A SIMPLE CRITERION FOR THE DETECTION OF ANOMALIES IN THE ORBITS OF SPECTROSCOPIC BINARIES

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It has often been remarked that discoveries are oftenest made through close examination of observed exceptions to well-known laws or wide generalizations. Anomalies in the field of stellar motion are as important as elsewhere, but when the moving stars constitute a spectroscopic binary it is not always very easy to find out whether the observed velocity-curve is explainable by simple motion of two bodies under the law of gravity or whether some disturbing cause must be sought.

In looking for a particular type of apparent anomaly which was to be expected on a theory of light-velocity to be discussed elsewhere, the author had occasion to study the problem of two bodies, with a view to obtaining, if possible, a simple relation which was characteristic merely of motion under gravity and independent of the constants of any particular orbit.

A relatively simple relation was found which is best stated in terms of the curve which is the integral of the common radial velocity-curve and which I will call the "distance-curve." The distance-curve is easily obtained by plotting as abscissae the times and as ordinates the *area* of the velocity-curve between zero velocity and the time-ordinate chosen. Thus if in Fig. 1 v is the velocity-curve, then d is an approximate representation of the distance-curve. If the axis OP is so chosen that the area of the v-curve above it is just equal to the area below, then the ordinates of the d-curve give the distance of the star at any instant from the nearest point of the orbit, this point being of course considered as moving with the center of gravity of the system.

The relation to be proved is then between the width of the distance-curve, such as MN, at any point and its height LM at the same point.

It may be written thus:

$$\frac{w}{W}=\frac{1}{\pi}\left\{\cos^{-1}\left(2\frac{h}{H}-1\right)-2\left(2\frac{h_1}{H}-1\right)\sqrt{\frac{h}{H}\left(1-\frac{h}{H}\right)}\right\} \qquad (1)$$

where

h = height of distance-curve at any point;

w = width of distance-curve at same point;

h_1 = height of distance-curve at the abscissa corresponding to the maximum point of velocity-curve (KS, Fig. 1);

H = maximum height of distance-curve;

W = total width (i.e., length of base) of distance-curve.

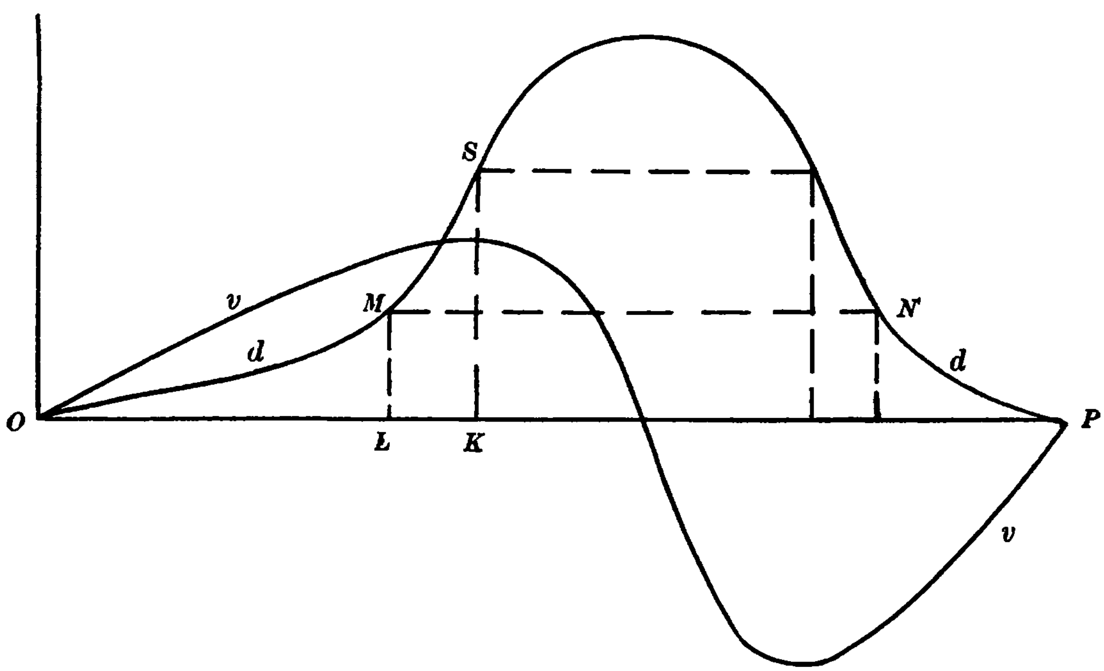

FIG. 1

PROOF OF RELATION

In order to prove equation (1) it will be convenient to state six theorems which are quite simple in themselves and on which this equation may be based.

Theorem I.—This is a geometrical theorem and states that if (Fig. 2) in any circle are drawn two parallel chords CB and MN, two tangents HRL and AVY parallel to these chords, and the diameter RV perpendicular to the tangent, and cutting the chords in P and Z; then if F is any point on the chord CB and the lines MF and FN are drawn, the following relation holds:

$$\frac{\text{Area } MFNBRCM}{\text{Total area of circle}} =$$

$$\frac{1}{\pi}\left\{\cos^{-1}\left(2\frac{VZ}{D}-1\right)-2\left(2\frac{VP}{D}-1\right)\sqrt{\frac{VZ}{D}\left(1-\frac{VZ}{D}\right)}\right\} \qquad (2)$$

where D is VR, the diameter of the circle.

This theorem can readily be proved by subtracting the area of $\triangle MFN$ from segment MRN.

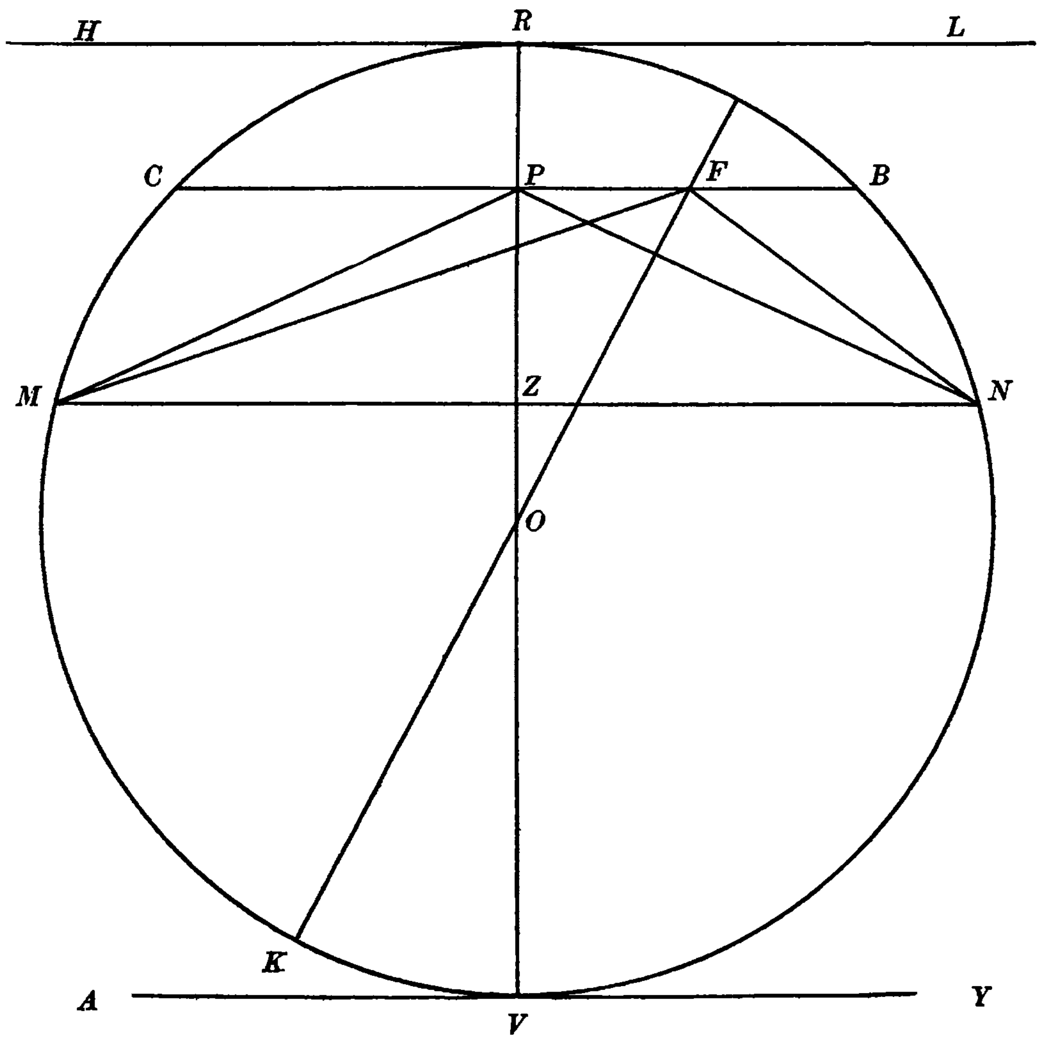

FIG. 2

Theorem II.—If four planes, each perpendicular to the line of sight, be imagined drawn in connection with the real orbit (Fig. 3), two tangent to the orbit, one through the focus, and one at any distance (z) from the nearest tangent plane, then the orbit, together with the intersection of these planes with the plane of the orbit, may be completely represented by the projection of a certain circle similar to that of Fig. 2, with the various lines as there drawn.

This is readily seen to be true if the point F and the direction of the lines HL, CB, etc., are properly chosen on the circle of Fig. 2. For clearness Fig. 3 is lettered similarly to Fig. 2.

Theorem III.—The relation of Theorem I holds also for the ellipse of Fig. 3, if the corresponding letters are substituted. That is,

$$\frac{\text{Area } M'F'N'B'R'C'M'}{\text{Total area of ellipse}}=$$

$$\frac{1}{\pi}\left\{\cos^{-1}\left(2\frac{V'Z'}{D'}-1\right)-2\left(2\frac{V'P'}{D'}-1\right)\sqrt{\frac{V'Z'}{D'}\left(1-\frac{V'Z'}{D'}\right)}\right\} \quad (3)$$

This follows from the fact that, since each area and line of Fig. 3 is a projection of the corresponding line or area of Fig. 2, we can write

$$\frac{\text{Area } M'F'N'B'R'C'M'}{\text{Total area of ellipse}}=\frac{\text{Area } MFNBRCM}{\text{Total area of circle}} \quad (4)$$

$$\frac{V'Z'}{D'}=\frac{VZ}{D} \quad (5)$$

$$\frac{V'P'}{D'}=\frac{VP}{D} \quad (6)$$

Theorem IV.—The ratio $\frac{V'Z'}{D}$ in the *right*-hand side of the relation of Theorem III may be found from an examination of the experimentally determined distance-curve of Fig. 1. In fact

$$\frac{V'Z'}{D'}=\frac{h}{H} \quad (7)$$

where h and H are the quantities in equation (1).

To prove this it is merely necessary to remember that the ordinates of the distance-curve (Fig. 1) give the distance of the body at any instant from the nearest tangent plane to the orbit. From this it is evident that h and H are simply projections of $V'Z'$ and D' on the line of sight.

Theorem V.—The ratio $\frac{V'P'}{D'}$ on the right of the relation in Theorem III may also be found from an examination of the experimentally determined distance-curve of Fig. 1. In fact

$$\frac{V'P'}{D'}=\frac{h_1}{H} \tag{8}$$

where h_1 and H are the quantities in equation (1).

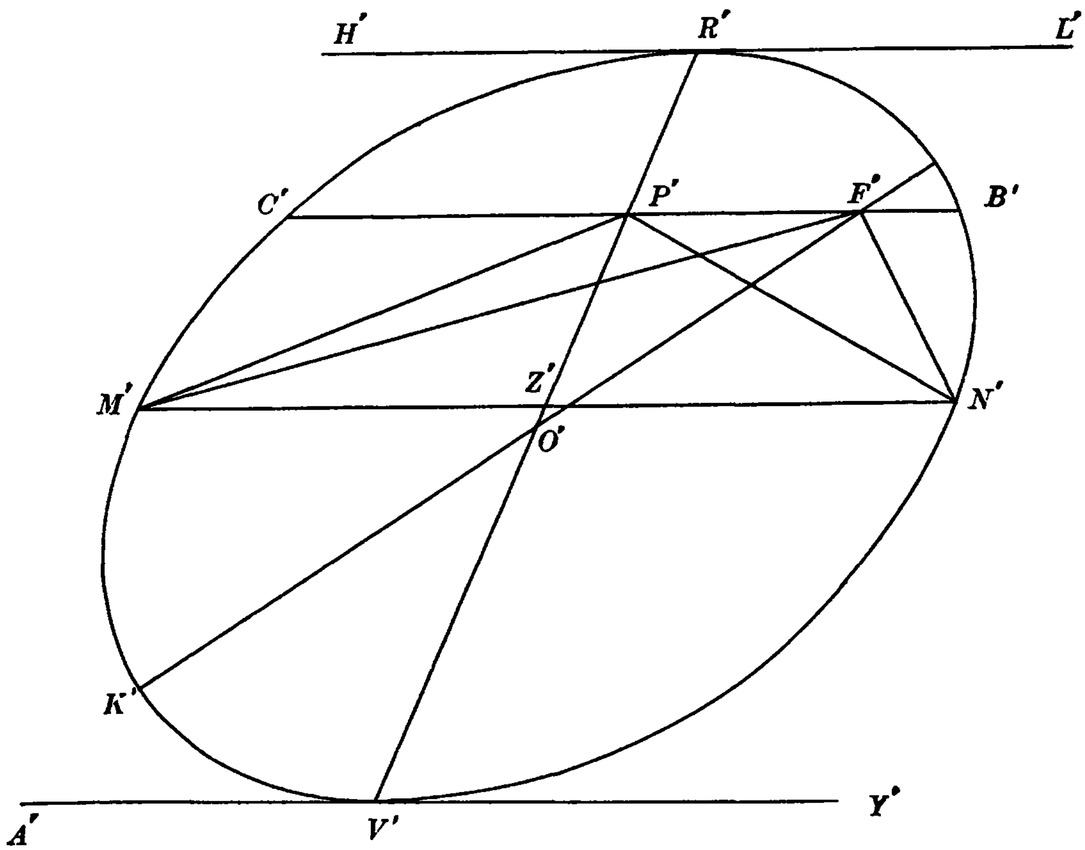

FIG. 3

This follows from the fact that h_1 was defined as the ordinate of the distance-curve which had the same abscissa as the maximum point of the radial velocity-curve. It follows, however, from the simplest expression of Newton's Laws as applied to central forces that the radial velocity is a maximum when the body is in a plane through the focus drawn perpendicular to the line of sight. Thus the height h_1 of the distance-curve represents the distance of the focus of the orbit from the nearest tangent plane.

Theorem VI.—The ratio which forms the *left*-hand side of the relation of Theorem III may be found from an examination of the experimentally determined distance-curve of Fig. 1. In fact

$$\frac{\text{Area } M'F'N'B'R'C'M'}{\text{Total area of ellipse}}=\frac{w}{W} \tag{9}$$

To prove this it is merely necessary to remember that from Kepler's Law the ratio

$$\frac{\text{Area } M'F'N'B'R'C'M'}{\text{Total area of ellipse}} \tag{10}$$

is equal to the time that it takes the body to go from M' to N' divided by the total time to go once around the orbit. By construction M' and N' are at equal distances from the nearest tangent plane and hence the positions M' and N' are represented by equal ordinates on the distance-curve. Hence the *width* of the distance-curve corresponding to the $M'N'$ ordinate, divided by the total width (the base) of the curve, is the time it takes the body to go from M' to N' divided by the periodic time and hence is equal to the above ratio (10).

It is evident that from Theorems IV, V, and VI the relation of equation (1) follows directly. We thus have

$$\frac{w}{W}=\frac{1}{\pi}\left\{\cos^{-1}\left(2\frac{h}{H}-1\right)-2\left(2\frac{h_1}{H}-1\right)\sqrt{\frac{h}{H}\left(1-\frac{h}{H}\right)}\right\} \tag{11}$$

SUMMARY AND CONCLUSION

The criterion for the detection of anomalies in the orbit of spectroscopic binaries which has been set up in this paper, may be outlined in words as follows:

If a plane be drawn through any simple double star orbit perpendicular to the line of sight, then the time during which one of the stars is on the far side of this plane is a simple function of the position of the plane along the line of sight, and of the position of the focus along the line of sight. All three of these quantities can be determined with great ease from the true radial-velocity-curve of one star.

If the time be expressed in terms of the periodic time, and the distance along the line of sight, measured from the nearest point of the orbit, be expressed in terms of the total length of the projection of the orbit on the line of sight, then the equation contains *no constant of the orbit except the distance of the focus along the line of sight expressed in the same way.*

This criterion is uniquely suited for the detection of any apparent anomaly in the motion of binary stars due to a possible dependence of the velocity of light on the velocity of the source. It was developed solely for this reason and the author is now using it with the purpose of detecting such an effect, does it exist. It was thought, however, that, since the criterion is so general and the relation itself so simple, compared with the complexity of the integral function connecting velocity and time, it might be found useful in other problems than the one for which it was developed.

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