The Motion of the Perihelion of Mercury deduced from the classical Theory of Relativity. By L. Silberstein, Ph.D., Lecturer in Natural Philosophy at the University of Rome.

It is well known that as early as 1845 Le Verrier found that the motion of the perihelion of Mercury, as derived from observations of transits, was greater by 38″ per century than it should be from the perturbation due to all the other planets of our system. A recent discussion of the subsequent investigations has shown the excess of motion to be about 5″ greater, viz., per century,

$$\delta\varpi = +42''\cdot 9.$$

Equally well known are the attempts of Newcomb and of Seeliger to account for this excess of motion of Mercury's perihelion. A discussion of Seeliger's results (which, broadly speaking, were very satisfactory) and a modification of his treatment have been given by H. Jeffreys (*M.N.*, vol. lxxvii. p. 112). On the other hand, a great sensation has been recently produced among astronomers by the surprising circumstance that Einstein's newest "*generalised* theory of relativity" has yielded for the said excess just its full value, *i.e.*, in round figures, 43″. In fact, Einstein gives in his recent paper,* for the angle ϵ through which the elliptic orbit of a planet is turned, in the direction of motion, per period T, the formula

$$\epsilon = 24\pi^3 \frac{a^2}{T^2 c^2 (1-e^2)} . \quad . \quad . \quad . \quad (G)$$

where a, e stand for the major semi-axis and the eccentricity of the orbit, and c is the velocity of light in empty space. Substituting $a = 0{\cdot}3871 . 1{\cdot}49 . 10^6$ km., $e = 0{\cdot}206$, $T \doteqdot 87{\cdot}97$ days, $c = 3 . 10^5$ km./sec., the reader will find, for $\delta\varpi$ per century, $43''\cdot 1$, which is the desired angle. The reason why I have denoted the above formula by (G) is, with all respect due to Einstein, that identically the same formula was given eighteen years earlier by Gerber,† whose investigation, entirely independent of any relativity ("old" or "new"), seems to have passed unobserved, most likely owing to its badly supported fundamental assumptions. To enter upon these latter would not answer the purposes of the present paper. It may, however, be interesting to notice that Gerber replaces Newton's potential $\frac{M}{r}$ by $\frac{M}{r\left(1 - \frac{1}{c}\frac{dr}{dt}\right)^2}$, where c is the "velocity of propagation of the gravitation potential";

* *Annalen der Physik*, vol. xlix., 1916, pp. 769–822. See also Professor de Sitter's papers in *M.N.*, vol. lxxvi., 1916, p. 699, and vol. lxxvii. p. 155.

† P. Gerber, *Zeitschr. math. Phys.*, xliii., 1898, pp. 93–104. A short account of Gerber's theory is given in *Enc. d. math. Wiss.*, vol. V. 1, pp. 49–51; a still shorter, and very unfair, account is given by Herr E. Gehrcke in *Annalen der Physik*, li., 1916, pp. 122–124.

rejecting the third, and the higher, powers of dr/cdt, Gerber obtains for any (isolated) planet in its motion round the central body the above formula. It is historically interesting that Gerber does not identify c with the velocity of light, but determines its value from the observed excess of the secular motion of the perihelion of Mercury, and finds $c = 305500$ km./sec., *i.e.* "surprisingly near the light velocity." Thus, whatever his theory, the formula (G), accounting for *the full excess* of Mercury's perihelion motion, will appropriately be called *Gerber's formula.*

Now, to repeat it, Gerber has deduced his formula from an untenable theory, or at least from one which has not been based upon well-established general principles. Einstein, eighteen years later, but undoubtedly without knowing Gerber's formula, has rediscovered it by deducing it from his "generalised" theory of relativity, which, in its turn, is again very far from being well established. In fact, notwithstanding its broadness and mathematical elegance, it certainly offers many serious difficulties in its very foundations, while none of its predictions of new phenomena, as the deflection of a ray by the sun, have thus far been verified. And even the fact that Einstein's new theory gives Gerber's formula, and therefore the *full* excess of $43''$ for Mercury, does not seem to be decisive in its favour. As far as I can understand from Jeffreys's investigation,* it would rather alleviate the astronomer's difficulties if the Sun by itself gave only a *part* of these 43 seconds.

Under such circumstances it has seemed worth while to investigate, in as general a way as possible, how much of that excess will be accounted for by the ordinary or, as Einstein now calls it, the "old" theory of relativity, retaining the *constancy* of the velocity of light, and, of course, its independence of any gravitational field. It will be remembered that this "old" theory is not only unobjectionable in its foundations and considerably simpler than the "new" theory of relativity, but that it has also the physical advantage of accounting for a good number of important facts: it harmonises with the null-effects of Michelson-Morley's and of Trouton-Noble's experiments, it stands the test of experience in the domain of the best observations on β-particles, and it represents Fresnel's dragging coefficient in the most direct way, viz. as a simple corollary of kinematics.

So much to justify a renewal of the investigation of planetary motion on the basis of the "old" theory of relativity, as laid down in Einstein's famous paper of 1905, and developed formally by Minkowski two years later. The planetary motion has been treated by Minkowski himself in an appendix to his memoir (*Grundgleichungen, etc.,* of 1907 Dec. 21), in which, however, he limits himself to a short sketch and concludes by saying that the proposed relativistic modification would hardly show any observable deviations from the results of ordinary Newtonian mechanics. In

* *Loc. cit.*, see especially p. 113, and the final paragraph of the paper, p. 118.

fact, his law of gravitation would correspond in our notation (*cf. infra*) to $N = M_0 m_0 / \gamma r^2$, and this gives, for a small planet (m_0/M_0 negligible),

$$\epsilon = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(M)}$$

In more recent times the problem of planetary motion has been treated relativistically by Dr. Wacker (*Dissertation*, Tübingen, 1909), whose assumptions amount to adopting the law of force $N = M_0 m_0 / r^2$, leading to

$$\epsilon = \tfrac{1}{6}(G), \text{ i.e. } \delta\varpi = 7''\cdot 2 \text{ for } ☿ \quad . \quad . \quad . \quad \text{(W)}$$

in words, to $\frac{1}{6}$ of Gerber's value quoted at the outset.* This is at the same time just $\frac{1}{6}$ of the full observed excess for Mercury. Now, there is nothing particular about these two special laws. We shall see that the problem can be treated more broadly, and so as to account for the full observed excess. In our opinion the retouching of Newton's gravitation law must be guided by experience (observation) and not by guesswork or predilection for particular integers. At the same time we shall use the opportunity of simplifying the mathematical treatment of the corresponding equation of motion, which is usually, and without any intrinsic reasons, being made formidably complicated.

Let $\mathbf{r}$ be the vector (x, y, z) drawn from the Sun to the planet, a system attached to the former being our reference system S. Let t be the S-time, and $\mathbf{v} = d\mathbf{r}/dt$ the (ordinary) vector velocity of the planet with respect to S. Write

$$\gamma = (1 - \beta^2)^{-\frac{1}{2}}, \ \beta = v/c \quad . \quad . \quad . \quad (1)$$

where c is the velocity of light, assumed to be constant and invariant, of course. Let m_0 be the *rest-mass* of the planet, a constant by assumption, and

$$m = m_0 \gamma \quad . \quad . \quad . \quad . \quad (2)$$

Finally, let $\mathbf{N}$, an ordinary three-dimensional vector (like $\mathbf{r}$ or $\mathbf{v}$) be the "Newtonian force" acting upon the planet. Then the equation of motion of the planet will be †

$$\frac{d}{dt}(m\mathbf{v}) \equiv \frac{d}{dt}\left(m\frac{d\mathbf{r}}{dt}\right) = \mathbf{N} \quad . \quad . \quad . \quad (3)$$

This can be read: The rate of increase of momentum, $m\mathbf{v}$, is equal to the force $\mathbf{N}$. The only departure from Newtonian mechanics is that the inertia-coefficient m is not a constant but a function of velocity, precisely as the familiar transversal mass of an electron.

* In revising the MS., I learn from Professor Eddington that de Sitter obtains, in *M.N.*, vol. lxxi., 1911, on Minkowski's and Lorentz's law of force, for Mercury, $\delta\varpi = 0$ and $\delta\varpi = 7''\cdot 15$ respectively, which are evidently Minkowski's and Wacker's results.

† The reader unacquainted with the "old" theory may consult the author's *Theory of Relativity*, Macmillan, 1914, pp. 192 *et seq.*, where m is identical with the present m_0.

If the reader does not like even the "old" relativity, but is progressive enough to admit the inertia of energy, he may write $m = m_0 + \frac{1}{2} m_0 \frac{v^2}{c^2}$, or $m = m_0(1 + \frac{1}{2}\beta^2)$, which to all purposes is the same thing as $m_0\gamma$. In fact, we shall ultimately retain but the second power of β, rejecting all higher powers of this exceedingly small fraction. Thus (3) as an equation of motion has nothing unfamiliar about it.

Next, without yet specifying the dependence of the force upon distance, let **N** be a central force, say,

$$\mathbf{N} = -\mathrm{N}\mathbf{u}, \qquad . \quad . \quad . \quad . \quad (4)$$

where **u** is a radial unit vector drawn away from the Sun, and N the absolute value of the force. With this symbol we shall have also $\mathbf{r} = r\mathbf{u}$. Under these circumstances the vector product $\mathrm{V}\mathbf{r}\mathbf{N}$ will vanish, and therefore by (3)

$$0 = \mathrm{V}\mathbf{r}\frac{d}{dt}(m\mathbf{v}) = \frac{d}{dt}(m\mathrm{V}\mathbf{r}\mathbf{v}).$$

Thus

$$m\mathrm{V}\mathbf{r}\mathbf{v} = \mathbf{k}$$

will be a constant vector both in size and in direction, relative to the system of reference S, of course. The planet moves in a plane normal to **k**, as it would according to Newtonian mechanics. The only difference is that m is slightly greater than its constant rest-value m_0. In other words, the *moment of momentum*, **k**, is a constant vector, as with Newton, but the inertial coefficient m is slightly variable.* Writing k for the absolute value of **k**, we have, in terms of the usual polar co-ordinates, r, θ,

$$mr^2\frac{d\theta}{dt} = k \qquad . \quad . \quad . \quad . \quad (5)$$

Using this equation in (3), we have, by (4),

$$-\mathrm{N}\mathbf{u} = \frac{k}{mr^2}\frac{d}{d\theta}\left(m\frac{d\mathbf{r}}{dt}\right) = \frac{k^2}{mr^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{d\mathbf{r}}{d\theta}\right).$$

Now, remembering that $\mathbf{r} = r\mathbf{u}$, the reader will easily verify the identity

$$\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{d\mathbf{r}}{d\theta}\right) = -\left(\frac{d^2\rho}{d\theta^2} + \rho\right)\mathbf{u},$$

where $\rho = \frac{1}{r}$. Thus the equation of motion gives the scalar differential equation for the orbit,

$$\frac{d^2\rho}{d\theta^2} + \rho = \frac{mr^2}{k^2}\mathrm{N}, \; \rho = \frac{1}{r} \qquad . \quad . \quad . \quad (6)$$

where N is the absolute value of the central attractive force.

* These somewhat lengthy remarks are intended for those readers who are not at all familiar with relativity.

Thus far we have only assumed that **N** is a central force, N**u**, without specialising its absolute value or intensity N. Now, according to Newton's gravitation law we should have, in appropriate units, $N = M_0 m_0 / r^2$, where M_0 is the constant and, as we shall assume, relativistically invariant *rest-mass* * of the Sun. Instead of this let us take

$$N = \frac{M_0 m_0}{r^2} f(\gamma) \quad . \quad . \quad . \quad . \quad (7')$$

where f is an *arbitrary function* of $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$, to be determined from experience, *i.e.* from astronomical observations. To begin with, we know only that, apart from second-order effects, Kepler's three laws, and therefore Newton's law of force, have to reappear. Thus we shall require $f(1) = 1$, and since the higher powers of β^2 are ultimately rejected, we can without any loss to generality write $f(\gamma) = \gamma^{n-1}$, that is,

$$N = \frac{M_0 m_0}{r^2} \gamma^{n-1} \quad . \quad . \quad . \quad . \quad (7)$$

where n is *any real number.* Its value (or the form of f) will be determined from the observed excesses of the motion of perihelia, which are second-order effects in the precise, technical sense of the word. Meanwhile we shall leave n undetermined.

As to relativistic requirements, the vector $\gamma \mathbf{N} = -\gamma N \mathbf{u}$ as given by (7′) or (7) is the space-part of a genuine four-vector or physical quaternion, as it should be. Thus the equation of motion (3), with (7), satisfies the Principle of Relativity. (For proof of the above statement see Note at the end of the paper.)

Introducing the law (7) into (6), we have, as the equation to the planetary orbit,

$$\frac{d^2\rho}{d\theta^2} + \rho = \frac{M_0 m_0^2}{k^2} \gamma^n \doteq \frac{M_0 m_0^2}{k^2}\left(1 + \frac{n}{2}\beta^2\right) \quad . \quad . \quad (8)$$

Next, multiplying the original equation of motion by **v**, scalarly, we have the equation of energy

$$c^2 m_0 \frac{d\gamma}{dt} = (\mathbf{N}\mathbf{v}) = M_0 m_0 \gamma^{n-1} \frac{d\rho}{dt},$$

whence, for any $n \neq 2$,† and denoting by E the integration constant, which is the total energy,

$$\gamma^{2-n} = (2 - n)\left(\frac{M_0 \rho}{c^2} + \frac{E}{m_0 c^2}\right),$$

* It will be noted that rest-masses may, but need not necessarily, be relativistic invariants. See, for instance, pp. 247 *et seq.* of my *Theory of Relativity.*

† And if $n = 2$, then $\log \gamma = \frac{M_0 \rho}{c^2} + \frac{E}{m_0 c^2}$, and, up to β^4, $\frac{1}{2}\beta^2 = \frac{M^0 \rho}{c^2} + \frac{E}{m_0 c^2}$, as in (9), apart from the meaning of the constant term.

or, up to β^4, and including the constant additive term $\frac{1}{n-2}$ in the energy constant,

$$\tfrac{1}{2}\beta^2 = \frac{M_0\rho}{c^2} + \frac{E}{m_0c^2} \quad . \quad . \quad . \quad . \qquad (9)$$

Introducing this value into (8) and writing

$$g^2 = 1 - n\left(\frac{M_0m_0}{kc}\right)^2, \qquad C = \frac{M_0m_0^2}{g^2k^2}\left(1 + \frac{nE}{m_0c^2}\right) \quad . \quad (10a)$$

(and zero instead of the last term in the case $n=2$), we have the differential equation of the orbit

$$\frac{d^2\rho}{d\theta^2} + g^2\rho = Cg^2 \quad . \quad . \quad . \quad . \qquad (10)$$

where $\rho = 1/r$. Thus if θ is reckoned from the perihelion

$$\frac{1}{r} = \left(\frac{1}{r_0} - C\right)\cos g\theta + C,$$

whence also, for $\theta = \frac{\pi}{2}$, $1/C = r_{\pi/2} = a(1-e^2)$, while $r_0 = a(1-e)$.

Finally, therefore, the orbit becomes

$$\frac{1}{r} = \frac{1 + e\cos g\theta}{a(1-e^2)} \quad . \quad . \quad . \quad . \qquad (11)$$

i.e., if $e < 1$, *an ellipse with moving perihelion.* At the same time we have, by (10*a*), and since $C = 1 : r_{\pi/2}$,

$$\frac{1}{a(1-e^2)} = \left(1 + \frac{nE}{m_0c^2}\right)\frac{M_0m_0^2}{g^2k^2} \quad . \quad . \quad . \qquad (11a)$$

The angle ϵ through which the perihelion moves (or the orbit is turned round) in the direction of motion, *per period* of revolution, is, by the first of (10*a*),

$$\epsilon = 2\pi\left(\frac{1}{g} - 1\right) = 2\pi\left[1 - n\left(\frac{M_0m_0}{kc}\right)^2\right]^{-\frac{1}{2}} - 2\pi\,;$$

that is, with the rejection of the fourth power of the small fraction M_0m_0/kc,

$$\epsilon = n\pi\left(\frac{M_0m_0}{kc}\right)^2.$$

Here we can write, with more than sufficient accuracy, by (5), $k = mr^2\dot{\theta} \doteqdot m_0r^2\dot{\theta} = 2\pi abm_0/T$; and, by (11*a*), $M_0 = 4\pi^2a^3/T^2$, where T is the period of revolution and $b = a\sqrt{1-e^2}$, the minor semi-axis.

Ultimately, therefore, the motion of the perihelion, per period, becomes

$$\epsilon = 4n\pi^3\frac{a^2}{T^2c^2(1-e^2)}\,* \quad . \quad . \quad . \qquad (12)$$

* The two cases treated by Minkowski and by Dr. Wacker, as mentioned before, correspond to $n=0$ and $n=1$ respectively. Gerber's formula (G) follows for $n=6$.

Thus the law of force (7),

$$N = M_0 \frac{m_0}{r^2}\gamma^{n-1} = M_0 \frac{m}{r^2}\gamma^{n-2},$$

or its four-dimensional equivalent, satisfying the Principle of Relativity, gives the motion of perihelion (12), where n is *any* real number. The *numerical value* of n, it would seem, *is to be decided upon the basis of astronomical observations.* If it is true that Mercury's excess of 43″ per century has to be thrown *entirely* upon the Sun, then make

$$n = +6;$$

and if only in part, then make it less than six. There is, however, a strong presumption that the whole excess is to be attributed to the Sun's gravitation. If so, then n, determined from Mercury's $\delta\varpi$, is just (or very nearly) *six*, and gives for Venus the value $\delta\varpi = 8''\cdot 6$. The required law of force would then be

$$N = M_0 \frac{m_0}{r^2}\gamma^5 = M_0 \frac{m}{r^2}\gamma^4 \quad . \quad . \quad . \quad (7a)$$

where $m = m_0\gamma \doteqdot m_0\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)$ is, according to the well-established proportionality of heavy and inert mass, the *heavy* mass of the planet. Why n is just six, or the exponent of γ in the latter formula just four, I am sure I do not know. But as little do we know "why" the exponent of r is "just" or exceedingly nearly equal -2.

Such a naturalistic method of improving Newton's law of gravitation seems a great deal safer than those based on fantastic constructions or rash generalisations. Since the capital part of that great law, the value of the exponent of r, has (through Kepler's empirical laws) been ultimately derived from and solely supported by experience, viz. by the observations on the shape of the orbits, it seems but natural that the exponent of a retouching factor, such as γ, should be determined from refined experience, viz. from the observed true excess of the secular rotation of the axes of those orbits. At any rate, the "old" Principle of Relativity has turned out to be broad enough to give us a (very simple and) sufficiently general form of the retouching factor of Newton's law, as required for the said secular excesses.

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Note to page 507.—In order to see that the law of force (7′), and therefore also (7), satisfies the Principle of Relativity, let us return to the general equation of motion (3). If τ be the proper time of the planet, then $d/dt = d/\gamma d\tau$, so that (3) becomes

$$\gamma\mathbf{N} = m_0\frac{d}{d\tau}(\gamma\mathbf{v}) = m_0\frac{d^2\mathbf{r}}{d\tau^2}.$$

Now, m_0 is, by assumption, a relativistic invariant, and $d^2\mathbf{r}/d\tau^2$ is the space-part of a relativistic four-vector or (as I call it) of a physical quaternion, viz. $d^2q/d\tau^2$, where $q = \mathbf{r} + ict$; and τ, as is well known, is an invariant. Thus the relativistic requirement is that $\gamma\mathbf{N}$ (not $\mathbf{N}$ itself) *should be the space-part of a four-vector* (or the vector of a physical quaternion) X, viz.

$$\gamma\left[\mathbf{N} + \frac{i}{c}(\mathbf{Nv})\right] = \mathrm{X} = \frac{d^2q}{d\tau^2} \quad . \quad . \quad . \quad . \quad (13)$$

Now, let $\mathrm{Y} = dq/d\tau$ be the four-velocity of the planet, and similarly $\mathrm{Y}_s = dq_s/d\tau_s$ that of the Sun, and write $\mathrm{R} = q - q_s = \mathbf{r} - \mathbf{r}_s + ic(t - t_s)$, so that Y, Y_s, R are all four-vectors. With these symbols, and denoting by ( ) the "scalar product" of two four-vectors, write

$$\mathrm{X} = \frac{\mathrm{F}(\mathrm{YY}_s)}{(\mathrm{RY}_s)^3}\{(\mathrm{Y}_s\mathrm{Y})\mathrm{R} - (\mathrm{YR})\mathrm{Y}_s\} \quad . \quad . \quad (14)$$

where F is any function of (YY_s). Then, the bracketed expression being a four-vector, and its scalar coefficient even an invariant, X will be a genuine four-vector as required. In fact, the above X is but a large sub-case of the four-vectors constructed by Poincaré (*Rend. del Circolo Mat. di Palermo*, xxi., fasc. i., 1906). In comparing with his memoir notice that Poincaré's C is our $-\frac{1}{c^2}(\mathrm{YY}_s)$. Now, the space-part of (14) is precisely of the form of the law of force (7′). To see this, remember that our reference system S has been the rest-system of the attracting centre (the Sun), and this will be a legitimate system provided that m_0/M_0 is small.* This means that $\mathbf{r}_s = 0$, $\mathbf{v}_s = 0$, and therefore $\mathrm{Y}_s = ic\gamma_s = ic$, and $c(t - t_s) = -r$, *i.e.* $\mathrm{R} = \mathbf{r} - ir$. Under these circumstances we have

$$(\mathrm{YY}_s) = -c^2\gamma, \quad (\mathrm{RY}_s) = cr,$$

and therefore for $\gamma\mathbf{N}$, or the vector of the quaternion (14),

$$\gamma\mathbf{N} = \mathrm{F}(-c^2\gamma) \, . \, \frac{\mathbf{r}}{c^3r^3} = \frac{\mathrm{F}(-c^2\gamma)}{c^3}\frac{\mathbf{u}}{r^2},$$

or N equal an arbitrary function of γ divided by r^2, as in formula (7′), of which the adopted law of force (7) is but a particular case.

* In the case of Mercury this is at least as small as 1 : 5,600,000, and, according to Newcomb, even as 10^{-7}. It is also sufficiently small for Venus, etc. For a double star, of course, (14) itself would have to be used, without the simplification which has been possible in our case.