Theory of the Double Fizeau Toothed Wheel

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I

IN a discussion of the theory of relativity, R. W. Wood[1] has considered a modification of Fizeau's method for determining the velocity of light, in which two toothed wheels are mounted at the ends of a long axle, and light is sent in one direction only. If such an apparatus, rotated while stationary at such a speed that light passing through a tooth in the near wheel is transmitted by a given tooth in the distant wheel, is then set in motion in the direction of the axis of rotation, the velocity of light relative to the apparatus will be changed so that transmission will no longer take place, unless some alteration in dimensions or shape occurs. A Fitzgerald contraction of length, by the factor $(1-v^2/c^2)^{\frac{1}{2}}$, where v is the translational velocity, will not restore transmission, since the changed velocity of light is $c\pm v$. Wood concludes that there must be an additional compensating factor, namely a twist of the axle. F. E. Hackett[2] has discussed the same problem more in detail, and considers the "twist" as analogous to the twist of a steel tube placed in a coaxial spiral magnetic field (Wiedemann effect).

It is the purpose of this paper to point out that this "twist" is not a phenomenon peculiar to rotation, or supplementary to the Fitzgerald contraction; it is a necessary result of that contraction when any uniformly moving body is constrained to a definite orientation. The Fitzgerald contraction (in conjunction with the Larmor-Lorentz change of rate of a clock in motion) suffices in fact to elucidate all problems connected with the behavior of rotating disks set in uniform translational motion.

Consider the double Fizeau toothed wheel of Wood, which is shown schematically in plan view in Fig. 1. Here W_1 and W_2 are the two toothed wheels, mounted on the axle A. The tops of the wheels, which are presented to the observer, are in motion toward the right, with the velocity $V=r\omega$. The apparatus as a whole is moving with the velocity v in the direction parallel to the axle. We have at once that a rectangular element of the cylinder bounding the wheels, a_1, a_1', b_1, b_1', is moving at the instant depicted, with the velocity $R=(v^2+V^2)^{\frac{1}{2}}$ in a direction inclined to the axle by the angle $\theta=\tan^{-1} V/v$. The immediate problem is to determine the shape taken by the elementary rectangle a_1, a_1', b_1, b_1', due to the Fitzgerald contraction.

This problem may be profitably studied in two steps. As the first step we consider the element a_1, a_1', b_1, b_1', as a rectangle symmetrically disposed about a line xx', at right angles to the direction of motion and maintaining its orientation unchanged when in motion (Fig. 2). For purposes of clear exposition let us take the components of velocity large as compared with the velocity of light, e.g. $v=0.8$ c, $V=0.4$ c. The resultant shape of the elementary rectangle is obtained by reducing the normal distances of all points from the line xx' in the ratio $(1-R^2/c^2)^{\frac{1}{2}}:1$. This gives us the figure, a_2, a_2', b_2, b_2'; the rectangle becomes a rhomboid whose sides are inclined both to the axle and to the planes of the wheels.

This shape cannot represent the true condition in the toothed wheel system of Fig. 1, since the horizontal sides of the element are actually mechanically constrained to be perpendicular to

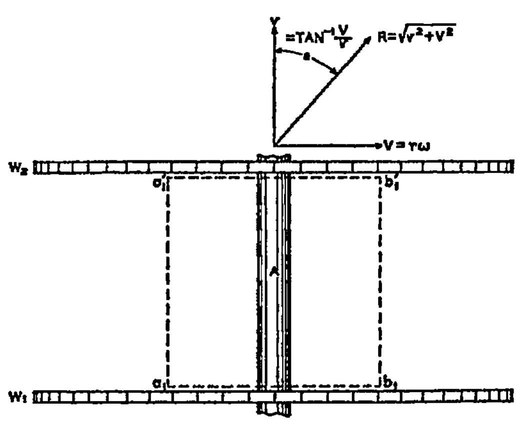

FIG. 1. Plan view of double Fizeau toothed wheel.

[1] R. W. Wood, *Physical Optics* (Macmillan, second edition, 1911), p 690.
[2] F. E. Hackett, Phil. Mag. 44, 740 (1922).

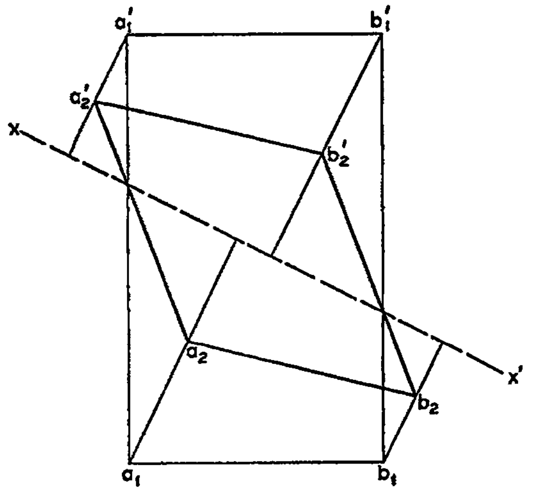

FIG. 2. Fitzgerald contraction of rectangle.

the axle. We must therefore find the contracted shape conforming to this condition. To do this we follow the construction shown in Fig. 3. We have as before the axis xx', normal to the direction of motion and contraction, which we select as passing through a corner b_1, of the rectangle. We take any point p' on the toothed wheel W_1 whose perpendicular distance from xx' is s', and note that before contraction its perpendicular distance from xx', which we will call s, must be $s'/(1-R^2/c^2)^{\frac{1}{2}}$. This gives us the point p, which is a point on the side of the rectangle which, in motion, because of the contraction, will be on W_1. We now proceed to erect on the line determined by the fixed corner of the rectangle b_1 and p, an undistorted, uncontracted rectangle, similar to a_1, a_1', b_1, b_1', which we may identify by the same letters with the subscript 2, namely a_2, a_2', (b_1), b_2'. We next subject this rectangle to the contraction $(1-R^2/c^2)^{\frac{1}{2}}$, normal to xx', by the same procedure as before, giving us finally the rhomboid, a_3, a_3', b_1, b_3', a figure which has the shape and the orientation assumed by our original rectangle under the Fitzgerald contraction when mechanically constrained to have its base coincide with W_1.

It is now necessary to show that the rhomboid a_3, a_3', b_1, b_3', owing its shape solely to the Fitzgerald contraction under mechanical constraint, possesses the exact shape and dimensions to insure invariance in the behavior of the double Fizeau toothed wheel,—in other words that its shape is equivalent to a Fitzgerald contraction in the direction of motion of the system of value $(1-v^2/c^2)^{\frac{1}{2}}$, plus a "twist" of the right value.

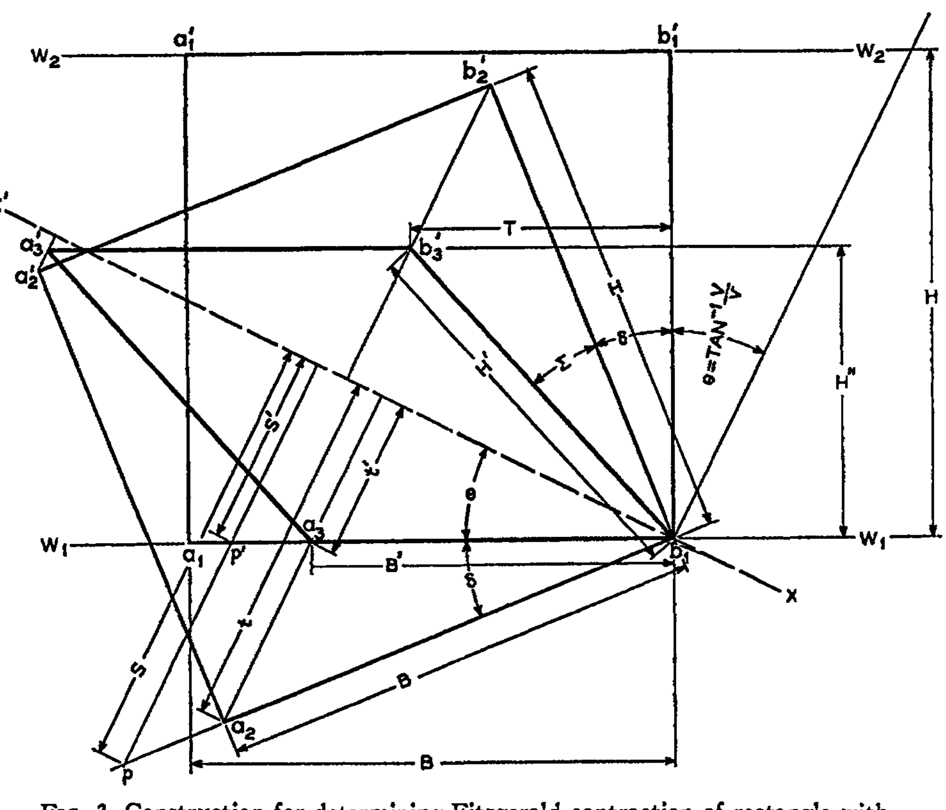

FIG. 3. Construction for determining Fitzgerald contraction of rectangle with base constrained to definite orientation.

In order to carry through this proof we first apply distinguishing designations to the various distances and angles in Fig. 3, as follows:
the base of the original rectangle, $a_1-b_1=B$,
the altitude of the original rectangle, $a_1-a_1'=b_1-b_1'=H$,
the base of the final rhomboid, $a_3-b_1=B'$,
the side of the final rhomboid, $b_3-b_1=H'$,
the altitude of the final rhomboid $=H''$,
the displacement of b_3' from the original side, $b_1-b_1'=T$,
the angle between the tilted rectangle, a_2, a_2', b_1, b_2' and $W_1=\delta$,
the angle between b_1-b_3' and $b_1-b_1'=\delta+\epsilon$,
the uncontracted normal distance of a_2 from $xx'=t$,
the contracted normal distance of a_3 from $xx'=t(1-R^2/c^2)^{\frac{1}{2}}=t'$.

Proceeding now first to the determination of B' we have, from the oblique triangle a_2, a_3, b_1

$$B'=B\frac{\cos(\theta+\delta)}{\cos\theta} \tag{1}$$

and

$$\frac{\tan(\theta+\delta)}{\tan\theta}=\frac{t}{t'}=\frac{1}{(1-R^2/c^2)^{\frac{1}{2}}}. \tag{2}$$

From (2) we get by elementary trigonometric transformations, the value of $\cos(\theta+\delta)$, giving finally

$$B'=B\frac{(1-R^2/c^2)^{\frac{1}{2}}}{\cos\theta(1-(R^2/c^2)+\tan^2\theta)^{\frac{1}{2}}}. \tag{3}$$

By a series of similar operations, involving intricate but straightforward transformations, which need not be repeated here, we obtain

$$H'=H\left(\frac{\tan^2\theta+(1-R^2/c^2)^2}{\tan^2\theta+(1-R^2/c^2)}\right)^{\frac{1}{2}}, \tag{4}$$

$$H''=H\left(\frac{\tan^2\theta+(1-R^2/c^2)^2}{\tan^2\theta+(1-R^2/c^2)}\right)^{\frac{1}{2}} \cdot\frac{\tan^2\theta+(1-R^2/c^2)}{([\tan^2\theta+(1-R^2/c^2)]^2+R^4/c^4\tan^2\theta)^{\frac{1}{2}}}, \tag{5}$$

$$T=H\left(\frac{\tan^2\theta+(1-R^2/c^2)^2}{\tan^2\theta+(1-R^2/c^2)}\right)^{\frac{1}{2}} \cdot\frac{(R^2/c^2)\tan^2\theta}{([\tan^2\theta+(1-R^2/c^2)]^2+R^4/c^4\tan^2\theta)^{\frac{1}{2}}}. \tag{6}$$

These formulae, upon substituting $\tan\theta=V/v$, and $R^2/c^2=V^2/c^2+v^2/c^2$ and arranging terms, take the simple forms:

$$B'=B\left(1-\frac{V^2}{c^2(1-v^2/c^2)}\right)^{\frac{1}{2}}, \tag{7}$$

$$H'=H\left(1-\frac{v^2}{c^2}+\frac{V^2v^2}{c^4(1-v^2/c^2)}\right)^{\frac{1}{2}}, \tag{8}$$

$$H''=H(1-v^2/c^2)^{\frac{1}{2}}, \tag{9}$$

$$T=H\frac{Vv}{c^2(1-v^2/c^2)^{\frac{1}{2}}}. \tag{10}$$

We are now ready to compare the behavior of the double-toothed wheel system when stationary, and when moving in the direction of its axle with the velocity v.

Taking first the stationary case, we have on putting V_s for the peripheral velocity of the disks, the simple relation

$$V_s/c=d/H, \tag{11}$$

where d is the distance from the leading side of the rectangle that a light beam from b_1, projected normally to W_1, will hit. Substituting $r\omega$ for V_s we get

$$d=Hr\omega/c. \tag{12}$$

For the double Fizeau-disk system in motion we have the following relations

$$H''+vt=ct, \tag{13}$$

$$d'=V_mt-T, \tag{14}$$

where V_m is the peripheral velocity with the system moving; and t is the time required for the light signal to traverse the moving rectangle; from which,

$$d'=VH''/(c-v)-T. \tag{15}$$

Using (9) and (10) this gives

$$d'=\frac{HV_m}{c(1-v^2/c^2)^{\frac{1}{2}}}. \tag{16}$$

We now consider the significance of the terms $V/(1-v^2/c^2)^{\frac{1}{2}}$ in (7), (8), (9) and (10). Remembering that when the system is set in motion with the velocity v the clock by which we measure the angular velocity of the disk goes slower in

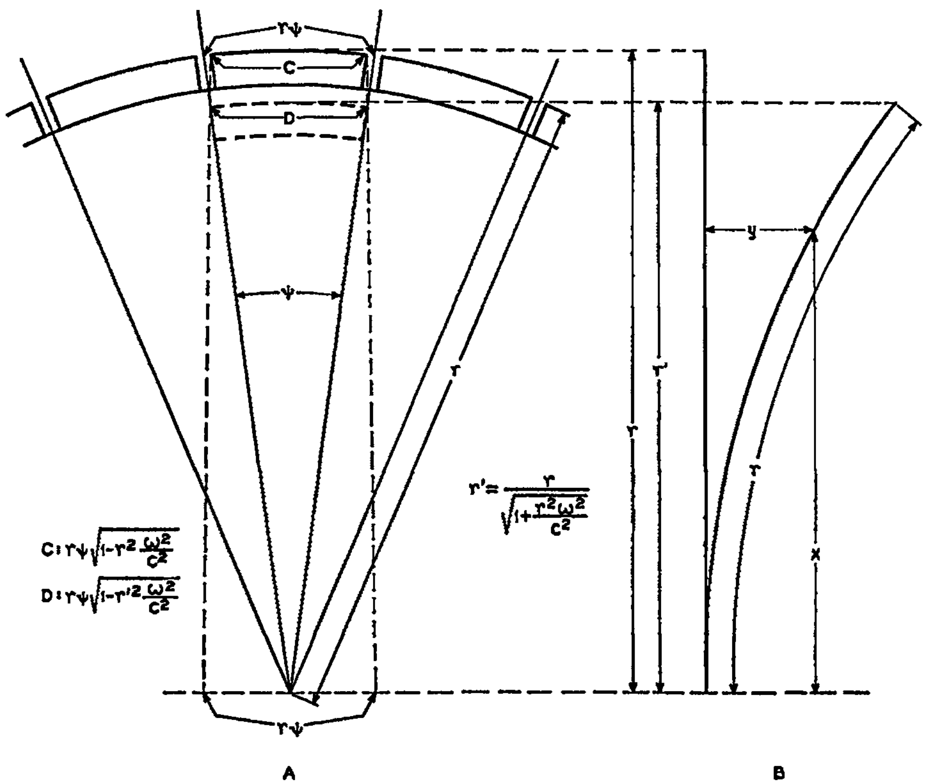

FIG. 4. "Dishing" of a rotating disk subject to Fitzgerald contraction.

the ratio $(1-v^2/c^2)^{\frac{1}{2}}$, we see at once that $V_m = V_s(1-v^2/c^2)^{\frac{1}{2}}$ when the angular velocity is set at the same apparent value. Hence (16) becomes:

$$d' = HV_s/c = Hr\omega/c, \tag{17}$$

so that by comparison with (12)

$$d' = d. \tag{18}$$

We find therefore that the Fitzgerald contraction is of itself sufficient to give both the contraction in the direction of motion (9) and the "twist" (10) necessary to insure the invariance of behavior of the apparatus when in motion at right angles to the plane of the toothed wheels.

II

The above solution of the Fizeau double-toothed wheel problem in terms of Fitzgerald contraction alone prompts the discussion along similar lines of some associated problems in the behavior of rotating disks, which are considered in this and the succeeding sections.

From the above discussion it follows that the two toothed wheels suffer a contraction of peripheral length. This contraction must occur for each wheel, and, by placing $v=0$, we see that for a single stationary rotating disk the periphery must be contracted in the ratio $(1-r^2\omega^2/c^2)^{\frac{1}{2}} : 1$. A difficulty here arises, pointed out by Ehrenfest, that a radius of the disk, which is moving transversely to its length, will suffer no contraction, thus conflicting with the elementary relation between the diameter and the periphery of a circle. This problem has been considered by Lorentz, and by Eddington,[3] who decide by methods involving the general theory of relativity, that both periphery and radius must contract alike, by the factor $(1-\frac{1}{4}v^2/c^2)^{\frac{1}{2}}$. This solution appears incompatible with the results of Section I.

A solution in terms of the simple Fitzgerald contraction is possible if we abandon a condition which is usually tacitly assumed in the Ehrenfest paradox, namely that the disk must remain *flat*. The nature of this solution is at once evident on

[3] A. S. Eddington, *Mathematical Theory of Relativity* (Cambridge University Press, 1923), p. 112.

recalling the shape which is given to a circular saw blade in order that, when in rotation, it shall resist the outward stresses set up by the high peripheral velocity. The saw blade is made in a "dished" shape, and becomes flat at high speeds. We have with the Fitzgerald contraction an inverse condition, the disk is under stress to become smaller instead of larger. The solution is for it to be flat when not rotating, to "dish" as it rotates.

We may calculate the configuration of the rotating disk by the construction shown in Fig. 4, where a section of the disk is shown in plan at A, in profile at B. Referring to A, a peripheral element of the disk of length $r\psi$ assumes, because of the Fitzgerald contraction, the length $r\psi(1-r^2\omega^2/c^2)^{\frac{1}{2}}$. By so doing it becomes shorter than the section of the (uncontracted) periphery which it occupied when stationary. In order to bring all the so shortened peripheral elements into contact again, each element must be moved toward the center. In being so moved however an element increases in length (by decreasing its Fitzgerald contraction) due to the decreased velocity as the radial distance becomes smaller. Denoting the radius at which the moved element just touches the next as r', we express the relationship resulting as follows:

$$r'\psi = r\psi\left(1-\frac{r^2\omega^2}{c^2}\right)^{\frac{1}{2}}\left[\frac{(1-r'^2\omega^2/c^2)^{\frac{1}{2}}}{(1-r^2\omega^2/c^2)^{\frac{1}{2}}}\right]$$

or (19)

$$r' = r(1-r'^2\omega^2/c^2)^{\frac{1}{2}},$$

from which

$$r' = \frac{r}{(1+r^2\omega^2/c^2)^{\frac{1}{2}}}. \tag{20}$$

Remembering that the length of the radius r is unaffected by the rotation, we see from B that the prescribed configuration of the radius is a curve of length r for which the relationship (20) is obeyed. This is shown in Fig. 4B.

In order to determine the equation of this curve, let y be the distance of any point on the curve from the stationary trace, and $x(=r')$ the distance of the point from the axis. Calling $\omega^2/c^2=\alpha^2$,

$$r = \frac{x}{(1-x^2\alpha^2)^{\frac{1}{2}}},$$

$$dr = \frac{dx}{(1-\alpha^2x^2)^{\frac{3}{2}}},$$

$$\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} = \frac{dr}{dx} = \frac{1}{(1-\alpha^2x^2)^{\frac{3}{2}}},$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{(1-\alpha^2x^2)^3} - 1,$$

$$dy = \frac{\alpha x(3-3\alpha^2x^2+\alpha^4x^4)^{\frac{1}{2}}}{(1-\alpha^2x^2)^{\frac{3}{2}}}dx,$$

the integral of which is

$$y = \frac{\sqrt{3}}{2} - \frac{((3+\alpha x^2)(1-\alpha x))^{\frac{1}{2}}}{2} + 2\left[\tan^{-1}\left(\frac{1-\alpha x^2}{3+\alpha x^2}\right)^{\frac{1}{2}} - \tan^{-1}\frac{1}{\sqrt{3}}\right]. \tag{21}$$

When ω is small

$$dy/dx \cong \sqrt{3}\alpha x,$$

$$y \cong \frac{\sqrt{3}}{2}\alpha x^2 = \sqrt{3}\frac{r'^2\omega}{c}, \tag{22}$$

or the disk assumes an approximately paraboidal shape.

The contraction in diameter, and the dishing, produced by the Fitzgerald contraction, are measures of *rotation with respect to the ether*—not of *relative* rotation of material bodies. Two similar disks in relative rotation cannot each be contracted in diameter with respect to the other. It follows that by rotating a disk at such a speed that it exhibits maximum diameter we establish that it has no rotational motion with respect to the ether.

III

In the problem of the double Fizeau toothed wheel as treated in Section I the effect of the translational motion of the system is relatively simple, since the axial symmetry of the disks is unaffected by motion at right angles to their

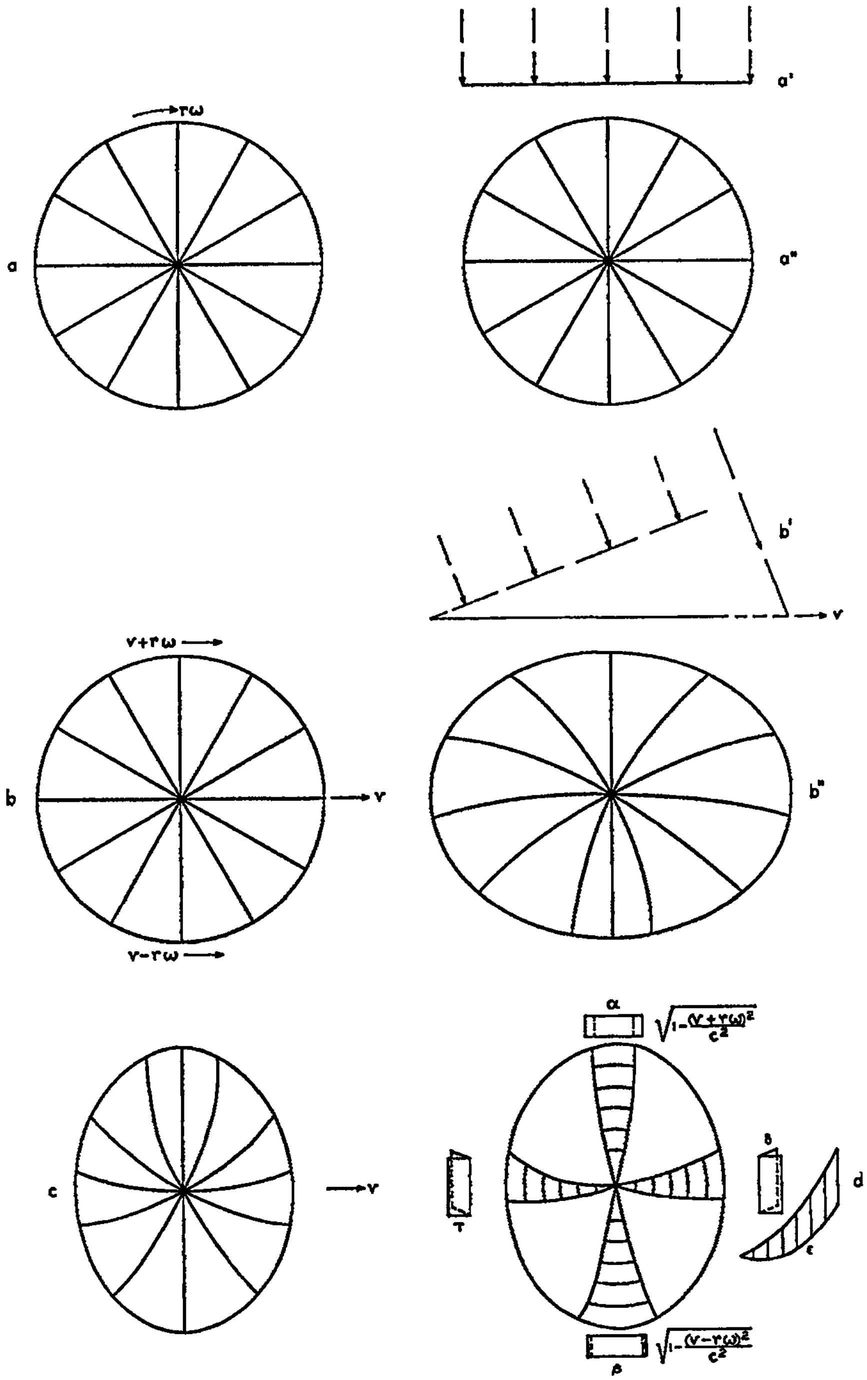

FIG. 5. Fitzgerald contraction of rotating disk moving in its own plane.

planes. A more complicated problem is presented when the translational motion is at some other angle to the plane of a rotating disk, for instance parallel to that plane. In this case, (Fig. 5b) the velocities of elements of the periphery range from $v+r\omega$ to $v-r\omega$, calling for different Fitzgerald contractions at different points.

I shall not present a detailed treatment of this case, but will confine myself to showing that the Fitzgerald contractions are of the required character to produce a configuration of the disk which will mask the effects of the translational motion. I shall attack this problem in the reverse direction from that followed in Section I, taking up first the results to be expected from optical observation without the occurrence of any form of contraction, deduce from these what the contractions need to be to offset the optical effects introduced by motion, and compare these with the contractions as predicted by the analysis used above in connection with Fig. 3.

Assume a flat opaque disk (Fig. 5a) rotating with the angular velocity ω. Suppose it to be perforated with a number of radial slits. Now let a light flash be originated from a distant point on the upwardly produced axis (Fig. 5a′) and, passing through the disk, fall upon a light sensitive surface, such as a photographic plate. The trace so obtained will be (Fig. 5a″) an exact copy of the disk.

Let us next set the disk in motion to the right (Fig. 5b) with the velocity v. Owing to the motion the light flash from the distant point on the axis no longer meets the disk normally, but at the angle $\sin^{-1} v/c$; (Fig. 5b′) the leading edge of the disk is recorded later than the following edge, it will have traveled a distance greater than the diameter, and the radial slits will, during this interval, have turned about the axis. As a result the photographic trace of the disk will be a lengthened ellipse; the traces of the slits will be curved lines, as shown in Fig. 5b″.

If the rotating disk in motion is to yield an optical record which shall be indistinguishable from the record of the stationary disk (which is, broadly speaking, what the Fitzgerald contraction was invented to do) the disk must assume a configuration of a character inverse to b″. It must be contracted in the direction of motion; its slits must have the opposite curvature to those of the trace b″. The necessary disk configuration is that shown in Fig. 5c.

It now remains to show that the distorted disk 5c with its curved slits is the figure that the original circular disk with radial slits would assume as a result of the Fitzgerald contraction. Referring to Fig. 5d we see how this can come about: At α and β are shown peripheral elements of the circular disk moving parallel to the direction of translation, as they would be contracted due to their velocities, α showing greater contraction than β because of its greater velocity ($v+r\omega$ as against $v-r\omega$). At γ and δ are shown peripheral elements moving transverse to the direction of translation. These, constrained on their sides toward the center of the disk, are warped, according to the analysis of Section I into rhombs, whose transverse diameter is reduced by the factor $(1-v^2/c^2)^{\frac{1}{2}}$. Finally consider, at ϵ, the configuration taken by a series of radial elements starting at the center of the disk. These will be increasingly warped with increasing radial distance, and being constrained to remain in contact, will take the horn-shape shown. It is clear, without more detailed analysis, that the configuration c is that produced by the Fitzgerald contraction.[4]

[4] Both stationary and moving rotating disks are subject also to the dishing effect already discussed.