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The Emission Theory of Electromagnetism

BY

LEIGH PAGE

Professor of Mathematical Physics in Yale University

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HISTORICAL INTRODUCTION

The subject of electromagnetism exhibits a remarkable dualism which is absent from other branches of physics. Not only do electric and magnetic fields appear as essentially distinct though related entities, but in the study of the effects produced by one electric charge on another it is necessary to distinguish between positive and negative electricity and in the case of magnetism between north and south-seeking polarity.

Naturally the first systematic investigations in the subject were limited to the phenomena of magnetostatics and electrostatics, in which the forces between stationary magnets or electric charges are the objects of study. As early as 1600 Gilbert explained the directive tendency of a freely suspended magnetic needle as due to the magnetization of the earth, combined with the general principle that like poles repel and unlike attract. However, it was not until two centuries later that Coulomb, by means of the torsion balance, demonstrated that the law of force, both as regards magnetic poles and electric charges, was the same inverse square law that Newton had found to hold in the province of gravitation.

At the time of Coulomb's death no connection was known between the phenomena of magnetism and electricity. A magnet exerted forces on other magnets in its neighborhood, but produced no effect whatever on an electrified pith-ball, and conversely. It was quite by chance that Oersted, in 1819, observed that a magnet is deflected by an electric current in such a manner as to set itself at right angles to the wire. During the succeeding year Ampère formulated in quantitative form the laws of action between a current and a magnet, or one current and another. It was found that a closed circuit carrying a steady current behaves in all particulars like a magnetic shell of uniform strength having the circuit as its periphery, so that the subject of magnetostatics was at once extended by the inclusion in its domain of circuits carrying constant currents.

The next great step was made almost simultaneously by Faraday and Henry. Faraday had long felt that as the flow of electricity through a conductor creates a magnetic field about the wire, it should be possible to produce a current by the presence of a magnetic field. His first experiments gave a negative result, and it was somewhat by accident that he noticed a momentary current through his galvanometer while a magnet was approaching or receding from the circuit. Further investigation led to the discovery of current induction; whenever the magnetic field through a circuit changes, a current is induced which persists so long as the change in the field continues. Henry had discovered the fundamental phenomenon of current induction even earlier than Faraday, but circumstances were such that he was unable to publish his results until the account of Faraday's work was in print. Hence Faraday rightly receives credit for one of the greatest discoveries of science.

Maxwell's Equations

The formulation in mathematical language of the discoveries of Coulomb, Ampère, and Faraday was undertaken by Maxwell, whose "equations," slightly modified in form by Larmor and Lorentz, have been confirmed by every test which experiment offers. In order to appreciate the significance of these equations, it is necessary to define precisely the quantities involved. An electric field (region through which electric forces act) is conveniently specified in terms of the ***electric intensity*** **E**. This is a vector function of position in space and time, which is everywhere equal in magnitude and direction to the force which would be exerted by the charges producing the field on a unit positive test charge[1] at rest relative to the observer. Similarly the ***magnetic intensity*** **H** in a magnetic field is defined as the force exerted by the magnets or currents producing the field on a hypothetical unit north pole. If, then, ρ stands for the electric charge per unit volume, v for the velocity with which it is moving, and **F**

[1] More precisely, the test charge should be very small compared to the charges producing the field, so as not to disturb the latter by its presence. Then the *electric intensity* is the ratio of the force exerted on the test charge to its charge.

for the electromagnetic force per unit volume on the matter contained therein, the equations of electromagnetism take the form:

$$\nabla \cdot \mathbf{E} = \rho \qquad (1) \qquad\qquad \nabla \cdot \mathbf{H} = 0 \qquad (3)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\dot{\mathbf{H}} \qquad (2) \qquad \nabla \times \mathbf{H} = \frac{1}{c}(\dot{\mathbf{E}} + \rho\mathbf{v}) \qquad (4)$$

$$\mathbf{F} = \rho(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}) \qquad (5)$$

in the rational C. G. S. units suggested by Heaviside and Lorentz, and now used almost universally in theoretical discussions. The letter c denotes the speed of light, $3(10)^{10}$ centimeters a second.

The symbol ∇ stands for a differential vector operator, and its appearance in the first four equations indicates that they are differential equations. (1) and (3) are scalar equations, (2) and (4) vector equations, and the four together constitute the equations of the electromagnetic field. By means of them may be calculated the values of **E** and **H** produced by any arbitrarily assigned distribution of charge ρ and current $\rho\mathbf{v}$. It is to be noted that while electrical entities, i. e. charges, appear explicitly in these equations, there are present no magnetic entities, i. e. poles. This is in accord with the experimental fact that while electrons (negatively charged particles) and protons[2] (positively charged particles) may be extracted from material atoms, no such thing as a magnetic pole has ever been isolated. Ampère himself conjectured that the magnetic properties of matter might be due to currents of electricity circulating without resistance in the atom or molecule of a magnetic substance, and that therefore the postulation of a magnetic particle was superfluous in explaining the known facts of magnetism. Indeed, as will appear later, it is possible to go even further and treat the magnetic intensity **H** as merely a convenient symbol for a product of yet more fundamental vectors.

Before proceeding to a detailed examination of the electromagnetic equations, it will be advantageous to introduce the concepts of *line of force* and *tube of force* as used by Faraday. Consider an electric field produced by charges (at rest or moving)

[2] The word electron will be used hereafter in a generic sense to designate the ultimate charged particle of either sign.

in the neighborhood. If, now, a very small positive test charge is placed in the field, it will be repelled from neighboring positive charges and attracted toward negative charges. If free to move, it will describe a path. If its motion is constrained so as to be very slow[1] (so that it will be unaffected by magnetic fields), the path described is what is known as a *line of electric force.* To be sure, if the charges producing the field are moving, the field is continually changing, and different portions of the line of force described by a slowly moving test charge will correspond to different times. This introduces no real difficulty, however, as a number of such test charges may be supposed to trace out, at the same instant, different short portions of a single line of force.

From the definition contained in the preceding paragraph, it is clear that a line of electric force has everywhere the direction of the electric intensity **E**. In fact, it might have been defined by that property. Suppose, now, that an electric field has been mapped out by means of lines of force, each line being provided with an arrow to indicate the direction in which the test charge has traced it out. Such lines will obviously diverge from positive charges, and converge at negative charges. The direction of the electric intensity at any point will be indicated by the direction of the line of force through that point. It is possible, however, to go further, and map out the field in such a manner as to indicate the magnitude of **E** by the density of the lines of force. However, as a line of force is discrete whereas the field appears to be continuous, it is necessary first to introduce the idea of *tube of force.* A tube is defined as a bundle of M lines, where M is any sufficiently large number. Now, in mapping out a field, the lines of force are drawn in such density that the number of tubes per unit cross section is numerically equal to **E** at every point in the field. By making M large enough the number of lines of force per unit cross section can be kept large in even the weakest fields, whereas if the convention of drawing a number of lines of force per unit cross section equal to the electric intensity had been adopted, a difficulty would have arisen in representing graphically fields in which **E** is numerically small.

Magnetic as well as electric fields can be represented by lines of force. A *line of magnetic force* is defined as a line having everywhere the direction in which a small hypothetical north pole at rest relative to the observer would be urged by the field. Lines of force are drawn in such density as to make the number of tubes of force everywhere equal to the magnetic intensity **H**.

To return now to Maxwell's equations, it is found that their interpretation is greatly simplified by the concept of line of force. The quantity on the left hand side of (1) is known as the *divergence of* **E**. In terms of tubes of force, the divergence of **E** is the excess of the number of tubes of electric force passing out of a unit volume over the number entering that volume. Consider a region where no charges are present. Then ρ vanishes, and the divergence of **E** is zero. This means that as many lines of force leave each unit volume as enter it. Hence the lines of force are *continuous* through those portions of space where no charge is present. Where charges are located, the divergence of **E** is equal to the density of charge. If the charge is positive, a number of tubes of force equal to the magnitude of the charge originate on it; if the charge is negative, a number of tubes equal to its magnitude end on it. Hence each line of electric force starts from a positive charge and stretches without break to a negative charge. By thinking of the lines of force as stretched elastic bands, Faraday was able to see in a qualitative manner how the charges present tend to move under the action of the field. In fact, equation (1) is precisely the mathematical statement of Coulomb's law that the electrostatic force exerted by one charge on another varies inversely with the square of the distance between the two. For, consider a single charge. Symmetry requires that the lines of force diverging from it shall spread out uniformly in all directions, somewhat as the spokes of a wheel from the hub, and, as the lines are continuous, the number of tubes per unit cross section will vary inversely with the square of the distance from the charge. Hence the electric intensity and the force exerted on a second charge will vary with the distance according to the law discovered by Coulomb.

Equation (3) states that the divergence of **H** vanishes. Therefore the lines of magnetic force are continuous closed curves. This would not be the case if magnetic poles existed *per se;* in that case lines of magnetic force would run from north to south poles just as lines of electric force run from positive to negative charges. On the electron theory of matter, however, the magnetic properties of the molecule are supposed to be due to rings of electrons revolving about a positively charged nucleus and thus forming a current circuit. The lines of magnetic force thread this circuit from the south to the north side, returning outside the circuit.

On the left hand side of (2) appears a vector derivative known as the *curl of* **E**. This quantity represents the work necessary to carry a unit positive charge once around the periphery of a unit area, so oriented as to make the work a maximum, and is directed at right angles to this area. The dot over the **H** in the right hand member denotes differentiation with respect to the time, and the factor c standing for the velocity of light $[3(10)^{10}\text{cm/sec}]$ appears on account of the choice of units. The equation as a whole expresses Faraday's law of current induction for stationary circuits: the electromotive force per unit area equals $1/c$ times the rate of decrease of the magnetic intensity.

The remaining field equation (4) expresses in mathematical language Ampère's law together with an important addition introduced by Maxwell. Ampère's law alone would state that the work done in carrying a unit north pole once around the periphery of a unit area ($\nabla \times \mathbf{H}$) is equal to $1/c$ times the current density ($1/c\ \rho\mathbf{v}$). In this form, however, the electromagnetic equations fail to satisfy the law of continuity. This law states that

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$$

The left hand side is the rate of increase of charge per unit volume per unit time. Now, on the electron theory of matter, the only way the charge can increase is by more charge flowing into the specified unit volume than flows out of it. The right hand member of this equation gives the excess of the inward flow over the outward flow. Hence the equation of continuity must of necessity be fulfilled by any satisfactory set of field equations. If now, equation (1) is differentiated with respect to the time,

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \dot{\mathbf{E}}$$

and if the operator $\nabla .$ is applied to (4),

$$-\nabla \cdot (\rho \mathbf{v}) = \nabla \cdot \dot{\mathbf{E}}$$

since $\nabla \cdot \nabla \times \mathbf{H}$ is identically zero. Hence the law of continuity is satisfied if, and only if, the term $\dot{\mathbf{E}}$ is added to Ampère's current density $\rho\mathbf{v}$. Maxwell considered that this added term represents a current due to displacement of the ether. Its inclusion in (4) enabled him to show that electromagnetic waves should be propagated through space with the velocity of light. So long as the electric field remains unchanged, $\dot{\mathbf{E}}$ vanishes. Hence in the treatment of steady currents this term does not appear, and unless a current is oscillating very rapidly, it is unimportant. So it is not surprising that Ampère did not include it in his theory.

Physical measurement is confined to the investigation of the effect of one charged particle on the motion of another. Electromagnetic theory, however, describes the effect through the agency of an intermediate concept—the electromagnetic field. If it is desired to compute the force exerted by charge A on charge B, the first step in the process is to compute the field produced by A. This is done by means of the four field equations, which determine the values of the electric and magnetic intensities produced by the charge density ρ of A and the current density $\rho\mathbf{v}$ due to A's motion. At this stage, however, the problem is but half solved. How are the **E** and **H** due to A going to affect B? This question is answered by equation (5). **F** is the force per unit volume on B; ρ the density of charge in B; **v** is B's velocity; and **E** and **H** the field intensities at B due to A. The first term in the right hand member specifies the force due to A's electric field **E**. It has the direction of **E** and is the same whether B is at rest or in motion. The second term gives the force due to the magnetic field **H** produced by A. This component of the force exists only when B is moving, and is greater the greater B's velocity. Also it is at right angles to both **H** and **v** (the cross product $\mathbf{v} \times \mathbf{H}$ is a vector at right angles to the plane of **v** and **H** in the sense of advance of a right handed screw rotated from **v** to **H** through the smaller angle between their positive directions). This second term specifies the induced electromotive force produced in a wire, moved so as to cut lines of magnetic force; it is responsible for the action of electric generators and motors.

From the four field equations can be deduced two wave equations; one involving **E**, the other **H**. Both represent transverse waves traveling through space with the velocity **c** of light. The electric and magnetic vectors (at points not too near the source) are equal, in phase, and perpendicular to one another as well as at right angles to the direction of propagation. The two together constitute an electromagnetic wave which, according to its wave length, may be classified as Hertzian wave, heat radiation, light, X ray or γ ray. How do these waves, carrying light and heat from the sun to the earth, traverse ninety million miles devoid of matter? From time immemorable it has been felt necessary to postulate a medium to carry these waves, and this medium, which must be thought of as filling all space and even permeating the interstices between electrons in matter, has been called the ether. From the very start the concept of an ether has led to serious difficulties. In the first place it must have the properties of an elastic solid in order to transmit transverse waves. But how can the earth and other planets travel through a rigid medium without encountering appreciable resistance? The logical extension of the ether concept to many of the more complicated phenomena of optics led to serious inconsistencies, and finally the Michelson-Morley experiment showed the utter futility of an ether which, like a material body, was to be considered as having a discrete structure.

If the rate of doing work on all the charges contained in a region τ is computed, the electromagnetic equations lead to a relation which may be brought into consonance with the principle of conservation of energy provided $1/2(E^2 + H^2)$ is interpreted as the electromagnetic energy per unit volume of the field, and $c(\mathbf{E} \times \mathbf{H})$ as the flux of energy (energy passing through unit cross section in unit time) through the surface enclosing τ. Similarly the law of conservation of momentum may be preserved provided $1/c(\mathbf{E}\times\mathbf{H})$ is interpreted as the electromagnetic momentum per unit volume of the field. It must be emphasized, however, that energy (the classical electrodynamics is here under consideration and no reference is intended to the still mysterious quanta) is not discrete in the sense that material particles are. Thus it is ideally possible to follow one and the same electron throughout the course of time but energy, on account of its continuous nature, fails to retain its identity. A joule of energy may disappear from one locality, and a joule appear in another, but there is no means of determining whether the first and second joule are one and the same entity. Indeed, all that can be said is that the total amount of energy inside a closed rigid impenetrable envelope remains unchanged as time elapses.

The electromagnetic equations which have been discussed in the preceding paragraphs have been built up little by little as experiment has revealed new laws of nature. As has been noted, they describe in quantitative form a number of apparently independent experimental discoveries. The first object of this paper is to show, in non-mathematical language, how Maxwell's equations in their entirety can be deduced from a single, simple, basic concept. Its second object is to provide a substitute for the ethers of Green and Kelvin which is in accord with the relativity principle and avoids the difficulties of the older theories.

THE EMISSION THEORY OF ELECTROMAGNETISM

Maxwell's greatest contribution to science was his demonstration that light is an electromagnetic phenomenon. Hertz's detection of the electromagnetic waves proceeding from an oscillating electric charge and his success in showing that these waves could be refracted and diffracted like light furnished an unanswerable confirmation of Maxwell's theory. Now the field in an electromagnetic wave is in no essential way different from any other electromagnetic field. Hence if such a wave travels in a straight line with the velocity of light, it is difficult to doubt that every electromagnetic field is in motion with this velocity. But what is moving? The lines of force, if continuous along their length, can exhibit motion only at right angles to themselves. Certainly the lines of force constituting the field of a single stationary point charge can have no motion at right angles to their length. Hence we conclude that in this case the motion must be along the lines. But there is no meaning to the statement that a line is moving along itself except in so far as the line represents a locus of points, and it is the points which are in motion. Therefore we are forced to conclude that a line of force is to be considered as a locus of points each of which is moving in a straight line with the velocity of light. These points will be named *moving elements*. An essential characteristic of a moving element is that it may be identifiable as one and the same throughout the course of time. Otherwise there would be no meaning to the statement that it moves. As a moving element is supposed to continue to move indefinitely in a straight line with the velocity of light, it must be thought of as passing through matter without suffering deflection or diminution of speed. Hence it has no such properties as mass or energy associated with it. In fact, the representation of the field by means of moving elements is purely kinematical in character; in no sense dynamical.[3]

Consider now the electric field of a single stationary[4] point charge. As the lines of force are stationary, the moving elements of which they are composed must be moving either radially outward or radially inward along these lines. Likewise in the case of a moving charge the moving elements which carry its field must be traveling radially, either away from a point previously occupied by the charge, or toward a point to be occupied by it at some future time. Either of these alternatives is mathematically consistent with the electromagnetic equations, but the latter must be rejected on account of the teleological implications which are involved in it. For consider moving elements at a great distance from the charge under consideration. If moving inward toward the charge, their present direction of motion must be such as to make them converge at a point to be occupied by the charge at some future time. Thus the present nature of distant portions of the field is specified by the future location of the charge.

In view of this objection to inward motion, it will be assumed that the moving elements travel *outward* from a point charge, no matter whether the charge is positive or negative. Hence, as time elapses, the field originally in the neighborhood of the charge is carried farther and farther away from it. To restore the portions of the field which are thus removed, it must be assumed that the charge is continually emitting new moving elements. Hence we picture a point charge as a source of streams of moving elements shot out in all directions with the velocity of light. The locus of those moving elements emitted in a single direction is a line of force. The electric field of a stationary point charge, then, will consist of continuous lines of force stretching out radially in all directions from the charge to infinity.

Further justification for the assumption that all points of an electromagnetic field are moving relative to the observer with the velocity of light may be obtained from the relativity principle.

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[3] This statement may need modification as regards the effect of gravitational fields. The recently discovered deflection of a ray of light passing close to the limb of the sun indicates that moving elements are affected by gravitational forces.

[4] The words "stationary" and "moving" always refer to the system of the observer.

According to this principle the velocity law of moving elements should be the same for all inertial systems. But the only velocity which has the same value for all such systems is that of light. Therefore all moving elements must be in motion with the velocity of light.

Before proceeding further it must be noted that, in the theory of the electromagnetic field which is being developed, the term line of force is used in a somewhat different sense from that employed by Faraday. Faraday's line of force referred to the resultant field produced by the attractions and repulsions of perhaps a very large number of charges. Now the complex field produced by an aggregate of charges may be thought of as the resultant of simple fields each of which is due to an individual charge, or element of charge. It is such simple fields that are under consideration here, and the term line of force is to be understood as referring to the simple field of a single elementary charge. By an elementary charge is meant any charge which may be treated as a unit in so far as its field is concerned. In calculating the electromagnetic mass of the electron, for example, each infinitesimal element of the electron's charge must be treated as an elementary charge. On the other hand, in computing the electromagnetic forces between the nucleus and the electrons in an atom, the distances between the charged particles involved are so great compared to their linear dimensions that it is not necessary to consider anything smaller than the electron as the elementary charge. Again, in dealing with the electromagnetic waves emitted by a charge oscillating back and forth across a spark gap, the entire charge, even though it consist of billions of electrons, may be treated as an elementary charge and its field as a simple field.

A complex field, then, will be treated as the resultant of a number of simple fields. Each simple field will have its own lines of electric force and of magnetic force determining **E** and **H** by their directions and their density. The electric or magnetic intensity of the resultant complex field is then the vector sum of the corresponding quantities for all the simple fields which extend to the point in question. For example, the electrostatic field produced by a number of stationary elementary charges A, B, C, etc., is represented by groups of straight lines of force, the lines of each group diverging uniformly from one of the charges and extending to infinity. Each group of lines constitutes a simple field and specifies the electric intensity produced by the elementary charge to which it belongs. Two lines of force belonging to different groups may cross one another, and will in general intersect at some point in their course, but obviously no two lines of the same simple field may cross. A simple field is specified by three vector functions of position in space and time, **E**, **H**, and the velocity **c** of the moving elements which carry the field. Of these the last varies only in direction, as its magnitude is the constant speed of light. It will appear later that these three vectors are related in such a manner that it is possible to eliminate either **c** or **H** from the field equations. Any equation deduced for a simple field which is linear in **E** and **H** and does not contain **c** will apply equally well to the resultant of a number of such fields. For the resultant intensity is merely the vector sum of the intensities of the overlapping simple fields.

For the present, attention will be confined to lines of electric force. Later on it will be shown that the same moving elements which act as carriers of the electric field will serve as carriers of the intrinsic magnetic field, that is, that portion of the magnetic field which is independent of the inertial system in which the observer may happen to be located.

Coulomb's Law

Consider first Coulomb's law. The very fact that the field is represented by lines of force which are continuous everywhere except where charges are present insures the validity of this law. For imagine a closed surface surrounding a region τ of unit volume. If the unit of charge is defined as that charge from which M lines of force diverge (it will be remembered that M lines constitute a tube) then the excess of the number of tubes leaving τ over the number entering this region will be equal to the charge contained therein, that is, to the density of charge. But this is just the first of the field equations.

Some Simple Fields

Consider now the effect of motion on the field of a point charge. Figure 1 shows a single line of force of a stationary charge e. The moving elements 1, 2, 3, 4, 5 were emitted respectively 1, 2, 3, 4, 5 seconds earlier than the time for which the diagram is drawn. The locus e-1-2-3-4-5 is the line of force. The next figure (2) pictures the formation of a line of force in the field of a charge e moving with constant velocity to the right. The moving element 1 was emitted one second earlier than the time for which the diagram is drawn. At the time of its emission the charge was at e_1, and during the intervening second the moving

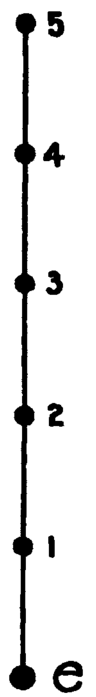

FIGURE 1.
Stationary Charge.

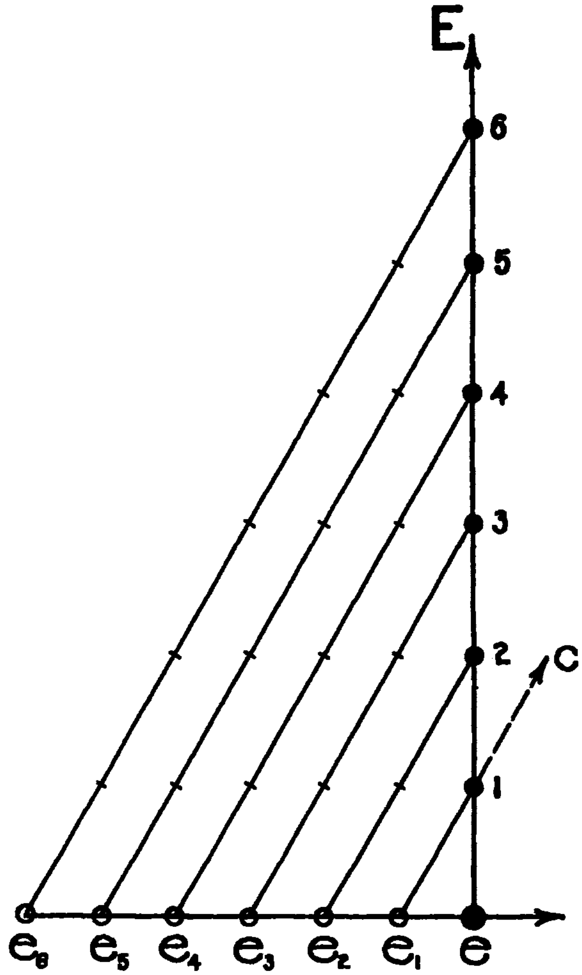

FIGURE 2.
Charge moving with Constant Velocity ½ c.

element has described the path e_1-1. Its velocity **c**, then, is represented by a vector having the magnitude $3(10)^{10}$cm/sec and the direction e_1-1. Similarly 2 was emitted from e_2, 3 from e_3, etc. The line of force at the instant considered is the locus e-1-2-3-4-5-6. As time elapses, the moving elements will continue to move obliquely upwards and to the right, carrying the line of force forward as they proceed. The diagrams picture only a single line of force; others must be supposed to extend from the charge in other directions.

From figures (1) and (2) it is seen that the direction of motion of the moving elements coincides with the direction of the line of force which they constitute only in the case of a static field. When the charge is moving with a constant velocity relative to the observer, as in fig. 2, the vectors **c** and **E** make an acute angle with each other which becomes ever greater as the velocity of the charge approaches the velocity of light. Now it is only when a charge is in motion that it has a magnetic field, and this field becomes stronger the greater the charge's velocity. Hence it might be suspected that the magnetic intensity depends upon the angle between **c** and **E**. Consider the vector product $\mathbf{c} \times \mathbf{E}$. This product is defined as a vector at right angles to the plane of **c** and **E** in the sense of advance of a right handed screw rotated from **c** to **E** through the smaller angle between their positive directions, and has a magnitude equal to the product of the magnitudes of **c**, **E**, and the sine of the angle between these two vectors. Dividing this product by the scalar magnitude of c, a vector is obtained that corresponds at least qualitatively to the magnetic intensity as observed experimentally. Let us define **H** by the relation

$$\mathbf{H} \equiv \frac{1}{c}(\mathbf{c} \times \mathbf{E})$$

It will appear subsequently that this vector plays the same role in the equations to be deduced that the magnetic intensity does in Maxwell's equations. Hence it must be identical with the magnetic intensity as ordinarily defined.

Electromagnetic Waves

Let us consider next the electromagnetic waves emitted by a charge oscillating up and down in a straight line with simple harmonic motion. Two lines of electric force coming from this charge are pictured in fig. 3, the moving elements being represented by black dots. The path of the oscillating charge is the line AB, and at the instant for which the diagram is drawn the charge has just reached the upper extremity B of this line. As the velocity of light is so very great, it is inconvenient in describing the formation of the field to use the second as the unit of time. Therefore we will employ a much smaller unit, which will be designated merely as a *time-unit*. The charge is supposed to make a complete oscillation in 4 time-units, and to acquire a velocity one half that of light at the mid-point of its path. Consider now the moving elements emitted upward. Their locus will be the straight line of force *BC*. No waves are emitted in

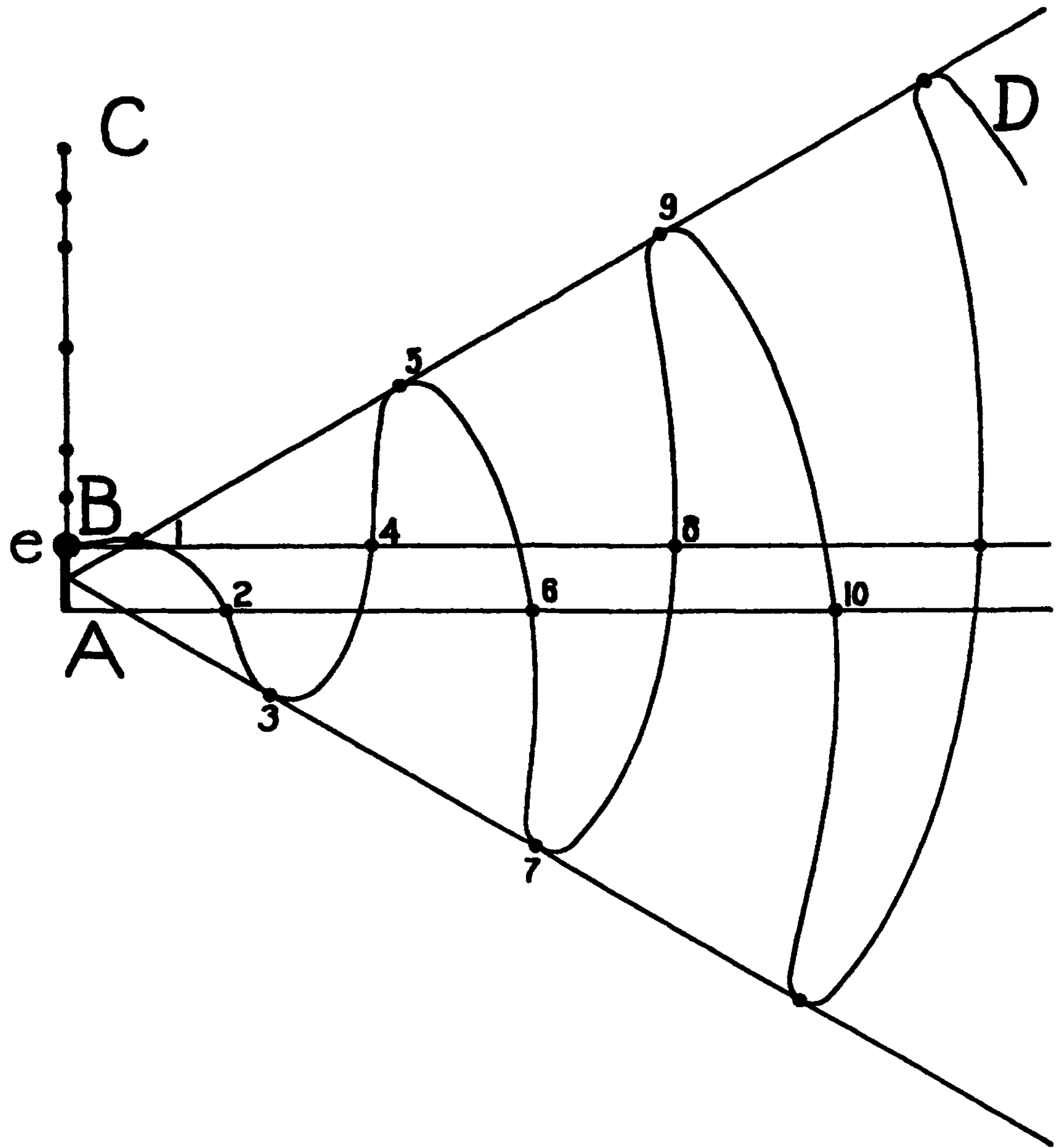

FIGURE 3.
Simple Harmonic Motion. Max. Vel. = ½ c.

this direction. In the horizontal direction, however, the state of affairs is somewhat different. The charge at rest at *B* is about to emit a moving element to the right. The line of force constituting the locus of this and other moving elements previously emitted in the same direction leaves the charge horizontally. In this case, however, it is necessary to define somewhat carefully the meaning of "same direction." As the charge is accelerated, it is continually passing from one inertial system to another. Now, a moving element, once emitted, is in no wise affected, in so far as its motion is concerned, by the future experience of the charge from which it came. Conversely, it would be expected that the charge, in emitting further moving elements belonging to the same line of force, would be uninfluenced by those which had been previously emitted.

Now consider two moving elements a and b belonging to the same line of force, the second of which is emitted a very short time dt later than the first. Let S designate the inertial system in which the charge is at rest when a is emitted. Then, on account of its acceleration, the charge will have passed from S to some other inertial system S' by the time b is emitted. Let α be the angle which the direction of emission of a makes with the direction of the velocity of S' relative to S, the angle being measured in S, and let α' be the corresponding angle in the case of b, α' being measured in S'. Then a and b are said to be emitted in the *same direction* if the planes[5] of their velocity vectors are the same, and α equals α'. This definition of "same direction" is perhaps as natural and simple a definition as could be given; its justification lies in the fact that it leads to correct results. Strictly speaking, the space-time transformation of the restricted relativity theory must be used in comparing the angles α and α', although in cases where the velocity of the moving charge is never comparable with that of light, the Galilean transformation constitutes a sufficiently close approximation.

To return to fig. 3, consider the moving element which was emitted horizontally to the right one time-unit earlier than the instant for which the diagram is drawn. At the time of its emission the charge was at the middle of its path and moving upward with a velocity equal to half that of light. The moving element under consideration was emitted horizontally as seen by an observer going along with the charge. Hence, to an observer stationary with respect to the diagram, its velocity is obtained by compounding the velocity of the charge with the horizontal velocity $3(10)^{10}$cm/sec which it has relative to the system of the

[5] By the plane of the velocity vector is meant the plane determined by this vector and the direction of motion of S' relative to S.

charge. Since the velocity of the charge is comparable with that of light in this case, the compounding must be done in accord with the space-time transformation of relativity. This gives a velocity $3(10)^{10}$cm/sec (for the speed of light is an invariant of the relativity transformation) in the direction 1-5-9. Hence, at the end of one time unit—that is, at the instant for which the diagram is drawn—the moving element is at 1. In the same way it is easily seen that moving elements emitted 2, 3, 4, 5 . . . time units earlier are at the points labeled 2, 3, 4, 5 . . . at the instant for which the diagram is drawn. Connecting up these moving elements the line of electric force extending from B to D is obtained. As time goes on, the moving elements advance along the straight lines on which they lie, carrying the wave with them.

In some ways this kinematical representation of an electric field bears a resemblance to Newton's corpuscular theory of light. There is this fundamental difference, however. Newton's corpuscles themselves possessed energy and by their impacts they could exert pressure on those bodies upon which they impinged; in fine, they were dynamical in their characteristics. Moving elements, on the other hand, are purely kinematical in nature, and as they cannot be deflected or slowed down by matter, they cannot of themselves impart energy or pressure to the bodies through which they pass. They act merely as carriers of the lines of force, and as a vehicle for the transmission of energy in the form of radiation they perform the functions of the elastic ether of prerelativity days. The forces exerted by the electromagnetic waves which they carry are due to the lines of force. If the wave pictured in fig. 3 should pass over an electron, this electron would be urged alternately up and down by the oppositely directed portions of the line of force.

It is clear from the figure that the wave becomes more and more nearly transverse as it proceeds outward from the oscillating charge. Furthermore, the angle between $\mathbf{c}$ and $\mathbf{E}$ approaches 90°, and the magnetic intensity, which is at right angles to the plane of these two vectors, approaches the electric intensity in magnitude. In the diagram no lines of magnetic force are drawn; it will be shown later that the intrinsic magnetic field is carried by the same moving elements as carry the lines of electric force.

It has been noted that moving elements are unaffected by their passage through matter. The same must therefore be true of the lines of force which they carry. Suppose, for instance, that the wave pictured in fig. 3 impinges on a slab of matter opaque to radiation of its particular frequency. Then the electrons in the atoms of the slab are set in oscillation, and in consequence emit waves of the same frequency themselves. These new waves, it may be shown, interfere with the exciting wave in such a manner as to produce a zero ***resultant*** field on the far side of the material, and in general, to give rise to a certain degree of reflection. But the simple field of the exciting wave has been in no wise affected by the presence of the slab.

Magnetic Field Due to Circular Current

It may be of interest to examine the interpretation on the emission theory of the magnetic field set up by a steady current in a circular coil of wire. In fig. 4 the current due to the con-

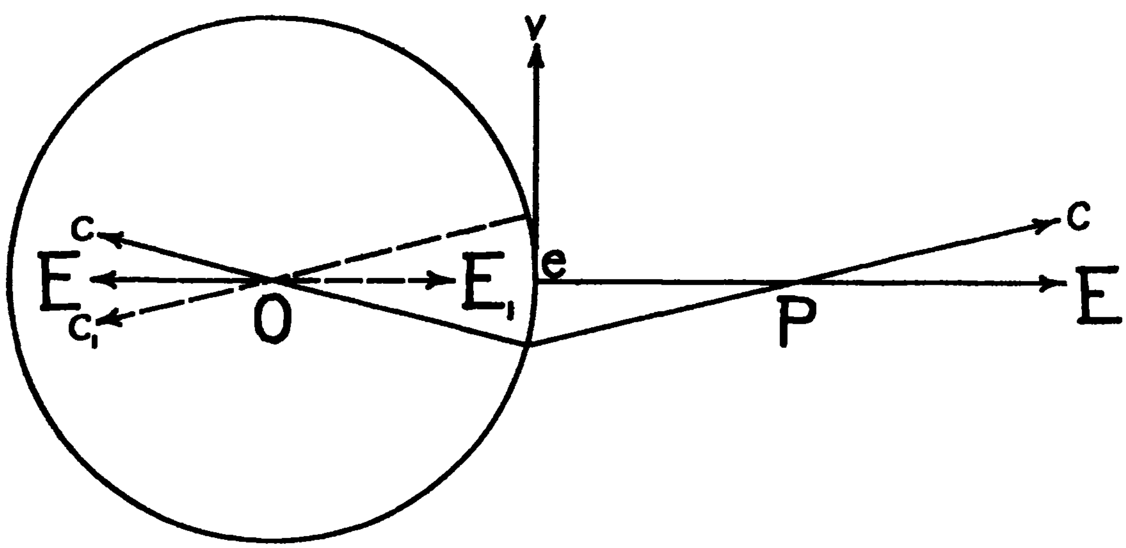

Figure 4.
Magnetic Field due to Circular Current.

vection of electrons around the circular wire is supposed to be flowing in the counter-clockwise sense. Consider the electron *e*. It has a velocity **v** upwards, and as this velocity is presumably very small compared to the velocity of light, the field at nearby points will be very closely that of an electron traveling with constant velocity along the tangent to the circle at *e*. This field has already been pictured in fig. 2. Hence **c** and **E** at the center *O* of the coil will be as indicated in the diagram, and the magnetic intensity, having the direction of $\mathbf{c} \times \mathbf{E}$, will be directed out from the page. An electron of opposite sign at e will travel around the wire in the opposite sense; its field at O is indicated by the vectors $\mathbf{E}_1$ and $\mathbf{c}_1$. While $\mathbf{E}_1$ annuls $\mathbf{E}$, $\mathbf{c}_1 \times \mathbf{E}_1$ has the same direction as $\mathbf{c} \times \mathbf{E}$ and therefore adds to the magnetic field of the first electron. While the magnetic fields produced by the two electrons under consideration may be so weak as to be insignificant, all the electrons in the coil give rise to magnetic forces in the same direction at O and hence the resultant effect may be large. The electric fields, on the other hand, cancel one another in such a manner as to leave no appreciable resultant.

It is easily seen from the diagram that at a point P outside the coil the magnetic intensity has the direction opposite to that at O.

Deduction of the Field Equations

While the illustrations which have been discussed above are instructive applications of the emission theory, the real test of its validity lies in the deduction from it of the well-tested electromagnetic equations. It has already been shown that Coulomb's inverse square law is a direct consequence of any representation of a field by continuous lines of force. Let us next consider the Ampère-Maxwell law expressed by equation (4). This is a relation between the time rate of change ($\dot{\mathbf{E}}$) of the electric intensity, and the space rates of change ($\nabla \times \mathbf{H}$) of the magnetic intensity. Consider a simple electric field $\mathbf{E}$ at the point P at time t. At a time dt later, the lines of force which passed through P initially will have been carried away by the moving elements of which they are the loci, and a new set of lines of force, which had been located at some neighboring point Q, will have moved up to take their place. In moving up from Q to P, however, these new lines may (a) crowd together due to a variation in the direction of $\mathbf{c}$ from line to line, (b) twist on account of a variation in the direction of $\mathbf{c}$ from point to point along a single line. Taking all these effects into account, a simple calculation shows that

$$\dot{\mathbf{E}} = -\mathbf{c}\nabla\cdot\mathbf{E} + \nabla\times\{\mathbf{c}\times\mathbf{E}\}$$

Putting $\mathbf{H}$ for $\frac{1}{c}(\mathbf{c}\times\mathbf{E})$ this becomes

$$\dot{\mathbf{E}} = -\mathbf{c}\nabla\cdot\mathbf{E} + c\nabla\times\mathbf{H}$$

If there are charges in the neighborhood of P, the lines of force carried by the moving elements emitted during the time dt have to be taken into account. These are responsible for a contribution to $\dot{\mathbf{E}}$ amounting to

$$(\mathbf{c}-\mathbf{v})\nabla\cdot\mathbf{E}$$

Adding this to the expression obtained above, and replacing the divergence of $\mathbf{E}$ by its equivalent ρ,

$$\dot{\mathbf{E}} = -\rho\mathbf{v} + c\nabla\times\mathbf{H}$$

which is identical with the fourth of Maxwell's equations. The interpretation, however, is very different. Here the equation is established as a purely kinematical relation between the space and time derivatives of the vectors $\mathbf{E}$ and $\mathbf{c}$ which, as all observations are made from a single system, does not even involve the relativity transformation. There is no reference to a hypothetical ether or to displacements of the same.

The two remaining field equations are obtained most easily by first obtaining expressions for the electric and magnetic intensities due to a point charge. For this simple field it may be shown that the divergence of $\mathbf{H}$ vanishes, and the curl of $\mathbf{E}$ equals $1/c$ times the time rate of decrease of $\mathbf{H}$. As any complex field may be considered as the resultant of the overlapping simple fields due to a number of point charges, these equations are valid for all fields. Their deduction, it may be noted, involves the space time transformation of the restricted relativity theory.

The Force Equation

Having deduced the four field equations, the next step is to establish an equation which will specify the effects of the electric and magnetic fields produced by any given distribution of charges and currents on a charge moving through these fields. This equation involves the transformations which give the values of $\mathbf{E}$ and $\mathbf{H}$ in one system in terms of their values as observed in some other system. Consider, for example, an observer in inertial system S who is investigating the simple field produced by a point charge permanently at rest in some other system S'. As the charge is moving relative to him it has the properties of a current element as well as those of an electrified particle. Therefore he perceives a magnetic as well as an electric field. On the other hand, to an observer going along with the charge in system S', the field is purely electrostatic. To both observers the total number of lines of electric force issuing from the charge is the same, but the first perceives in addition a magnetic field which is absent to the second. The reason for this magnetic field is immediately apparent from the point of view of the moving element theory. As seen by the observer in S', the lines of force are as pictured in figure 1. Since **c** and **E** have the same direction, there is no magnetic field. As viewed from system S, however, **c** and **E** have different directions (as in figure 2) and therefore $\mathbf{c} \times \mathbf{E}$ does not vanish. Hence a magnetic field is present.

Consider, now, the most general type of simple field. The values of **E** and **H** as measured by an observer in S are given; it is desired to deduce an expression for the electric intensity $\mathbf{E}'$ in S'. The problem is purely kinematical, involving the space-time transformation of relativity in its solution. As **E** and **H** in S are known, the direction and density of the lines of electric force and the direction of motion of the moving elements which carry them are given. Hence it is only necessary to pass to system S' and map out the same lines of force in terms of the space and time standards of this system. A few lines of algebra show that the component of the electric intensity in the direction of the relative motion of the two systems is the same in each, while at right angles to the relative motion,

$$\mathbf{E}'_{\perp} = \frac{1}{\sqrt{1-\beta^2}} \left\{ \mathbf{E}_{\perp} + \frac{1}{c} (\mathbf{v} \times \mathbf{H})_{\perp} \right\}$$

where β stands for the ratio of the relative velocity of the two systems to the velocity of light, and the subscript $\perp$ refers to those components of the vectors to which it is attached which lie at right angles to the relative velocity. The term in the right hand member of this equation which involves **H** arises from the time transformation of relativity; if the Galilean relation $t' = t$ had been used it would have been absent.

It is necessary here to make a very fundamental assumption, which is due to Lorentz. Before doing so, it may be well to review briefly the logical development of the theory up to this point. First, we have defined a moving element as a discontinuity which moves through space in a straight line with the velocity of light **c** and whose motion is unaffected by material obstructions. Secondly, we have defined a charge as a source of moving elements emitted uniformly in all directions. The third step has been to define a line of electric force as the locus of all moving elements emitted in the same direction, the term same direction being defined in a simple and natural manner. Fourthly, the electric intensity has been defined as a vector having the direction of the lines of electric force and a magnitude proportional to their density, and the magnetic intensity has been defined by the vector product $1/c(\mathbf{c} \times \mathbf{E})$. Then the four field equations have been shown to be purely kinematical relations imposed by the nature of the space-time in which we live; indeed they are no more mysterious than the equation of continuity in hydrodynamics.

It will be noted that up to this point the development of the theory has consisted in little more than setting down definitions, and making certain logical deductions from them. By itself such a procedure can never lead to a theory capable of accounting for observable phenomena. Indeed the field we have built up is purely conceptual; in order to connect it up with physical realities it is necessary to postulate a law of action of the field on a charge moving through it. The necessary postulate, which may be called the *dynamical assumption,* involves another definition which will now be considered.

In the following, we will consider electrons which are momentarily at rest relative to an observer in system S. Let de be an infinitesimal portion of the charge of such an electron. Then the product $\mathbf{E}\, de$, where $\mathbf{E}$ is the electric intensity due to all charges other than de, is defined as the *electromagnetic force* on de. The dynamical assumption, then, states that the *resultant electromagnetic force on every electron vanishes.* Consider, for instance, an electron at rest relative to the observer in an impressed field $\mathbf{E}_1$. The dynamical assumption requires that the impressed force $e\mathbf{E}_1$ shall be balanced by a back pull on the electron due to its own field. But the electron's field exerts no resultant force on it unless the electron is accelerated. If, however, an acceleration $\mathbf{f}$ exists, calculation shows that the resultant electromagnetic force on the electron due to its own field is given very closely by[6] $-m\mathbf{f}$, where m is the so-called electromagnetic mass of the electron. Applying, then, the dynamical assumption, we get

$$e\mathbf{E} = m\mathbf{f}$$

where $\mathbf{E}$ refers now to the impressed field alone.

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[6] This is only the first term of a series in $\mathbf{f}$ and its time derivatives. However the succeeding terms are generally quite negligible compared with the first.

This equation applies only to an electron momentarily at rest relative to the observer. It is, however, a simple matter to extend it to the case of a moving electron. For, whatever velocity less than light the electron may have relative to system S, it will be momentarily at rest in some other inertial system S'. So, relative to an observer in S'

$$e\mathbf{E}' = m\mathbf{f}'$$

where $\mathbf{E}'$ and $\mathbf{f}'$ are the values of the electric intensity and the acceleration as measured by the observer in S'. Since the magnitude of e is a discrete number (the number of tubes of electric force emerging from the electron), it is of necessity the same no matter from what system it may be observed. Making use, now, of the transformation just deduced for $\mathbf{E}'$ and the relativity transformation for $\mathbf{f}'$, we find that

$$e\mathbf{E}_{\parallel} = \frac{m}{(1-\beta^2)^{3/2}}\mathbf{f}_{\parallel}$$

in the direction of the electron's motion, and

$$e\left\{\mathbf{E}_{\perp} + \frac{1}{c}(\mathbf{v}\times\mathbf{H})_{\perp}\right\} = \frac{m}{\sqrt{1-\beta^2}}\mathbf{f}_{\perp}$$

at right angles thereto. These two scalar equations may be combined in the single vector equation

$$e\left\{\mathbf{E} + \frac{1}{c}(\mathbf{v}\times\mathbf{H})\right\} = \frac{d}{dt}(m_t\mathbf{v}) \qquad (6)$$

where

$$m_t \equiv \frac{m}{\sqrt{1-\beta^2}}$$

is the transverse mass of the electron. Now, on the classical dynamics the mass of a particle is taken to be independent of its velocity, whereas if we identify the time rate of change of $m_t\mathbf{v}$ with the force on the electron it is seen that the mass increases with the velocity, becoming infinite as the velocity of light is approached. Experiments with fast moving beta rays have verified the formula given above for the mass; in fact, the reason that the mass has been assumed constant in dynamics is that such experiments as could be carried out with gross matter have involved speeds quite insignificant as compared with the speed of light.

Identifying the right hand member of (6) with the force on the electron, and referring the equation to a unit volume, the fifth of the electromagnetic equations is obtained. However, equation (6) is more specific than (5) in that it does not involve the intermediate concept of force but gives at once a relation between the electric and magnetic intensities and the motion of the electron on which they act.

It has been noted that the term involving **H** on the left hand side of (6) is introduced by the time transformation of the restricted relativity theory. Now the two essential kinematical consequences of this theory are the following: (1) the measured length of a rod moving relative to the observer in the direction of its length is less than the length of the same rod when at rest, (2) the measured rate of a moving clock is less than the rate of the same clock when at rest. The first of these consequences is proved by the Michelson-Morley experiment, but this experiment offers no test of the second. On the basis of the emission theory, however, the second is abundantly verified by the presence of the second term on the left hand side of (6). For this term is responsible for the action of every dynamo and every motor in this age of electricity.

Intrinsic Magnetic Fields

The moving elements emitted from an elementary charge with the velocity of light have served as carriers of the lines of electric force in the theory which has been developed. The magnetic intensity has played a subsidiary role, and no attempt has been made to account for the magnetic field on the same basis as the electric field. The fact that the divergence of the magnetic vector is always zero shows that, at any one instant, the magnetic field can be represented by continuous lines of force which specify by their direction and density the direction and magnitude respectively of the magnetic intensity. The question therefore arises: can the same moving elements that carry the lines of electric force be strung together in such a way as to carry the lines of magnetic force as well? It is perfectly obvious that suitable loci of the moving elements can be found which will represent the lines of magnetic force at a specified instant relative to a specified observer. The determination of such loci, in fact, is contained in the definitional equation of the magnetic intensity. Our purpose, however, is much broader; we wish to investigate the possibility of connecting the moving elements in such a manner as to give lines of force which shall represent correctly the magnetic field at all times and relative to all observers.

A brief consideration shows that this is not possible for that part of the magnetic field of a point charge which is due to its velocity alone. Consider a charge permanently at rest in system S'. Relative to an observer in this system, its field is purely electrostatic; no lines of magnetic force are present. Relative to an observer in some other inertial system S, however, the charge is moving, and therefore it has a magnetic field. Hence the representation used for the more fundamental lines of electric force cannot be applied to this type of magnetic field. For a change in the observer's state of motion merely changes his point of view, altering, perhaps, the density and direction of the lines of force but annulling none of them. Such a field as we are considering here will be termed an *apparent field,* as it can be wiped away by a change in the observer's state of motion.

When an elementary charge is accelerated, a magnetic field is produced which depends primarily upon the acceleration of the charge. It is this field which, together with the part of the electric field due to acceleration, determines the irreversible radiation from the charge. This field, which will be designated as the *intrinsic magnetic field,* cannot be annulled by transferring the observer to some other inertial system. Now it is found that this field can be represented by lines of magnetic force which are carried by the same moving elements that carry the lines of electric force. Consider all moving elements emitted at the same instant in directions making the same angle with the direction of acceleration, the angle being measured in the system in which the charge is momentarily at rest. The locus of these moving elements constitutes a line of force of the intrinsic magnetic field for all times and all observers. As these circular[7] lines of magnetic force are carried out by the moving elements which compose them, their diameters grow larger and the density of the lines

[7] Circular relative to the system in which the charge was momentarily at rest at the instant of emission.

becomes less, falling off inversely with the distance from the charge. Figure 5 pictures the combined electric and intrinsic magnetic lines in the field of a charged particle moving with constant acceleration. The moving elements are represented by black dots, the lines of electric force are labeled **E**, and those of magnetic force **H**. At the instant for which the diagram is drawn

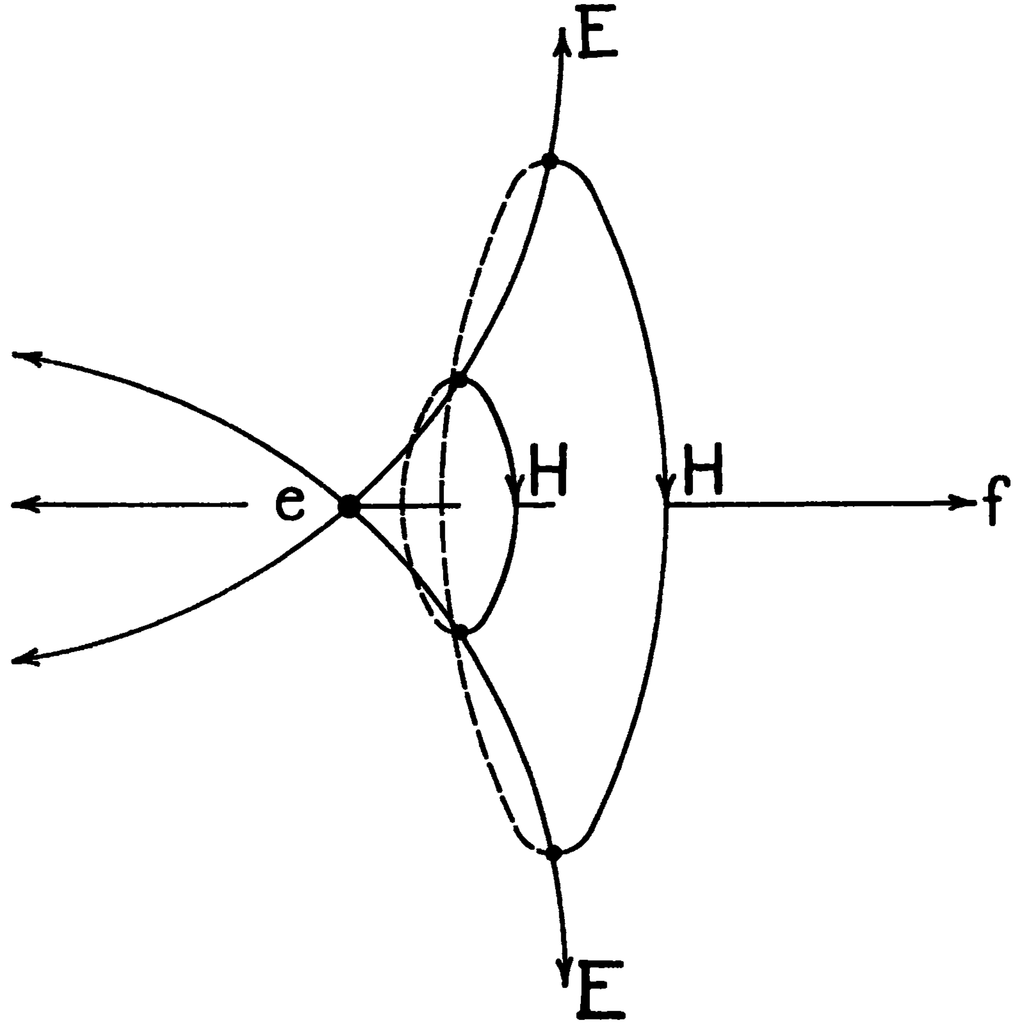

Figure 5.
Electric and Intrinsic Magnetic Fields of an Accelerated Charge.

the charge is at rest relative to the observer and the moving elements are moving along the tangents to the lines of electric force. In this case both sets of lines of force are circles the diameters of which increase the farther away they are carried from the charge.

While the intensity of the apparent magnetic field falls off with the inverse square of the distance from the charge, that of the intrinsic field falls off with the inverse first power. Hence, at a considerable distance from the charge, the intrinsic field alone persists. In an electromagnetic wave it constitutes the sole magnetic field. In the wave pictured in figure 3 the lines of magnetic force through the moving elements 8, 10 . . . are circles lying in planes at right angles to AC and with centers on this line. As time goes on, the circumferences of these circles are carried farther away from the charge, thus increasing their diameters and diminishing their density.

It would seem desirable to relate the magnitude of the magnetic intensity to the density of the lines of magnetic force in the same manner as in the corresponding electrical case. Thus, in an intrinsic magnetic field, there should be M lines of force to a tube, and the number of tubes per unit cross section should equal the magnetic intensity. Furthermore, it would be expected that every moving element should lie at the intersection of an electric and a magnetic line. It is quite possible to satisfy these two objectives, and their fulfillment leads to a relation between the frequency of emission of moving elements and the number M of lines to a tube. As, however, this relation is not at present susceptible of experimental test, it is of no particular interest.

A remarkable symmetry is seen to exist between the linking up of moving elements required to give lines of magnetic force and that required to give lines of electric force. For if a point charge momentarily at rest in system S at the time t passes into system S' at a time dt later on account of its acceleration, two moving elements emitted respectively at the times t and $t + dt$ link up to form an infinitesimal section of a line of electric force if their velocity vectors make the same angles with the direction in which S' is moving relative to S and lie in the same plane with the direction of relative motion. On the other hand two moving elements whose velocity vectors make the same angles with the direction of the motion of S' relative to S but do not lie in the same plane link up to form a line of magnetic force if emitted at the same time. A line of electric force is the locus of moving elements emitted at different times but in the same plane, whereas a line of magnetic force of the intrinsic field is the locus of moving elements emitted in different planes but at the same instant.

The emission theory described in this paper gives a kinematical picture of electromagnetic fields which leads unambiguously to the equations of classical electrodynamics and at the same time provides a substitute for the ether which is entirely in accord with the relativity principle. Instead of being propagated through an elastic medium in the manner of material waves, electromagnetic waves are to be thought of as carried by discontinuities emitted from the source with the velocity of light. The distinction between the two points of view is essentially the distinction between the type of wave which may be propagated by an elastic cord whose ends are fixed, and that which is carried by the drops of water emerging from a garden hose which is given a simple harmonic motion at right angles to the nozzle. As the hose is waved up and down, the water spouts out in varying directions, and the locus of the drops emerging from it differs from our picture of the electric wave emitted by an oscillating charge only in that the drops have smaller velocities than moving elements.

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