Secretary-Treasurer, OSCAR S. ADAMS, U. S. Coast and Geodetic Survey.
Member of the Executive Committee, Prof. CLARA L. BACON, Goucher College.

After the program the visitors were shown about the beautiful grounds and buildings or were taken to see some of the various athletic events that were in progress.

OSCAR S. ADAMS, *Secretary.*

THE GROWTH OF THE SOLAR SYSTEM.

By WILLIAM DUNCAN MACMILLAN, University of Chicago.

Since the sun with its attendant planets is journeying through space with a speed of about 12 miles per second and since there are large regions of space which are visibly nebulous it follows that at rare intervals the sun must pass through such nebulosity and add to its mass. This nebulosity varies from a relatively great density such as occurs in Andromeda and regions in Orion to a density so low that it can be detected only by photographic means, as for example, the region of the Pleiades. The direct evidence ends, perhaps, with the photographs of these regions of low visibility, but the apparently great variation in nebular density compels us to admit the existence of nebulosity below the bounds of visibility even photographically; and from the physical nature of these objects we cannot doubt that regions of very low density are much more extensive than regions of higher density.

In addition to nebulosity which is self-luminous the phenomenon of dark regions with apparently luminous back grounds is interpreted by Barnard and others as indicating the existence of nebulosity which is not self-luminous. How extensive these regions of dark nebulosity are we have but little means of knowing for it would be only rarely that they would be projected upon luminous backgrounds, but such evidence as there is indicates that they are rather extensive.

In addition to matter in a distinctly nebulous form one can scarcely escape postulating the existence of isolated solid fragments, for it is easy to see how such solid fragments can occur. In the course of time—very long, perhaps—some star will pay us a visit, with the result of considerable disturbance to our present orderly system. How serious this disturbance will be will depend, naturally, upon the particular circumstances under which it occurs, and the character of the system which does the visiting. A not impossible result would be to throw one or more of the planets out into space, and it seems almost certain that some of our comets and planetoids and some of our very numerous meteors would receive that kind of treatment. Our sun would doubtless reciprocate upon the other system, so that one of the final results of such a visit would be some loose fragments in space.

Indeed it is not easy to avoid the idea that comets and their resultant meteors are fragments which the sun has picked up and attached to our system at some epoch in the sun's very ancient history. They are certainly fragments from a once larger body—thereby testifying to the fact that large solid bodies are sometimes broken up and their remains scattered. It seems certain that there must be more or less of this material in space.

How much of this material will the sun and planets pick up as they move along? It will simplify matters to suppose that this material is initially stationary and that the particles move only under the sun's gravitation. This hypothesis is doubtless contrary to the facts but inasmuch as their motions will be at random with respect to the sun this hypothesis will give at least a first approximation to the actual facts. The particles, then, will describe hyperbolas about the sun, and every particle whose perihelion distance is less than the radius of the sun will fall into the sun and add its mass to the sun's mass. All of the others will pass by and be of no consequence.

Imagine the sun moving along a straight line with a speed s, and consider a particle at a distance D from this line while the sun is still very remote. Since the particle is initially at rest it has a speed s with respect to the sun along a line parallel to the sun's motion. If k^2 is the gravitational constant, M the sun's mass and a the semi-axis of the particles' orbit about the sun, then the velocity of the particle with respect to the sun at a distance r is given by the formula

$$v^2 = k^2 M\left(\frac{2}{r} + \frac{1}{a}\right).$$

When $r = \infty$ then $v = s$, from which it follows that

$$a = \frac{k^2 M}{s^2}. \qquad (1)$$

From the law of areas, or moment of momentum, we have also

$$xy' - yx' = \sqrt{k^2 Ma(\epsilon^2 - 1)},$$

where ϵ is the eccentricity of the hyperbolic orbit. But when very remote from the sun $y' = 0$, $y = D$, $x' = -s$, and therefore

$$D^2 s^2 = k^2 Ma(\epsilon^2 - 1). \qquad (2)$$

Eliminating a between (1) and (2) and then solving for ϵ we have

$$\epsilon = \sqrt{1 + \frac{s^4 D^2}{k^4 M^2}}, \qquad (3)$$

so that the perihelion distance is

$$a(\epsilon - 1) = \sqrt{\frac{k^4 M^2}{s^4} + D^2} - \frac{k^2 M}{s^2}.$$

If now the radius of the sun is R, then in order that the particle may strike the sun we must have

$$\sqrt{\frac{k^4M^2}{s^4} + D^2} - \frac{k^2M}{s^2} < R,$$

from which it is seen that

$$D^2 < \frac{2k^2MR}{s^2} + R^2. \tag{4}$$

In the case of a body having the mass, speed, and size of the sun the term R^2 is small as compared to $2k^2MR/s^2$, so that if we take a cylinder of radius

$$\rho = \frac{\sqrt{2k^2MR}}{s}, \tag{5}$$

we can say that all of the particles within this cylinder will be swept up by the sun, and the sun's *effective* radius is ρ. For the sun its numerical value is 14,000,000 miles.

Since in one unit of time the sun moves through a distance s, it follows that the rate at which material is swept up is

$$\frac{dM}{dt} = \pi\rho^2 s\delta = \pi\delta\left(\frac{2k^2MR}{s} + R^2 s\right), \tag{6}$$

where δ is the average density of matter in space. This expression has a minimum for $s^2 = 2k^2M/R$, which for our sun gives $s = 383$ miles per second, and for this minimum value $\rho = \sqrt{2}R$. For such speeds as that of the sun the last term of (6) might be neglected in which case we would have the result that the rate at which the mass increases is inversely proportional to the speed.

If we knew the value of δ, which is the amount of matter in a unit volume of space we could compute the rate at which the sun is growing. Without doubt δ varies from one region of space to another. In a recent article in the *Astrophysical Journal*[1] the estimate was made that one particle one one-hundredth of an inch in diameter for every 560 cubic miles of space would account for the blackness of the night sky. Taking this value of the density and the present speed of the sun it would require 1.4×10^{17} years for the mass of the sun to double. If, however, the sun should enter a nebulous region in which the density is as low as 10^{-12} times the density of air at sea level its mass would double in one billion years. A density of this tenuity would be obtained if one cubic foot of air were so expanded as to fill 8 cubic miles. We have of course no means of knowing what the densities of the nebulas are but it is evident that if the sun should penetrate such a region as that of the Orion Nebula its mass would be materially increased in a relatively short time. But independently of such special regions it is clear that not only the sun but all of the stars are gathering in material unto themselves sometimes rapidly but usually slowly.

[1] MacMillan, *Astrophysical Journal*, volume 48, July, 1918.

A closely related subject is the effect of this process upon the planetary system. In order to simplify the discussion let us suppose that the sun and Jupiter in their Keplerian motion pass through a region of space filled with material which they gather in as they journey along, and seek the effect of this growth upon the system. Let ξ_1, η_1, ζ_1 and ξ_2, η_2, ζ_2 be the coördinates of Jupiter and the sun respectively with respect to axes fixed in space; m_1 and m_2 are their masses. The components of momentum at any instant will be $m_1\xi_1'$, $m_1\eta_1'$, $m_1\zeta_1'$; $m_2\xi_2'$, $m_2\eta_2'$, $m_2\zeta_2'$. If either or both of the bodies collide with a stationary mass or particle their masses will be increased and their velocities diminished, but their instantaneous momenta will be unaltered. The differential relations between the changes of velocities and masses are therefore

$$(7)\qquad \begin{aligned} d\xi_1' &= -\frac{\xi_1'}{m_1}dm_1, & d\eta_1' &= -\frac{\eta_1'}{m_1}dm_1, & d\zeta_1' &= -\frac{\zeta_1'}{m_1}dm_1, \\ d\xi_2' &= -\frac{\xi_2'}{m_2}dm_2, & d\eta_2' &= -\frac{\eta_2'}{m_2}dm_2, & d\zeta_2' &= -\frac{\zeta_2'}{m_2}dm_2. \end{aligned}$$

If $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ are the coördinates of the center of mass and M is the sum of the masses, the integrals for the motion of the center of mass are

$$(8)\qquad \begin{aligned} M\bar{\xi}' &= m_1\xi_1' + m_2\xi_2' = \alpha_1 = Ms\cos\lambda_1, \\ M\bar{\eta}' &= m_1\eta_1' + m_2\eta_2' = \alpha_2 = Ms\cos\lambda_2, \\ M\bar{\zeta}' &= m_1\zeta_1' + m_2\zeta_2' = \alpha_3 = Ms\cos\lambda_3, \end{aligned}$$

where s is the speed of the center of gravity and $\lambda_1, \lambda_2, \lambda_3$ are the direction angles of the path of the center of gravity. It is evident that α_1, α_2, and α_3 are unaltered by the collision.

We will take a new set of axes parallel to the fixed axes but with the origin at the sun. Then the coördinates of Jupiter with respect to the sun will be

$$(9)\qquad \begin{aligned} x &= \xi_1 - \xi_2, & y &= \eta_1 - \eta_2, & z &= \zeta_1 - \zeta_2, \\ x' &= \xi_1' - \xi_2', & y' &= \eta_1' - \eta_2', & z' &= \zeta_1' - \zeta_2'. \end{aligned}$$

By differentiation of the second set of (9) we get

$$dx' = d\xi_1' - d\xi_2', \qquad dy' = d\eta_1' - d\eta_2', \qquad dz' = d\zeta_1' - d\zeta_2';$$

and by substitution of values from (7)

$$(10)\qquad \begin{aligned} dx' &= -\left[\frac{\xi_1'}{m_1}dm_1 - \frac{\xi_2'}{m_2}dm_2\right], \\ dy' &= -\left[\frac{\eta_1'}{m_1}dm_1 - \frac{\eta_2'}{m_2}dm_2\right], \\ dz' &= -\left[\frac{\zeta_1'}{m_1}dm_1 - \frac{\zeta_2'}{m_2}dm_2\right]. \end{aligned}$$

After solving (8) and (9) for $\xi_1', \eta_1', \zeta_1'$; $\xi_2', \eta_2', \zeta_2'$ and substituting in (10) we have

$$
\begin{aligned}
dx' &= -\frac{\alpha_1}{M}\left(\frac{dm_1}{m_1}-\frac{dm_2}{m_2}\right)-\frac{x'}{M}\left(\frac{m_2}{m_1}dm_1+\frac{m_1}{m_2}dm_2\right),\\
dy' &= -\frac{\alpha_2}{M}\left(\frac{dm_1}{m_1}-\frac{dm_2}{m_2}\right)-\frac{y'}{M}\left(\frac{m_2}{m_1}dm_1+\frac{m_1}{m_2}dm_2\right),\\
dz' &= -\frac{\alpha_3}{M}\left(\frac{dm_1}{m_1}-\frac{dm_2}{m_2}\right)-\frac{z'}{M}\left(\frac{m_2}{m_1}dm_1+\frac{m_1}{m_2}dm_2\right).
\end{aligned}
\tag{11}
$$

The change in the relative velocity $v^2 = x'^2 + y'^2 + z'^2$ will be

$$vdv = x'dx' + y'dy' + z'dz',$$

and by means of (11) this expression becomes

$$vdv = -\frac{1}{M}(\alpha_1 x' + \alpha_2 y' + \alpha_3 z')\left(\frac{dm_1}{m_1}-\frac{dm_2}{m_2}\right)-\frac{v^2}{M}\left(\frac{m_2}{m_1}dm_1+\frac{m_1}{m_2}dm_2\right). \tag{12}$$

If μ_1, μ_2, μ_3 are the direction angles of v we have

$$
\begin{aligned}
x' &= v\cos\mu_1, & y' &= v\cos\mu_2, & z' &= v\cos\mu_3,\\
\alpha_1 &= Ms\cos\lambda_1, & \alpha_2 &= Ms\cos\lambda_2, & \alpha_3 &= Ms\cos\lambda_3;
\end{aligned}
$$

and if ω is the angle between the direction of motion of the center of gravity and the direction of the relative motion of Jupiter we have

$$\cos\omega = \cos\lambda_1\cos\mu_1 + \cos\lambda_2\cos\mu_2 + \cos\lambda_3\cos\mu_3$$

and therefore, on removing a factor v, equation (12) becomes

$$dv = -s\cos\omega\left(\frac{dm_1}{m_1}-\frac{dm_2}{m_2}\right)-\frac{v}{M}\left(\frac{m_2}{m_1}dm_1+\frac{m_1}{m_2}dm_2\right). \tag{13}$$

If the direction of motion of the center of gravity were perpendicular to the plane of relative motion the $\cos\omega$ would always be zero. If it is not perpendicular then $\cos\omega$ varies, as Jupiter completes a circuit about the sun, from a certain positive value to the same negative value; the extreme limits being ± 1 when the direction of motion of the center of gravity lies in the plane of relative motion. If we suppose the added material to be gathered in slowly at all points of the orbit the average value of $\cos\omega$ will be zero, and in the long run this term will be inappreciable. Furthermore it would be exactly zero if the material gathered in by the sun and Jupiter was proportional to their own masses. This term gives the change in relative velocity due to a displacement of the center of gravity. Under the conditions we are discussing its effect is very small and it will be neglected. It is found then that (13) can be written

$$\frac{dv}{v} = \frac{1}{\dfrac{1}{m_1}+\dfrac{1}{m_2}}\, d\left(\frac{1}{m_1}+\frac{1}{m_2}\right),$$

so that if A is the constant of integration we have

$$v = A\left(\frac{1}{m_1}+\frac{1}{m_2}\right).$$

If we suppose that in the accretion of material the ratio of m_1 to m_2 is not altered we will have

$$Mv = B, \tag{14}$$

where B is a constant not altered by the accretion.

The expression for v in the relative motion is

$$v^2 = k^2M\left(\frac{2}{r}-\frac{1}{a}\right).$$

On multiplying through by M^2 we have

$$B^2 = k^2M^3\left(\frac{2}{r}-\frac{1}{a}\right).$$

If a collision occurs r is not altered, nor is B. Hence by differentiation with respect to M and a

$$3M^2\left(\frac{2}{r}-\frac{1}{a}\right)dM+\frac{M^3}{a^2}\,da = 0,$$

which shows how the major axis is affected by the collision. If the process of accretion is a very slow one as we suppose is the case so that all parts of the orbit share alike in the collisions then we can replace $1/r$ by its average value, viz.,[1] $1/a$, after which this differential relation reduces to

$$3\frac{dM}{M}+\frac{da}{a} = 0,$$

from which it follows that

$$M^3a = C, \qquad \text{or} \qquad a = \frac{C}{M^3}, \tag{15}$$

where C is a constant not altered by accretion. The expression for the period $P = 2\pi a^{3/2}/k\sqrt{M}$ becomes

$$P = \frac{2\pi C^{3/2}}{kM^5}. \tag{16}$$

Stated in words these results are that during a process of slow growth in the sun and Jupiter due to stationary material picked up by them in their journey through space, their mean distance apart varies inversely as the cube of the sum of their masses, and their period varies inversely as the fifth power of the sum of their masses.

[1] The average value of $\frac{1}{r}$ for a complete revolution is $\frac{1}{P}\int_0^P \frac{dt}{r} = \frac{1}{a}$.

For example, suppose the masses of the sun and Jupiter were each increased to 5 times their present mass, then we would have $P_1 = 3125P_2$. That is the period of Jupiter, 11.86 years, would be reduced to $33\frac{1}{4}$ hours, and its distance would be reduced from 500 millions to 4 millions of miles. It is needless to discuss the fate of the terrestrial planets.

If it is true that the energies of the sun and stars are maintained by the subatomic energies of this added material and if the yielding up of its subatomic energies implies a loss in mass, then it is not necessary that the mass of the sun be increasing even though it is gathering in new material. It may even be declining at the present time. Since Jupiter is not wasting much energy in radiation it would seem as though the mass of Jupiter must certainly be growing. It requires but little imagination to see the possibility of Jupiter eventually rivalling the sun in mass and even brilliancy, and the two together constituting a double star.

To be sure, such a conception involves an enormous range in time, but it will never be possible to understand the astronomical forms which are now presented to our vision in the wide expanses of the heavens until we not only understand the physical processes now at work but also extend their logical consequences to such intervals of time as are necessary to establish a cycle, and undoubtedly such an interval is very great.

CUSPIDAL ROSETTES.[1]

By WILLIAM FRANCIS RIGGE, Creighton University, Omaha, Neb.

The rose, rosette, rosace, Rosenkurve or multifolium, or whatever other name it may have, is a periodic polar curve whose equal sectors may have any angular magnitude. Its general equation, as usually given, $\rho = a + b \sin n\theta$,[2] supposes the tracing point to move with a simple harmonic motion of n cycles along a radial line through the pole, at the same time that it makes one revolution about this pole with uniform angular speed. A simple instance of such a polar curve is the trifolium, Fig. 1 (p. 324), which is drawn by having a tracing pen move with simple harmonic motion of amplitude b in a radial line over a uniformly rotating disk in such a way that the pen just touches its centre without passing beyond, and the disk makes one revolution in three cycles of the pen. The equation is then $\rho = a(1 - \sin 3\theta)$, as is seen by inspection in Fig. 2.

If in the general equation a is greater than b, the pen does not reach the center as in Fig. 3, and if a is less than b, the pen passes beyond it as in Fig. 4 and draws

[1] See "Concerning a new Method of Tracing Cardioids" by William F. Rigge, in the January, 1919, MONTHLY and "On the Construction of Certain Curves Given in Polar Coördinates" by R. E. Moritz, in the May, 1917, MONTHLY.

[2] The two terms may have unlike signs and sin be replaced by cos.