Solving the Pioneer Anomaly According to the General Relativity theory

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Abstract

In this paper I'll solve the Pioneer anomaly according to the General Relativity Theory of Einstein. I depend in my explanation on the method used by Einstein to explain the light bending by gravity [4].

The Theory

The original form of the Schwarzschild metric involves anisotropic coordinates, in terms of which the velocity of light is not the same for the radial and transverse directions (pointed out by A S Eddington [5]). Eddington gave alternative formulations of the Schwarzschild metric in terms of isotropic coordinates (provided $r \geq 2GM/C^2$ [6]).

In isotropic spherical coordinates, one uses a different radial coordinate, r_1, instead of r. They are related by

$$r = r_1\left(1+\frac{GM}{2C^2 r_1}\right)^2$$

Using r_1, the metric is

$$C^2 d\tau^2 = \frac{\left(1-\frac{GM}{2C^2 r_1}\right)^2}{\left(1+\frac{GM}{2C^2 r_1}\right)^2} C^2 dt^2 - \left(1+\frac{GM}{2C^2 r_1}\right)^4 \left(dr_1^2 + r_1^2 d\theta^2 + r_1^2 \sin^2\theta \; d\phi^2\right)$$

Then for isotropic rectangular coordinates x, y, z, where

$x = r_1 \sin\theta \; \cos\phi$, $y = r_1 \sin\theta \; \sin\phi$, $z = r_1 \cos\theta$, and $r_1 = \sqrt{x^2+y^2+z^2}$

The metric then becomes

$$C^2 d\tau^2 = \frac{\left(1-\frac{GM}{2C^2 r_1}\right)^2}{\left(1+\frac{GM}{2C^2 r_1}\right)^2} C^2 dt^2 - \left(1+\frac{GM}{2C^2 r_1}\right)^4 \left(dx^2 + dy^2 + dz^2\right)$$

In the terms of these coordinates, the velocity of light at any point is the same in all directions, but it varies with radial distance r_1 (from the point mass at the origin of coordinates), where it has the value

$$C' = \frac{\left(1-\frac{GM}{2C^2 r_1}\right)}{\left(1+\frac{GM}{2C^2 r_1}\right)^3} C \qquad (1)$$

For weak gravitational field and large distances, eq. (1) will be reduced to

$$C' = \frac{8r_1 - \frac{4GM}{C^2}}{8r_1 + \frac{12GM}{C^2}} C$$

Thus $C - C' = \dfrac{\dfrac{16GM}{C^2}}{8r_1 + \dfrac{12GM}{C^2}}\; C$, thus for large $r >> \dfrac{2GM}{C^2}$, we get $C - C' = \dfrac{2GM}{C^2 r}\; C$. Thus from that

$$C' = \left(1 - \frac{2GM}{C^2 r}\right) C \qquad (1)$$

According to Einstein's solution for the light bending by gravity [4], he treated the problem in the weak field approximation, according to the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m}{r}\right) C^2 dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dl^2$$

Where $m = \dfrac{GM}{C^2}$, $dl = (dx^2 + dy^2 + dz^2)^{1/2}$. The speed of light in these coordinates is found by setting $ds = 0$, thus from that we get

$$C' = \frac{dl}{dt} = \left(1 - \frac{2GM}{C^2 r}\right) C \qquad (2)$$

Where eq. (2) is the same as eq. (1), in the case of weak gravitational field and large distances

In the Schwarzschild geometry a radially infalling particle falling from infinity with vanishing initial velocity moves on a path described by[7]

$$\left(1 - \frac{2m}{r}\right) \frac{dt}{d\tau} = 1 \quad and \quad \left(\frac{dr}{d\tau}\right)^2 = \frac{2m}{r}$$

C (speed of light) is taken to be 1 in the equations above. Thus, from these equations, we finally find that

$$\frac{dr/d\tau}{dt/d\tau} = \frac{dr}{dt} = \sqrt{\frac{2GM}{r}} \left(1 - \frac{2GM}{C^2 r}\right) [7] \qquad (3)$$

Where $\sqrt{\dfrac{2GM}{r}}$ is the speed of the particle. Finally, from eqs. (1), (2), (3), we find that; both the speed of light and the speed of any particle will be varied with radial distance *r* (from the point mass at the origin of coordinates) with the gravitational field by the same factor $\left(1 - \dfrac{2GM}{C^2 r}\right)$ (weak field approximation).

From that, we can conclude the Pioneer relativistic velocity in the weak gravitational field to be as

$$V' = \left(1 - \frac{2GM}{C^2 r}\right) V \qquad (4)$$

V' is the measured Pioneer velocity for the earth observer, V is the real Pioneer velocity, G is the gravitational constant, M is the Sun mass, C is the speed of light in vacuum, and r is the distance between the probe and the Sun.

Thus from eq. (4), in the case of Pioneer 10/11, and since they are going away from the Sun, we can conclude that their velocities should be less than their real velocities for us on the earth. Therefore that must produce a slight blue shift on top of the larger red shift [1].

The JPL analysis of unmodeled accelerations used the JPL's Orbit Determination Program (ODP) [1,2]. Over the years the data continually indicated that the largest systematic error in the acceleration residuals is a constant bias of $a_p \approx (8 \pm 3) \times 10^{-10} m/s^2$ directed toward the Sun to within the beam-width of the Pioneers' antennae[1].

According to the special relativity theory, the equation that describes the clock motion of the Pioneer probe comparing with the clock on the earth for an observer on the earth is given according to the equation

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{V^2}{C^2}}} \tag{5}$$

$\Delta t'$ is the reading of Pioneer's clock for the earth observer

Δt is the reading of the earth clock for the earth observer

We can get from eq. (5) that the Pioneer's clock motion will be slower than the earth clock motion for the earth observer. That is referring to the time dilation in relativity. But since the velocity of the Pioneer probe is less than the real velocity for the earth observer according to eq. (4), where

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{\left(1 - \frac{2GM}{C^2 r}\right)^2 V^2}{C^2}}} \tag{6}$$

The slowing rate of the clock of the Pioneer probe according to eq. (6) is less than according to eq. (5) for the earth observer. Thus, the earth observer will think that there is some force that is pulling the probe back toward the Sun, that is because of the effect of the gravitational field of the Sun. Also according to eq. (6) there is a slight blue shift on top of the larger red shift. According to eq. (5), for low velocities comparing to the speed of light, the difference between the predicted frequency and the reference frequency ν_0 as the result of the red shift is $\Delta\nu_{model}$ given as

$$\frac{\Delta \nu_{model}}{\nu_0} = \frac{V}{C} \tag{7}$$

But the observed frequency difference $\Delta\nu_{obs}$ that is given according to eq. (6)

$$\frac{\Delta \nu_{obs}}{\nu_0} = \frac{\left(1 - \frac{2GM}{C^2 r}\right) V}{C} \tag{8}$$

Thus from eqs. (7) and (8) we get

$$\left[\Delta \nu_{obs} - \Delta \nu_{model}\right] / \nu_0 = -\left(\frac{2GM}{C^2 r}\right)\frac{V}{C} \tag{9}$$

From eq. (9) we get the observed difference frequency is less than the predicted. That means there is a slight blue shift. According to the Pioneer team calculations, the observed, two-way anomalous effect by a DSN antenna can be expressed to first order in V/C as [1]

$$\left[\Delta\nu_{obs} - \Delta\nu_{model}\right]_{DSN} / \nu_0 = -\frac{2\ a_p t}{C} \tag{10}$$

By DSN convention [1], $\left[\Delta\nu_{obs} - \Delta\nu_{model}\right]_{usual} = -\left[\Delta\nu_{obs} - \Delta\nu_{model}\right]_{DSN}$

Thus from that and from eq. (9) we get

$$-\left(\frac{2GM}{C^2 r}\right)\frac{V}{C} = \frac{2a_p t}{C}$$

Thus the probe acceleration a_p, where $t = \dfrac{r}{C}$

$$a_p = -\frac{GM}{r^2}\frac{V}{C} \tag{11}$$

We can compute directly a_p from eq. (4). Since the probe is going away from the sun, thus we get

$$a_p = -\frac{dV'}{dr}C = -\frac{2GM}{r^2}\frac{V}{C} \tag{12}$$

Whereas the clock acceleration a_t in the case of the probe is going away from the Sun is given as

$$a_t = -\frac{dV'}{dr} = -\frac{2GM}{r^2}\frac{V}{C^2} = \frac{a_p}{C} \tag{13}$$

The difference between eqs. (11) and (12) is the factor of 2 that is used in eq. (10), because Pioneer team used two-and three-way data[1]. Where, we can neglect it and consider $\dfrac{\Delta\nu}{\nu_0} = \dfrac{a_p t}{C}$ and reach to eq. (12).

If we used eq. (12) to compute a_p by considering $G = 6.67\times10^{-11} m^3/kg.s^2$, $M = 1.99\times10^{30} kg$, are respectively gravitational constant and mass of the Sun. NASA data [3] show that in the very middle part (1983–1990) of the whole observational period of Pioneer 10 its radial distance from the Sun changes from $r \cong 28.8\ AU = 4.31\times10^{12} m$ to $r \cong 48.1\ AU = 7.2\times10^{12} m$, while year-mean radial velocity varies from $V_p = 15.18\times10^3 m/s$ to $V_p = 12.81\times10^3 m/s$. Respective values of the "relativistic deceleration" values for this period computed with the help of eq. (12) vary from $a_p = -7.26\times10^{-10} m/s^2$ to $a_p = -2.46\times10^{-10} m/s^2$. It is interesting (and surprising as well) that these results are very close in order to anomalous "Doppler deceleration" of the probe $a_p = -(8\pm3)\times10^{-10} m/s^2$ cited in[1]. Analogous computations for Pioneer 11, as checking point, show the following. Full time of observation of Pioneer 11 is shorter so observational period is taken from 1984 to 1989, with observational data from the same source [3]. Radial distances for beginning and end of the period are $r \cong 15.1\ AU = 2.26\times10^{12} m$, $r = 25.2\ AU = 3.77\times10^{12} m$. Respective year-mean radial velocities are $V_p = 11.86\times10^3 m/s$, $V_p = 12.80\times10^3 m/s$. Computed "relativistic deceleration" values for this period are the $a_p = -20.06 \times 10^{-10} m/s^2$, $a_p = -10.04 \times 10^{-10} m/s^2$: this is even in agreement with experimental value of a_p.

Conclusion

According to eq. (4), since the Pioneer probe is going away from the sun, there must produce a slight blue shift on top of the larger red shift [1], and the relativistic acceleration is going toward the Sun. But in the case of the probe is directed toward the Sun, eq. (4) indicates us that there must produce a slight red shift on top of the larger blue shift, and the relativistic acceleration is going upward the Sun.

References

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