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Extended Analysis of the Erroneous Use of the Virial Theorem for Elliptical- and Disc Galaxies, and for Galaxy Clusters, which leads to Dark Matter

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The virial theorem is a method that is used to evaluate the mass of a dynamic system under certain conditions. It is used in astronomy beyond its scope of validity, as it will be shown here. Moreover, it will be shown that the derivation for the evaluation of the system's mass is plain wrong as well due to erroneous physical attributions to the mathematical equations, especially in the frame of gravitational systems, including rotating and non-rotating systems. For a studied case of a simple disc galaxy, values of potential energy and mass are found that are up to 4.31 times as large as when estimated by the Virial Theorem. Due to that, "dark matter" has been erroneously postulated for gravitational systems, which indeed include the elliptical- and the disc galaxies.

Keywords: velocity, virial theorem, gravity, elliptical galaxies, disc galaxies, kinematics, kinetic energy, potential energy

1. What is the Virial Theorem?

Several authors quote the virial theorem as follows:
Generally, in gravity, the Virial Theorem relates the gravitational potential energy of a system to the kinetic energy and provides an insight into the stability of the system [1]. The equation is further used to estimate the mass of systems like galaxies and clusters.
The Virial Theorem equation is given by

$$2\overline{T} + \overline{\Omega} = 0$$

where $\overline{T}$ is the time averaged kinetic energy and $\overline{\Omega}$ is the time averaged potential energy.

However, authors say that there are a number of conditions needed before this can be used as such. The validity is said to be restricted to a stable, self-gravitating, spherical distribution of equal mass objects [2].

In practice however, it is used for pulsing systems and rotating galaxies, including elliptical- and disc galaxies as well.

In the next section, I will show what the derivation is, and I comment the different steps, including the three major errors. In the same section, I will discuss the consequences for gravitational systems and especially for elliptical- and disc galaxies.

The virial theorem is used to estimate the total mass of a system, and it will be shown that the assumptions are erroneous.

The plain text is taken from the references or in the spirit of these references, and the text in bold characters are remarks from the author.

2. The usual (official) derivation of the Virial Theorem

Here follows the derivation, which is similar in the several references that are annexed to this paper. The text in bold characters gives the critical analysis of the derivation.

The authors define:

$$G = \sum_i \mathbf{p}_i \bullet \mathbf{r}_i = \sum_i m_i \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} \bullet \mathbf{r}_i = \frac{1}{2} \sum_i m_i \frac{\mathrm{d}\left(\mathbf{r}_i \bullet \mathbf{r}_i\right)}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_i m_i\, r_i^2\right) \tag{1}$$

For a collection of N point masses the *scalar* moment of inertia about the origin is given by

$$I = \sum_{i=1}^{n} m_i\ \mathbf{r}_i^2 = \sum_{i=1}^{n} m_i\ r_i^2 \tag{2}$$

where m_i and $\mathbf{r}_i$ are the mass and position of the i^{th} particle.

<u>Remark 1</u>**: The eq.(1) means that for the angular component of the momentum, the left side is zero because the vector dot**

product of the momentum with respect to the position vector of the origin is zero. For the radial component, it is not zero.

The authors write the time derivative of the virial as

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \mathbf{p}_i \bullet \mathbf{r}_i = \sum_{i=1}^{n} \mathbf{p}_i \bullet \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} + \sum_{i=1}^{n} \frac{\mathrm{d}\mathbf{p}_i}{\mathrm{d}t} \bullet \mathbf{r}_i \tag{3}$$

in which

$$\sum_{i=1}^{n} \mathbf{p}_i \bullet \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \sum_{i=1}^{n} m_i \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} \bullet \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \sum_{i=1}^{n} m_i v_i^2 = 2T \tag{4}$$

where T is the total kinetic energy of the system with respect to the origin of the coordinate system.

Then, eq.(3) can be written as :

$$\frac{\mathrm{d}G}{\mathrm{d}t} = 2T + \sum_{i=1}^{n} \mathbf{F}_i \bullet \mathbf{r}_i \tag{5}$$

since $\mathrm{d}\mathbf{p}_i / \mathrm{d}t = \mathbf{F}_i$ is the net force on particle *i*.

The second term of eq.(5) is somewhat more complicated, according to the authors. One can see that the force $\mathbf{F}_i$ on i , from all the other particles involved, can be written as

$$\mathbf{F}_i = \sum_{j=1}^{n} \mathbf{F}_{ji} \tag{6}$$

where $\mathbf{F}_{ji}$ is the force applied by particle *j* on particle *i*.

After a number of steps, the authors come to the following identity:

$$\sum_{i=1}^{n} \mathbf{F}_i \bullet \mathbf{r}_i = \Omega \tag{7}$$

in which Ω is the potential energy of the whole system.

Remark 2: The eq.(7) has been derived for all the masses, two by two, with their mutual distances.

The authors conclude

$$\frac{\mathrm{d}G}{\mathrm{d}t} = 2T + \Omega \tag{8}$$

Remark 3: The left hand side of eq.(8) is exactly zero for ideally circular rotational gravitational systems due to the remark 1. Then, eq.(8) coincides with the gravity equation for circular orbits.

Now, the second part of the derivation concerns the averaging to the time of the eq.(8).

The authors average the left side by integrating over a period of time, and by dividing that result by the period.

Hence,

$$\frac{1}{t}\int_0^t \frac{\mathrm{d}G}{\mathrm{d}t}\mathrm{d}t = \frac{2}{t}\int_0^t T\,\mathrm{d}t + \frac{1}{t}\int_0^t \Omega\,\mathrm{d}t \tag{9}$$

which can be written as

$$\frac{1}{t}\big(G(t) - G(0)\big) = 2\overline{T} + \overline{\Omega} \tag{10}$$

The authors say that if the motion over the time is periodic, the left hand will vanish. For other systems, one can let the time period go to the limit of infinite, and the left side will vanish, resulting in

$$0 = 2\overline{T} + \overline{\Omega} \tag{11}$$

which is called the virial theorem.

Error 1: For the use of the mathematical trick of applying the infinite time, assumed to be applicable for the radial component of the momentum, there arise two problems. Indeed, the left hand side will fall to zero due to an infinite time. The integrals at the right hand side of eq.(9) will either diverge, or converge. If they converge, they will be divided by infinity and become zero as well; if they diverge, it might be possible that both the kinetic and the potential energy will mutually compensate without becoming infinite or zero, and in that case the virial theorem holds for pulsing systems.

However, since we talk of motion that is only radial, there are two physical possibilities: either the system converges to a singularity, or it expands to infinity.

The first case has indeed no solution, since a singularity is eventually obtained in a finite time; the second case will unbind the system and has no solution either. Indeed, the left hand side can only be set to zero when either of the physical situations has been reached.

A third solution however is that the system doesn't get unbound but that it becomes cyclic. This solution has already been treated above.

When all the masses of the system are moving radially, it might be possible to say that the kinetic energy equals half the potential energy, as far as can be assumed that the time derivative at the left hand side becomes small enough, and that the terms of the right side don't become zero or infinite. However, this decision doesn't follow from the derivation but from the physical behavior of the system, and from the precise mathematical derivation for the very system's equations.

Hence, in galaxies, the use of the virial theorem is correct until now for circularly rotating systems. For expanding or reducing systems however, it will be very uncertain to prove the sufficient validity of the virial theorem.

3. The (official) evaluation of the system's mass

The virial theorem is used to estimate the total mass from the observed system's velocities. The virial theorem says that

$$T = -\Omega / 2 \tag{12}$$

where the potential energy is given by [2]:

$$\Omega = -G\frac{N^2 m^2}{R_{\text{tot}}} = -\frac{1}{2}G\frac{M_{\text{tot}}^2}{R_{\text{tot}}} \tag{13}$$

In [2] it is explicitly stated: "We usually assume that all of the orbits travel on similar orbits that are isotropic, that is, are not flattened in any way and have no preferential direction; we say these are random orbits."

Compare this with [5], eq.(7.8) where the potential energy of an uniform sphere is given by:

$$\Omega = -\frac{3}{5}G\frac{M_{\text{tot}}^2}{R_{\text{tot}}} \qquad [5](7.8)$$

So, we get:

$$\frac{1}{2}M_{\text{tot}}v^2 = \frac{1}{4}G\frac{M_{\text{tot}}^2}{R_{\text{tot}}} \tag{14}$$

Eq.(13) results in

$$M_{\text{tot}} \simeq 2\frac{R_{\text{tot}}v^2}{G} \tag{15}$$

which more generally can be written as [1]:

$$M_{\text{tot}} \simeq 2\frac{R_{\text{tot}}v^2}{\alpha G} \tag{16}$$

with α a structure parameter of order unity.

The true overall extent of the system R_{tot} is taken, as well as the mean square of the velocities of the individual objects that comprise the system.

<u>Remark 4</u>: **For circularly rotating systems, we see that in eq.(14) the overall extend of the system R_{tot} has been taken. However, indeed in the eq.(4), the same radius was taken, as well as in eq.(5), but in eq.(6), the real distances between the masses, two by two, have been taken. In eq.(14) it is stated, with some accordance with eq.[5](7.8) that 5/6th of the value of the potential energy of an uniform sphere can be taken.**

4. An example with disc galaxies.

In the Appendix, a simple representative case was calculated for disc galaxies, with a limited number of stars, distributed in zones.

It is found that the real potential energy is 2.98 higher than expected from the virial theorem.

Another case in which a more probable mass distribution is shown, higher near the center, lower near the edges, gives a real potential energy, and consequently a real mass, of 4.31 times higher.

<u>Error 2</u>: **Prograde 2D disc galaxies have many stars that stay a long time together, and which are far from being on random orbits, at distances that are fractions of the radius. However, 3D spheres with random orbits will always have all the stars being at large distances from it. Hence, the average R in eq.(15) or the value of α in eq.(16) should be very different in both cases.**

However, if the overall radius is instead taken as in eq.(15), there is a huge difference in calculated mass, and both systems, prograde 2D and random 3D are not differentiated.

5. The approach of a direct Newtonian integration of disc galaxies.

Another approach to cope with the dynamics of disc galaxies is the direct integration of the gravity forces of a mass disc [6] [7].

Embang Li [6] performed a study where he integrated the mass of a disc, but where he simplified the calculus by stating that the mass outside the tangent chord of the considered orbit can be neglected. This is true for spherical systems only, but in the present case, the local error will decrease with increasing radius. Moreover, to the outside edges of the disc, the stars are decreasingly connected to it, and the structure becomes more 3-dimentional. Hence the approximation, of which the error is not cumulative, is negligible at the galaxy's outer edge. Li found that the gravitational attraction at the outer edge is a six-fold of the gravity that is found with a spherical shape. This is causing the false impression that there is dark matter needed in order to compensate the missing gravity.

The author of the present paper also calculated the velocity curve out of a disc galaxy with a mass distribution slope $\sim 1/R$ [7], while assuming a zero thickness of the hole disc, and found an increasing velocity with increasing radius, opposite to the expected Kepler's velocity for spherical galaxies.

6. Conclusion

In this paper, it is shown that the usual derivation of the virial theorem is wrongly applied for expanding or reducing systems. Only a specific derivation for the given system can give a final answer.

The virial theorem for circularly rotating gravity systems follows directly from the Newtonian gravity theory, but the equation supposing to evaluate the overall mass of the system is wrong for both rotating and pulsing systems, because the overall extent R_{tot} of the system is taken, while for the derivation of the virial theorem, the true distances of the masses, two by two were taken in order to find the potential energy of the system.

It can be concluded that the virial theorem is only valid for the angular component and for the radial cyclic component of the momentum.

The evaluation method of the system's mass however is not general. In a simple example of a disc galaxy, the real potential energy was found to be up to 4.31 times the estimated one by the

Virial Theorem. Consequently, the virial mass is underestimated by the same amount.

By calculating the velocities in given orbits directly from a plausible Newtonian mass distribution, one can straightforwardly find the observed velocity curves.

References

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APPENDIX

Consider a pattern of dots, representing stars of a disc galaxy. Consider a set of virtual concentric rings that can slide over the pattern in order to make calculations. The three annexed pictures will help us to define the potential energy of the system. In this case, the disc galaxy has a constant density.

CASE STUDY 1

In the Virial Theorem, one calculates the expected total potential energy of the system as following:

$$\Omega = -G\frac{N^2m^2}{R_{tot}} = -\frac{1}{2}G\frac{M_{tot}^2}{R_{tot}} \quad (13)$$

This is allegedly, the average of all the potential energies of the attraction two by two of the stars.

Let's have a look at a disc galaxy:

The 3 pictures show the stars, and concentric circles, positioned in three successive ways. From the third picture with the concentric circles also concentric with the galaxy, we see that the band widths from the center to the outer edge are L + L + ½ L.

Let's look at the three pictures, and apply the equation of the correct potential energy:

$$\sum_{i=1}^{n}\mathbf{F}_i \bullet \mathbf{r}_i = \Omega \quad (7)$$

in which

$$\mathbf{F}_i = \sum_{j=1}^{n}\mathbf{F}_{ji} \quad (6)$$

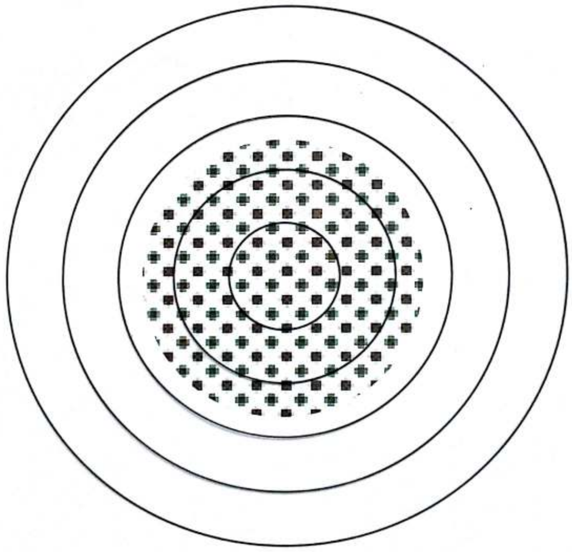
where $\mathbf{F}_{ji}$ is the force applied by particle *j* on particle *i*.

Eq.(7) has been estimated by the Virial Theorem as eq.(13).

1) In the first picture, the circles are concentric with the galaxy, and we get: also 20 stars in the center, 60 in the first band and 40 in the second band. The number of stars in an identical position are given by the stars in the central concentric circle: 20 stars.

Hence, the potential energy of the stars of this picture, two by two with the rest of the disc galaxy is:

20 G (20/(0.5 L)+60/(1.5 L)+40/(2.5 L))



1: # stars	Distance	Sum (stars/L)
20	0.50	40
60	1.50	40
40	2.50	16
Total : 120		96

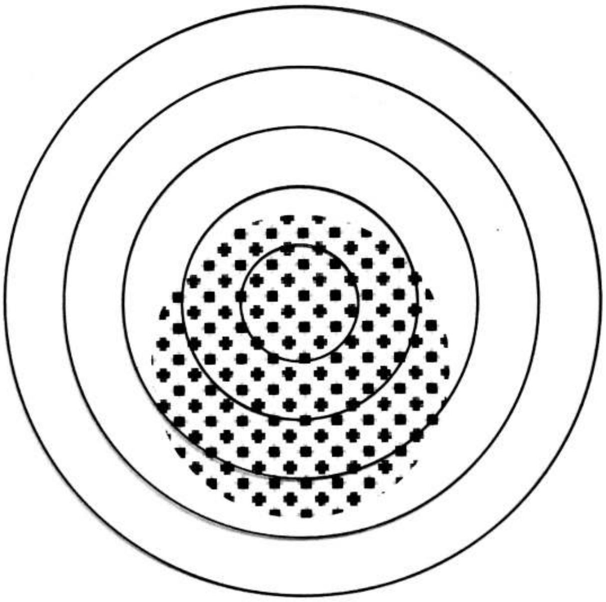
Potential Energy : 1955.56

2) In the second picture, the first concentric circle gives 20 stars, the second band gives 43 stars, the third 39, the last 18.

The center of the concentric circles is at a radius 1.5 L. The number of stars in an identical position are given by the second band of the case (1), which are all at 1.5 L from the center: 60 stars.

Hence, the potential energy of the stars of this picture, two by two with the rest of the disc galaxy is:

60 G (20/(0.5 L)+43/(1.5 L)+39/(2.5 L)+18/(3.5 L)).



2: # stars	Distance	Sum (stars/L)
20	0.50	40.00
43	1.50	28.67
39	2.50	15.60
18	3.50	5.14
Total : 120		89.41
Potential Energy :		5364.57

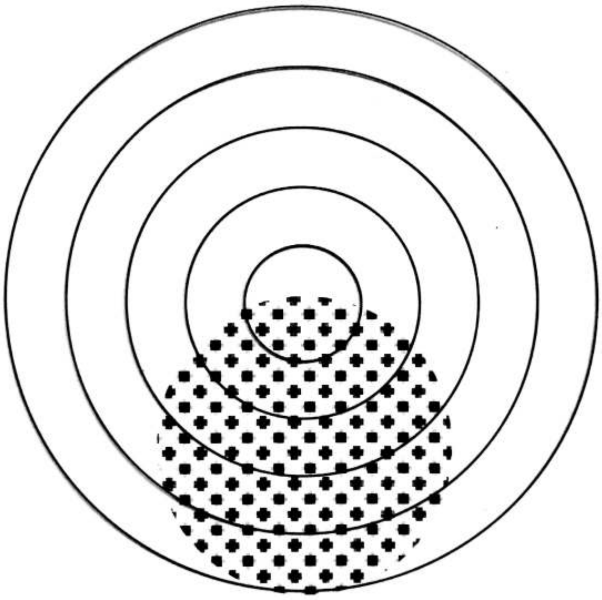
3) In the third picture, one measures the potentials two by two between a star at the edge and the other stars. Let's call the distance between the concentric circles L.

In the first concentric circle, one counts 11 stars, in the first band (between the first and the second circle), one counts 24 stars, the second band, 31 stars; the third band 32 stars; the last band 22 stars.

The number of stars in an identical position are given by the third band of the case (1), which are all at 2.25 L from the center: 40 stars.

Hence, the potential energy of the stars of the edge, two by two with the rest of the disc galaxy is:

40 G (11/(0.5 L)+24/(1.5 L)+31/(2.5 L)+32/(3.5 L)+22/(4.5 L)).



3: # stars	Distance	Sum (stars/L)
10	0.50	20.00
24	1.50	16.00
31	2.50	12.40
33	3.50	9.43
22	4.50	4.89
Total : 120		62.72
Potential Energy :		1254.35

SUM of Potential energy (calculated with star unity masses and per length L) : 1955.56 + 5364.57 + 1254.35 = 8574.48

$L = R_{tot}/2.5$

Potential energy, when expressed in terms of R_{tot} and total mass M^2 (with $G = R_{tot} = M_{Star} = 1$):

8574.48 x 2.5 = 21436.19

Since the equation of the Virial Theorem's Potential Energy is given by eq.(13) , the estimate here (with $G = R_{tot} = M_{Star} = 1$) is :

120 x 120 /2 = 7200

Hence, the real Potential Energy value is in reality 2.98 times larger than the estimated one by the Virial Theorem. Consequently, the according mass is also underestimated with a factor of 2.98.

CASE STUDY 2

When one takes a mass distribution that is larger around the galaxy's center (doubled in the central band) and lower at the edges (half the value in half a band width), one gets the real Potential Energy value 4.31 times larger than the estimated one by the Virial Theorem.

1: # stars	Distance	Sum (stars/L)
40,00	0,50	80,00
60,00	1,50	40,00
20,00	2,50	8,00
120,00		128,00
Potential Energy :		5120,00

2: # stars	Distance	Sum (stars/L)
25,00	0,50	50,00
53,00	1,50	35,33
33,00	2,50	13,20
9,00	3,50	2,57
120,00		101,10
Potential Energy :		6066,29

3: # stars	Distance	Sum (stars/L)
9,00	0,50	18,00
22,00	1,50	14,67
43,00	2,50	17,20
30,00	3,50	8,57
16,00	4,50	3,56
120,00		61,99
Potential Energy :		1239,87

SUM of Potential energy (per star and per length L) : 5120 + 6066.29 + 1239.87 = 12426.16

$L = R_{tot}/2.5$

Potential energy, when expressed in terms of R_{tot} and total mass M^2 (with $G = R_{tot} = M_{Star} = 1$):

12426.16 x 2.5 = 31065.40

Since the equation of the Virial Theorem's Potential Energy is given by eq.(13) , the estimate here (with $G = R_{tot} = M_{Star} = 1$) is :

120 x 120 /2 = 7200

Hence, the real Potential Energy value is in reality 4.31 times larger than the estimated one by the Virial Theorem. Consequently, the according mass is also underestimated with a factor 4.31.