

APPENDIX B - DERIVATION OF THE RELATIVISTIC ENERGY-MOMENTUM EQUATION

Reference [28] mentions on page 835 that simply combining equations (5.62) and (5.65) to each other should generate the complete relativistic energy-momentum equation (5.70), but they do not offer the detailed derivation of this equation: $E^2 = (pc)^2 + (mc^2)^2$.

Here is for convenience the complete step by step derivation of this famous equation:

$$E = \gamma m_0 c^2 \quad (5.62) \qquad p = \gamma m_0 v \quad (5.65)$$

$$\frac{E}{m_0 c^2} = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \frac{p}{m_0 v} = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\left(\frac{E}{m_0 c^2} \right)^2 = \frac{1}{1 - v^2/c^2} \qquad \left(\frac{p}{m_0 v} \right)^2 \frac{v^2}{c^2} = \frac{v^2/c^2}{1 - v^2/c^2}$$

$$\frac{E^2}{m_0^2 c^4} = \frac{1}{1 - v^2/c^2} \quad (B1) \qquad \frac{p^2}{m_0^2 c^2} = \frac{v^2/c^2}{1 - v^2/c^2} \quad (B2)$$

Subtracting momentum equation (B2) term for term from mass equation (B1), we obtain:

$$\frac{E^2}{m_0^2 c^4} - \frac{p^2}{m_0^2 c^2} \frac{c^2}{c^2} = \frac{1}{1 - v^2/c^2} - \frac{v^2/c^2}{1 - v^2/c^2} \quad (B3)$$

$$\frac{E^2}{m_0^2 c^4} - \frac{p^2 c^2}{m_0^2 c^4} = \frac{1}{1 - v^2/c^2} - \frac{v^2/c^2}{1 - v^2/c^2}$$

$$\frac{E^2 - p^2 c^2}{m_0^2 c^4} = \frac{1 - v^2/c^2}{1 - v^2/c^2} = \frac{\gamma}{\gamma} \quad (B4)$$

$$\frac{E^2 - p^2 c^2}{m_0^2 c^4} = \frac{\gamma^2}{\gamma^2}$$

$$\gamma^2 (E^2 - p^2 c^2) = \gamma^2 m_0^2 c^4$$

$$\gamma^2 E^2 - \gamma^2 p^2 c^2 = (m c^2)^2 \quad \text{where } m = \gamma m_0$$

$$\gamma^2 E^2 = p^2 c^2 + (m_0 c^2)^2 \quad \text{where } p = \gamma m_0 v \quad (\text{B5})$$

And finally, since γ is dimensionless, then $E = \gamma E_0$ and we obtain equation (5.70):

$$E^2 = (pc)^2 + (m_0 c^2)^2 \quad (\text{5.70})$$

With equation (B4), there is a strong temptation to simplify both occurrences of the Lorentz factor to 1 before proceeding, but this leads to the often encountered non-relativistic version: $E^2 = (pc)^2 + (m_0 c^2)^2$ which is Newtonian and adds only the translational half of the kinetic energy of a massive particle in motion, leaving out the electromagnetically oscillating half that converts to the relativistic mass increment (see Section 3.7).

The proper procedure is to square the mutually reducible γ factor occurrences, so they can be reunited with the two occurrences of m_0 as the development proceeds.

At face value, fusing the last occurrence of the squared γ factor with the energy ($\gamma^2 E^2$) may seem to be problematic, but considering that this factor is a dimensionless quantity, it can be multiplied with the energy component without any adverse effect for the integrity of the equation.

(Abstracted from [Electromagnetic Mechanics of Elementary Particles](#))