SOME MATHEMATICAL ASPECTS OF COSMOLOGY

(*Continued from page 63*)

II. COSMOLOGY

WE come now to a somewhat larger point of view. Cosmogony deals only with the mode of origin of the various celestial objects. But the mode of origin is of no more interest than the mode of dissolution, and both of these are but particular stages in a process of transformation that goes on unceasingly. The study of these transformations in their widest possible aspect is what I understand by the word *cosmology*. It does not belong to astronomy any more than it does to physics and chemistry, for cosmology is as much concerned with the life history of molecules, atoms and electrons and their inter-relations, as it is with the life history of planets, stars and galaxies. If it were a mature subject, instead of being, as at present, a mere infant, the crystal, the cell and the living organism would play a rôle which we might well call vital. To the cosmologist each of these things is a physical unit which comes into existence, plays its allotted rôle upon the stage of time, and passes out of existence. The mode of its organization is definite, its properties are specific, and its dissolution is liable to be more or less abrupt or catastrophic. Throughout all these transformations we recognize that there is something which persists, and that something we call *energy*. Energy itself is not defined, but it can be measured and with that measurement we must remain content, for the thing itself escapes us.

I am sure that I could not proceed much further without being assured by some one that I was taking a great deal for granted. It is necessary, therefore, for us to stop and to make some inquiries as to the nature of what we are trying to do. I take it that science aims to extend the boundaries of human experience to the utmost limits, and endeavors to coordinate the experience already acquired for the purpose that it may be available at command and that it may be used as a basis of prediction for the experiences which we anticipate. In doing this, it is merely extending in a purposeful and conscious manner, and intensifying, a process which begins with each individual in the first waking hours of infancy, but which frequently dies out during maturity, or even before maturity is reached. By the time we take up the process consciously we are a long way from the beginning, and it is a very difficult matter to get a correct perspective of our activities. We know that we are on our way, but we do not know, quite, where we are going.

It is in these difficulties that we turn once more to mathematics for aid; and not in vain. The geometers of Alexandria, some two thousand years ago, had trouble over the proofs of their theorems. They could not agree on what constituted a proof, for no two of them would start from the same "obvious" propositions. This situation led Euclid to attempt a unification of geometry; and for this purpose he laid down a system of definitions, axioms and postulates, once for all, to which he could appeal whenever necessary in the course of the argument. Doubtless this system of axioms and postulates covered the points which were of interest and dispute at that time, so that, although the system was by no means complete, it did bring unity and harmony into the science of geometry. The axioms are of the nature of logical statements, while the postulates are statements, supposed to be obvious, about the fundamental concepts of geometry. Evidently the first proof must rest upon propositions which are not proved, and new postulates are necessary whenever new aspects of the subject-matter are considered.

The rapid growth of mathematics during the seventeenth and eighteenth centuries was followed by a second period of sharp criticism early in the nineteenth century associated with names of Gauss, Cauchy, Abel, Riemann and many others. The foundations of arithmetic and geometry were carefully examined for the purpose of determining whether or not the structures built upon them were secure. The result of this scrutiny was that these subjects were removed from the domain of *nature* altogether. The real number system, for example, is a purely intellectual system. The first steps of its creation were taken unconsciously by rude, perhaps barbaric, people because it was a useful thing. Its completion, through the invention of the irrational numbers, was a definitely conscious operation; but a comprehension of nature of the system was not had until it was derived in a logical manner from a precise set of postulates relating certain undefined elements and undefined terms. There is nothing *obvious* about the postulates; and other number systems can be had by using other postulates. There is nothing *objective* about the real number system. It is simply a definite intellectual creation, which is interesting in itself and frequently useful in the many situations in which we find ourselves.

With slight changes of wording the same statements can be made with respect to geometry. The necessary postulates are different from the postulates of the number systems, because their subject-matter is different; but the development of a geometry from a system of postulates has the same abstract character as the development of a number system. Naturally, different geometries result from different systems of postulates, and there is nothing *objective* about any of them. The same thing is true about dynamics. It is idle to inquire whether the relativistic mechanics is true, or whether the classical mechanics is true. From the postulational point of view they are both true, if they are logically above reproach.

Indeed, having once risen to the level of the postulational method, the construction of intellectual edifices upon new systems of postulates becomes a fine game. Some systems of postulates will be found barren, for apparently nothing can be derived from them. Others are fertile, in the sense that at least a small body of theorems can be derived; while a very few others are extremely fertile, and so useful in their applications that we do not think of them merely as intellectual sports; they become sciences, such as algebra, geometry and mechanics. There is this interesting fact, however; so far as I am aware, no very fertile system has been built upon postulates which were not suggested more or less immediately by our common experiences in life.

On account of its philosophical bearing, I regard the development of the postulational method as the greatest achievement of the mathematicians of the nineteenth century. Not only has it made clear the nature of mathematics, but it has also thrown a flood of light upon the nature of the physical sciences, a fact which is well brought out by E. W. Hobson in his recent book, "The Domain of Natural Science." To him who would gain the widest possible point of view, that is to say, the cosmologist, it is a downright necessity.

There is a fundamental difference, however, between mathematics and the natural sciences. The pure mathematician is interested only in logical systems. He is, therefore, quite free from entanglements with observation and experiment; his postulates can be any consistent set of statements that his fancy dictates. The natural scientist is interested primarily in experience. Logical systems would have no interest to him whatever, if it were not for the extraordinary fact that he finds certain logical systems extremely useful. He is free in the choice of his postulates, therefore, only on those points with regard to which he can have no experience whatever, directly or indirectly. In order that I may speak the same language as the mathematician, I shall understand the word *postulate,* as used in cosmology, to refer only to statements about matters with respect to which we are and always will be entirely free from experience. Similar statements, which observations or experiments may show to be in harmony or in conflict with experience, I shall call *hypotheses.* Hypotheses have the nature of tentative postulates, and are therefore strange things to a mathematician. A mathematical system is closed in the sense that it contains only the assigned postulates and the theorems which are logically derivable from them. A cosmological scheme, which deals with experience, is necessarily an open one. One can not write down all the postulates, once for all, nor the undefined terms, for there is nothing to suggest that we have arrived at the outermost limits of experience, or even that such limits exist.

Notwithstanding the fact that each of us is free in the choice of his postulates, so that *no system of postulates merits the claim of exclusiveness,* still, on account of our common heredity and experience, it is true that certain postulates are commonly made, and have, therefore, something like a universal appeal to our esthetic sense. Let me write down a few of these postulates which seem to me to belong to a normal system:[36]

(1) *There exists a physical universe, external to myself, with which I have experience.*

I am not sure whether or not all the adherents of the modern theory of relativity use this postulate. At times it seems to me they do not. At any rate, there are people who seem perfectly happy with a mathematical formula. As for me, I am not happy unless I can see what lies behind the formula; that is to say, a qualitative understanding of a situation is of even greater importance than a quantitative one. Thus $\frac{x^2}{16}+\frac{y^2}{9}=1$ is an exact relation between the magnitudes x and y, whatever they may be. But it makes a great deal of difference whether x and y are to be interpreted as the cartesian coordinates of a point, or as the position and velocity of a particle in simple harmonic motion, or perhaps something else. A mathematical formula is not the goal of cosmology.

(2) *The geometry of the physical universe is euclidean.*

(3) *The time of the physical universe is newtonian.*

The purpose of postulates 2 and 3 is evident. Previous to the exposition of Einstein's doctrine of relativity they would doubtless have commanded universal assent, but the unusual character and the beauty of Einstein's system, together with the simplicity with which it enables us to anticipate certain very delicate phenomena in the domain of physics and astronomy, have won many adherents to it, so that the classical postulates 2 and 3, at least for the time being, are not universally adopted. Inasmuch

[36] See also MacMillan, "Some postulates of cosmology." *Scientia,* February, 1922.

as the relativists do not concern themselves with a physical basis for the transmission of radiant energy, their scheme being a purely mathematical one, I am not sure that they have any need for postulate 1. As a well-known physicist expresses it, they explain terms of the second order beautifully, but they do not explain terms of order zero at all. There are many of us who prefer the terms of order zero, and are unwilling to sacrifice our intuitions upon the altar of the terms of order two. Let us not forget that success or failure argues nothing for the truth or falsity of either system. The relativists have had great successes at certain points where the classic system has so far failed. That is all. This suggests that great discoveries are waiting for some one among the classicists, and the successes of the relativists should be stimulating.

(4) *The physical universe is not bounded in space.* Not all people, by any means, think of the universe as unbounded. I think I can safely say that nearly all mathematicians do, and many of the more abstract type of physicists and astronomers; nevertheless, it must be admitted that many scientists prefer to think of it as finite. There is no admitted agreement.

(5) *The physical universe is continuous in time.* Physical things do not disappear from one position in space, only to reappear at the succeeding instant at some distant position. Discontinuities of this type do not occur. Neither does any body act upon another and remote body instantaneously; which is equivalent to saying that energy is transmitted at a finite velocity. Furthermore, something does not become nothing, and nothing does not become something.

(6) *The distribution of matter throughout space is uniform if considered on a large scale,* by which I mean, the limit of the mean density of a spherical volume having any center tends towards a definite constant, different from zero, as the radius of the sphere increases indefinitely.

Consider a series of concentric spherical surfaces, the radii of which are proportional to the successive integers 1, 2, 3, . . . , and suppose n stars are placed upon the nth surface. We can regard such a system as a universe which is not bounded in space (postulate 4). The total number of stars is infinite, but the mean density of the volume of the nth surface is proportional to $\frac{n+1}{n^2}$, which has the limit zero as n increases. The distribution of matter in such a universe is not uniform. If, however, we place n^2 stars upon the nth surface, the mean density of the nth sphere is proportional to $\frac{(n+1)\ (2n+1)}{n^2}$ which has the limit 2 as n increases indefinitely. If the stars were scattered over the surfaces of the spheres at random, so as to avoid peculiar distributions, then we would say that the matter in this universe was uniformly distributed. (Considered on a small scale, matter is never uniformly distributed over any volume; even water is not uniform from this point of view).

If, however, all these stars radiate the same amount of light, and if the law of intensity of radiation is strictly the inverse square law, then the amount of light received at the center of the sphere is the same from each sphere, and since the number of spheres is infinite, the total amount of light received at the center is infinite; if, however, we allow for the occultation of one star by another, the entire sky is only as bright and hot as the disk of the sun. This result follows even for the universe in which n stars only are distributed over the nth sphere, for the amount of light received at the center from the nth sphere is proportional to $\frac{1}{n}$, and, as we know, the series $\Sigma\ \frac{1}{n}$ is divergent. We shall have occasion to return to this point later.

(7) *There exist physical units which, for a finite interval of time, preserve their identity and exhibit characteristic properties.*

(8) *The sequence of physical units is infinite both ways, like the positive and negative powers of a positive number.*

The term "physical unit" corresponds largely to the word "object." The smallest physical unit which we recognize at the present time is the positive electron, and the largest one is the galaxy. In ascending scale, we have electrons, atoms, molecules, ordinary masses, stars, star clouds, galaxies. We ourselves and the objects with which our thoughts are normally concerned belong to the class of ordinary masses, and the variety of the physical units which belong to this class is truly amazing. No two objects are exactly alike, yet resemblances are sufficiently strong to permit classification, and even to suggest the postulates on which the mathematician bases his number system.

Ordinary masses are built out of molecules; molecules are built out of atoms; atoms out of electrons. Likewise the stars are huge masses of gas; the star clouds are vast aggregations of stars; and the galaxy is an aggregation of star clouds. Each physical unit is built up of units of the next smaller order, and our method of accounting for the properties of objects is to recognize a differentiation in the parts of the object. If there existed a smallest physical unit there would be no differentiation, and hence it would have no properties.

It will be observed that ordinary masses are just in the center of our list of physical units. Shall we go back to the old notion that we are the natural center of the universe, or shall we regard this as a mere appearance, due to the fact that it is more and more difficult for us to have experience with those units which are more and more remote from us in the physical scale? We are at the center, because the center is everywhere. Two atoms of gold seem just alike because we are not very familiar with atoms of gold, and two electrons seem to be identical merely because of our profound ignorance. Super-galaxies exist though we have had no experience with them at all; likewise, hyper-super-galaxies,[37] and so on indefinitely. Things do not cease to exist merely because we are ignorant. We should beware of the tacit postulate, which often crops out, "Only those things exist with which we have had experience." Nature is much broader than experience, and we must have plenty of room for expansion.

(9) *The phenomena of nature occur always in such a way that certain relations remain invariant.*

This postulate asserts merely that science is possible, and the main purpose of science is to ascertain these invariants.

(10) *Every physical situation is definite and determined, both as to its extensions in space and its sequential states in time;* or, in simple language, nature is never in doubt.

This is not the case in mathematics. The value of a function at a point may be quite indeterminate, and the limit of the function as we approach a point may depend upon the mode of approach. Imagine all space filled with matter uniformly distributed. (I am speaking mathematically now), and that Newton's law of gravitation holds. What is the resultant acceleration on any given particle. Let p be the particle and let O be a point at a distance R from p. Let S be the sphere with center at O which passes through p, so that R is its radius. Then the attraction of this sphere upon the particle p is directed towards O and its magnitude is proportional to R. Take a second sphere S_1 with its center at O and its radius $R_1 > R$. Then the resultant attraction of the spherical shell between S and S_1 upon the particle p is zero however great R_1 may be. We conclude that the resultant attraction of the matter in all space upon the particle p is the same as the attraction of the sphere S_1, which is proportional to R and directed towards O. But as the point O is arbitrary, both as to distance and direction, the resulting attraction is completely undetermined. This is the Neumann-Seeliger proposition. Similarly, the attraction of a thin disk upon one of its own points is completely undetermined. But these are mathematical situations. According to the postulate, such situations do not arise in nature.

(11) *In every region of space, however small, there exists at least one physical unit.* The postulate denies the existence of empty space, and asserts on the contrary that every portion of space is infinitely complex.

(12) *The energy within a region of space does not increase or decrease, unless there is a corresponding decrease or increase in some other region of space.*

This is the doctrine of the conservation of energy to which physicists were led about the middle of the last century. It possesses some quality that appeals to the esthetic sense, for it has been adopted, almost universally.

(13) *The universe does not change always in any one direction.* Using figurative language—the universe is not like a stream which flows steadily from one unknown region to another. It is like the surface of the ocean, never twice alike and yet always the same. At the same time that the physicists were formulating the doctrine of the conservation of energy, which is sometimes called the first law of thermodynamics, they also formulated the second law of thermodynamics, which has been stated in various ways, but the essential idea is that energy is constantly being degraded into the form of heat and radiated away; the energy available for useful work is always diminishing, or in modern terms, the entropy is always increasing. Physicists and chemists have been very successful in predicting phenomena by means of this law, and it has a thoroughly reputable standing. Nevertheless, it has always met with violent opposition and dislike. It is out of harmony with the idea contained in postulate 13, and therefore it is unpopular. As I see it, the second law of thermodynamics is similar to the statement that under natural conditions water always flows down hill. This is true enough, but if it were the whole truth one could not avoid wondering why the water had not all gotten to the bottom of the hill long ago. The statement is true of water in the liquid state, but in the state of vapor it is equally natural for water to rise. We shall see later that the second law of thermodynamics states only one half of the complete process.

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(*To be continued*)

[37] This word is due to Moulton.

THE JOHN SCOTT MEDAL FUND

THE following is an extract from Power of Attorney to carry out certain provisions in the will of John Scott dated April 2, 1816.