent.*

Proof.-It is an immediate consequence of Theorems 1 and 6. □

Corollary 4 (of the Actual Infinite Sets) *All actual infinite sets are inconsistent.*

Proof.-It is an immediate consequence of Theorems 1 and 6. □

Corollary 5 (of Consistent Collections) *A set can be either a finite complete totality or a potentially infinite and uncompletable totality. Otherwise it is inconsistent.*

Proof.-It is an immediate consequence of Definition 5 and Corollary 4. □

Let us now recall the following definition:

Definition 10 (of Densely Ordered Sets) *If no element of a strictly ordered set has an immediate predecessor nor an immediate successor, the set is said to be densely ordered or to define a continuum.*

We can now prove the following:

Theorem 7 (of Inconsistent Dense Order)*Densely ordered sets are inconsistent.*

Proof.-Let X be a densely ordered set. Suppose X is finite. It will have a finite number of elements, say n. Let x_1 and x_2 be two elements of X such that x_2 is a successor of x_1. Since x_2 cannot be the immediate successor of x_1, there will exist between x_1 and x_2 at least one other successor x_3 of x_1. Since x_3 cannot be the immediate successor of x_1, there will exist between x_1 and x_3 at least one other successor x_4 of x_1. By repeating this argument $n-2$ times we will arrive at a successor x_{n-2} of x_1 that would have to be its immediate successor, which is impossible. Therefore, X cannot be finite. And being infinite it is inconsistent (Corollary 4). □

Corollary 6 (of the inconsistent $\mathbb{Q}$ and $\mathbb{R}$) *When considered as complete infinite totalities, the set $\mathbb{Q}$ of the rational numbers and the set $\mathbb{R}$ of the real numbers are both inconsistent.*

Proof.-It is an immediate consequence of Corollary 4, and also of Theorem 7, because they are densely ordered sets. □

Theorem 8 (of the Inconsistent Continuum) *The spacetime continuum is inconsistent.*

Proof.-The spacetime continuum is the Cartesian product (cross product) of sets $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, each of whose factors is the set $\mathbb{R}$ of real numbers. Consequently it is an inconsistent set (Corollaries 6 and 1). □

The above results on the inconsistency of the infinite sets, including the inconsistency of the continuum and of densely ordered sets, will change everything. So deconstructing the arguments that follow here and in the subsequent articles of this series of articles, will involve proving the falsity of Theorem 5 (and the falsity of each of the more than 40 independent proofs included in [37]).

Let us now consider the following:

Definition 11 (of Discrete Sets) *A set is discrete if it has a first element, a last element and each of its elements (except the first one) has an immediate predecessor and (except the last one) an immediate successor.*

And then, let finally prove the following

Theorem 9 (of Discrete Sets) *All discrete sets are finite.*

Proof.-Let A be any discrete set:

$$A = \{a, s_1(a), s_2(a), s_3(a) \dots\ s_v(a)\} \tag{13}$$

where $s_1(a)$ is the immediate successor of a; $s_2(a)$ the immediate successor of $s_1(a)$; $s_3(a)$ the immediate successor of $s_2(a)$; and so on. If an element $s_n(a)$ has a finite number n of predecessors, then its immediate successor $s_{n+1}(a)$ has also a finite number $n+1$ of predecessors: all n predecessors of $s_n(a)$ plus $s_n(a)$. And since the element $s_1(a)$ has a finite number of predecessors, just 1 predecessor, the element a, we can inductively conclude that all elements of A, including its last element, have a finite number of predecessors. Therefore, A has a finite number of elements. □

Theorem 10 (of the Strictly Ordered Sets) *Every strictly ordered set is discrete.*

Proof.-Let a be any element of any strictly ordered set A, and suppose A has not a last element. Since a is not the last element of A, there exist successors of a in A. Let us consider one such successors and denote it by a_1. For the same reasons as in the case of a, we can consider and denote by a_2 any successor of a_1 in A. For the same reasons as in the case of a_1, we can consider and denote by a_3 any successor of a_2 in A. For the same reasons as in the case of a_2, we can consider and denote by a_4 any successor of a_3 in A. We would thus have a sequence of successors of a: a_1, a_2, a_3, $a_4 \ldots$ in which there is not a last element. The bijection f between A and the ω-ordered set $\mathbb{N}$ defined by $f(a_i) = i$ proves that A, like $\mathbb{N}$, would be infinite, and therefore inconsistent (Corollary 4). Consequently, A has a last element. Exactly the same argument now referring to the predecessors of a, proves also that A has a first element. Let a now be any element of A other than the last element of A. Suppose that a has not an immediate successor. Let a_1 be any successor of a. Since a_1 is not the immediate successor of a there will exist another successor a_2 of a between a and a_1. Since a_2 is not the immediate successor of a there will exist another successor a_3 of a between a and a_2. The same argument above shows that the sequence of successors a_1, a_2, $a_3 \ldots$ of a is inconsistent. Therefore a has an immediate successor. The same argument now referring to any element b different from the first element of A proves that b has an immediate predecessor. Consequently, A is discrete (Definition 11). □

Theorem 11 (of Discrete Sets) *Every set is either discrete or discretely orderable.*

Proof.-Let A be any set. If it is strictly ordered, it is a discrete set (Theorem 10). If it is unordered and consistent, it will have a finite number n of elements. By a bijection f, each of its elements can be paired with a different natural number of the set $\mathbb{N}_n$ of the first n natural numbers in their natural order of precedence. The set A^* defined by f^{-1}:

$$A^* = \{f^{-1}(1), f^{-1}(2) \ldots f^{-1}(n)\} \tag{14}$$

is an ordered version of A, and therefore a discrete version of A (Theorem 10). □

As noted above, more than forty other different and independent arguments included in [37] reach the same conclusion about the inconsistency of the Hypothesis of the Actual Infinity subsumed in the Axiom of Infinity. This infinity is what Aristotle would surely call infinite by addition. In the next paper 4, it will be proved the inconsistency of the other Aristotelian infinitude: the infinite by division, which was the type of infinite involved in the formalized version of Zeno's Dichotomies I and II [11, 12, 67, 68, 62, 34, 69, 17, 42, 27, 28, 72, 29, 31, 30, 44, 43, 39, 40, 50, 2, 57, 62, 34, 64].

Physical models and theories work reasonably well (even very well) until the infinities appear. But physicists do not usually concern themselves with the formal consistency of the infinitist mathematics that they use in all their models and theories. Nor do they concern themselves with another problem essential to a consistent explanation of the physical world: the problem of the infinite regress (of proofs, definitions, and causes). As will be seen throughout this series of articles, it is possible to modify the infinitist models and theories used in physics by finitist and discrete versions in such a way that they remain compatible with all the accumulated empirical knowledge about the physical world. And, at the same time, they are much simpler, more physical and less extravagant than their infinitist counterparts.

5. Conclusion

If any one of the more than forty proofs of the inconsistent nature of the actual infinity given in [37] is right, then the Hypothesis of the Actual Infinity is inconsistent. One of those arguments has been reproduced here so that the reader can directly evaluate the possibility that, in fact, the Hypothesis of the Actual Infinity, and then the Axiom of Infinity were inconsistent. If so, we might draw our first two cosmological conclusions:

Theorem 12 (of the Finite Universe) *A consistent universe cannot contains an actual infinite number of physical objects.*

Proof.-It is an immediate consequence of Corollary 4. □

According to the Standard Model there exists a finite number of different elementary particles (six quarks, six leptons and five bosons), each with a different finite mass. Therefore, the following is also true:

Corollary 7 (of the Finite Mass-Energy) *The mass and the energy of the observable universe cannot be actually infinite.*

Proof.-It is an immediate consequence of the Standard Model, Theorem 12 and the mass-energy relation. □

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