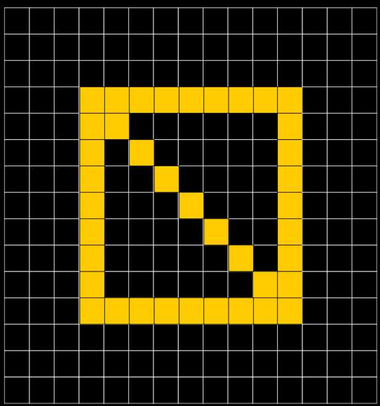
Antonio León Sánchez

The Discrete Reality of Physical Space

A discrete alternative to the infinitist spacetime continuum



Self-edition in Amazon's Kindle Direct Publications

Antonio León Sánchez

THE DISCRETE REALITY OF PHYSICAL SPACE

Collected published and unpublished works 2020-2025

Revised version. September 2025

This PDF contains the first 12 chapters of the 34 that make up the complete edition of the book.

Revised version: September 2025
First edition: June 2024
ISBN 9798327194533

Salamanca and Santiago del Collado (Ávila), Spain.

Table of Contents

1. Important warning

This book brings together a series of published and unpublished works written by the author between 2020 and 2023. Although they are not presented in the chronological order in which they were written, they are arranged to form a functional and coherent text on the real nature of physical space, it is what is called in Spanish a facticio[1]. For this reason, the reader will find several repetitions throughout the book. I could have eliminated these repetitions, but in exchange for modifying the original texts and their corresponding autonomy, which seems to me a worse solution. Besides, we learn by repeating.

As its main thesis, the book argues about the reality of physical space and its finite and discrete nature (although this is not the only thing it argues about). A reality that is already very difficult to deny, despite the opinion of certain relevant physicists who continue to deny that space is a real physical object. Indeed, the empirical detection of gravitational waves implies the empirical detection of the vibrations of space itself. And what does not exist simply does not have empirically detectable properties.

Although this is not a book on the history of science, it does include some chapters on the history of the concept of space, from the pre-Socratics to the relativistic spacetime continuum, which is an infinitist concept legitimated by the Axiom of Infinity. For this reason, chapter discussing the consistency of this axiom are also included. Although contemporary physicists usually do not pay attention to it, the Axiom of Infinity is the key to modern physics. Here, and in other works of the author, it is shown to be an inconsistent axiom, which will have very significant consequences in a good part of physical theories.

The inconsistency of the actual infinity serves, among other things, to demonstrate the discrete nature of space (and time). For this rea-

[1] Said of a book or volume: A collection of various books or printed matter (DRAE). I have not been able to find a satisfactory English translation.

son, the first three chapters of the book demonstrate the elements that constitute the formal setting for all the discussions that follow in the rest of the book. The fourth chapter sets out an ordered list of their respective statements. The discrete nature of space allows us to deduce some basic properties of the constituent substance of real physical space. Of course, the space substance must be different from, but not indifferent to, ordinary matter. The spatial substance might even be the generator of ordinary matter. In this sense, models similar to cellular automata are proposed here to initiate the discussion on the new discrete and finitist paradigm of the physical world.

Finally, the concept of complete totality is used throughout the book and, although the union of the two words, “complete” and “totality”, is sufficiently explicit, it can also be formally defined: A complete totality is *a set defined by comprehension in which every element that should be in the set, is in the set.*

2. Conventions and symbols

2.1 Conventions

P1 To facilitate explanations and discussions, some paragraphs of this book will be consecutively numbered (as this one). They will be referred to by the number that appear at the beginning of each paragraph, preceded by the letter P. For instance, P1 refers to this paragraph. As with the proofs, these numbered paragraphs end with the symbol □. For the same reason, all equations will also be consecutively numbered within each chapter, although in this case the numbers will be put in brackets on the right side of each equation:

$$f(i) = a_i \text{ (example of equation)} \tag{1}$$

Equations will be referred to by their corresponding numbers in brackets: the above equation would be referred to by (1). As usual, numbers in straight parentheses will indicate bibliographical references. In bibliographic references, the abbreviation p. will be used to indicate page or pages. □

Theorems, definitions, corollaries, etc. will be successively numbered. In most cases they will be named by proper names. The symbol "□" will be used to indicate the end of the demonstration of a statement when the demonstration follows the statement. To facilitate reading and minimize errors (related to punctuation) the initial letter of all substantives in the proper names of theorems, corollaries, definitions, principles, axioms and conclusions will be written in capital letters.

When the same explanation serves to two different alternatives, only one of the alternatives will be explained, adding in parentheses the word, or words, that would have to be changed in the given explanation to be the explanation of the other alternative. For example: If the first (last) item in the list is an even (odd) number, the list begins (ends) with an even (odd) number.

All symbols used in the book are listed at the end of this chapter. The ellipsis, symbolically represented by three dots ..., will often be used to denote the rest of the elements of a set or sequence that obviously follow the indicated elements. The logical expression "if, and only if" will be written "iff" when convenient. The expression "actual infinity" refers to one of the types of infinity, the other being the potential infinity. Both are introduced and explained in Chapter 3.

It will be inevitable the use of a few number of primitive concepts, i.e. concepts that cannot be defined in terms of other more basic concepts. That is the case, for instance, of point, line or set. The word "collection" will be used in a general sense to refer to sequences, sets, lists, tables, etc.

Definition 1 (of Complete Totality) *A complete totality is a set defined by comprehension in which every element that meets the definition of membership is in the set, so that to a complete totality of a certain type of elements, it is not possible to add new elements of that type because it already contains all of them.*

Most of the collections, mainly sequences and sets, will be ω-ordered (as the sequence 1, 2, 3, ... of the natural numbers in their natural order of precedence). In a few cases they will be ω^*-ordered (as in the case of the increasing sequence of negative integers ... -3, -3, -1). The sets used in the demonstrations, for example the real interval $(0, 1)$, or the set $\mathbb{Q}^+$ of the positive rational numbers, will always be the simplest possible in each occasion.

As usual, to put into a correspondence a set A with another set B means to pair off each element of the A set with an element of the set B. All correspondences will be injective, and in most cases surjective (bijections or one-to-one correspondences). Unless otherwise indicated, the sets $\mathbb{N}$ (natural numbers), $\mathbb{Z}$ (integer numbers), $\mathbb{Q}$ (rational numbers), $\mathbb{A}$ (algebraic numbers) and $\mathbb{R}$ (real numbers), and any of their subsets, will always be considered in their natural order of precedence, that is, ordered by their increasing magnitudes or values. In the case of $\mathbb{N}$, the natural order of precedence is the ω-order (a case of well-order defined in Chapter 20). In all the other cases, excluding $\mathbb{Z}$, the order of precedence is a dense order (see P2) that is not a well order.

In most cases, we will use the word "denumerable" to refer to the infinity of the set $\mathbb{N}$ of the natural numbers and to the infinity of any other set or sequence that can be put into a one to one correspondence with $\mathbb{N}$. The words "enumerable" or "numerable" can also be used with the same meaning. Although the word "countable" is also used to refer to finite or denumerable infinite sets, it will not be used here in order to avoid confusions. Finally, the terms "non-countable" or "non-denumerable" will be used to denote the infinities greater than the denumerable infinity.

Although formally unacceptable, Euclid defined two capital concepts in geometry: the concept of line [150, Definition 2, p. 153] and the concept of straight line [150, Definition 4, p. 153], being the second a particular case of the first; and being both of them currently assumed as primitive, undefinable, concepts. Languages maybe evolving from their most popular use that, unfortunately is not always the most correct one [130]. That could be the reason why in English, *line* and *straight line* came to mean the same thing, and now there is no English word to denote the original Euclidean concept of line, a universal concept that applies to all types of lines. For this reason, in the English edition of this book, the word "line*" will be used to refer to the general geometric object that Euclid called line. Thus, and still being a primitive concept, a line* (línea in Spanish) can be understood as any uni-dimensional continuum of points. Although it is possible to give a formally productive definition of straight line [196, 200], it will not be necessary to do so in this book, so that they can continue to be understood as a particular type of lines whose lengths are the shortest of all possible lines joining any two given points. No matter how redundant, straight lines will always be referred to by "straight lines". As usual, real and rational lines* and straight lines will be used to denote lines* and straight lines whose points represent respectively densely ordered sets (see P2) of real numbers and of rational numbers.

P2 In all discussions and arguments, time, distances and lengths will be assumed to be Euclidean and represented by real numbers and intervals of real numbers. As usual, a finite interval (a, b) is said finite if its extension $b - a$ is finite, even if the interval is infinitely dense, which means that between any two elements (points, instants, numbers) of the interval, the interval contains infinitely many different elements. This is the case of all intervals of rational and real numbers in their corresponding natural order of precedence. An element inside an interval will be an element of the interval different from its endpoints. □

Although supertasks will be introduced in Chapter 3, they will start to be used from the first chapters. A supertask consists of performing an infinite number of actions or tasks (for example counting numbers, or removing balls from a box containing balls) in a finite interval of time, which, unless otherwise indicated, will be the real interval (t_a, t_b). The successive actions a_1, a_2, $a_3, \dots$ of the infinite sequence of actions $\langle a_i \rangle$ will be supposed to be carried out in the successive instants t_1, t_2, $t_3, \dots$ of an ω-ordered, strictly increasing and convergent sequence of instants $\langle t_i \rangle$ within the interval (t_a, t_b), being t_b the limit of the sequence

$\langle t_i \rangle$. Every action a_i of $\langle a_i \rangle$ will be assumed to be performed in the precise instant t_i of $\langle t_i \rangle$, and all of them will be instantaneous.

Needless to say, all arguments in this book are of a conceptual nature, even when they make use of material artifacts as machines, boxes, balls and the like, all of which have to be understood as theoretical devices to illustrate the arguments and to facilitate discussions.

2.2 Symbols

The followings symbols and notations could be used in what follows:

MT: Modus Tollens

*: Thomson' lamp on.

o: Thomson's lamp off.

c: Thomson's lamp clicked.

$\mathbb{N}$: set of the natural numbers in their natural order of precedence.

$\mathbb{Z}$: set of the integer numbers in their natural order of precedence.

$\mathbb{Q}$: set of the rational numbers in their natural order of precedence.

$\mathbb{Q}^+$: set of the positive rational numbers in their natural order of precedence.

$\mathbb{A}$: set of the algebraic numbers in their natural order of precedence.

$\mathbb{R}$: set of the real numbers in their natural order of precedence, and real straight line.

$\mathbb{R}^+$: set of the positive real numbers in their natural order of precedence.

$\mathbb{R}^3$: Euclidean tridimensional space.

$\mathbb{R}^n$: Euclidean n-dimensional space.

$|A|$: cardinal of the set A.

$\dots$: ellipsis.

$\in$: belongs.

$\notin$: does not belong.

$\subset$: subset.

$\supset$: superset.

$\not\subset$: not subset.

$\cup$: union of sets.

$\cap$: intersection of sets.

$P(A)$: power set of the set A (set of all subsets of A).

$\aleph_0$: aleph-null, the smallest transfinite cardinal.

$2^{\aleph_0}$: power of the continuum.

ω: omega, the smallest transfinite ordinal.

2ω, 3ω, $\omega_1 \ldots$: ordinals greater than ω.

$2^{2^{\aleph_0}}$, $\aleph_1$, $\aleph_2 \ldots$: cardinals greater than $\aleph_0$.

∞: infinity, the improper real number.

(a, b): open interval or segment.

$[a, b]$: closed interval or segment.

$(a, b]$: right closed interval or segment.

$[a, b)$: left closed interva or segmentl.

I_o: 0-interval, interval whose left endpoint is 0.

$\langle q_n \rangle$, $\langle q_i \rangle \ldots$: ω-ordered sequence $q_1, q_2, q_3, \ldots$

$\sum_{i=1}^{n} x_i$: sum of n terms: $x_1 + x_2 + \cdots + x_n$.

$\sum_{i=1}^{\infty} x_i$: sum of infinite terms: $x_1 + x_2 + x_3 + \ldots$

$\lim_{n \to \infty} a_n$: limit of the sequence $\langle a_n \rangle$.

$\lim_n a_n$: limit of the sequence $\langle a_n \rangle$.

$\langle D_n(x) \rangle$: ω-ordered sequence of definitions of x.

$D_i(x)$: ith definition of x.

$\langle D_i(x) \rangle_{i=1,2,\ldots n}$: first n definitions of x.

${}^kS_i$: ith element of a collection at the kth definition of the collection.

$|x|$: absolute value of x.

$\min(a, b)$: least of the two values in brackets.

$\forall$: for all.

$\exists$: exists.

$\Rightarrow$: logic inference.

$\Leftrightarrow$: logic double inference.

iff: if, and only if.

$\neg$: logic negation.

$\vee$: logic or.

$\wedge$: logic and.

$\therefore$ therefore.

$\square$: end of a proof.

3. Finite versus infinite

3.1 Introduction

This chapter proves a fundamental result for the rest of the book:

The Hypothesis of the Actual Infinity is inconsistent.

A result that changes everything in mathematics and physics. I think that is why there is so much resistance to accepting it. The first demonstrations of this inconsistency have been available for more than twenty-five years, but so far their echo has been practically nil. Obviously, the infinitist mathematics of modern physics (which has never been tested) will be seriously affected by the inconsistency of the Hypothesis of the Actual Infinity (fortunately, experimental physics can only be finitist and discrete). Some of the mathematical and physical consequences of the inconsistency of the actual infinity will be discussed in this and the following chapters of the book. One such physical consequence will be proved in this chapter: In a consistent reality, there can only be a finite number of universes (if there are several), each with a finite number of physical objects.

3.2 Zeno, Aristotle and Cantor

Fortunately there is an abundant and excellent literature on the history of infinity (for instance: [376, 229, 318, 35, 300, 69, 223, 248, 251, 190, 191, 2, 252, 249, 66, 363, 22, 298]). The details of that story will not be necessary here, although three of its most relevant protagonists could be remembered as historical references:

a) Zeno of Elea (490-430 BC), a pre-Socratic philosopher that made use for the first time of the mathematical infinity when defending Parmenides' thesis on the impossibility of change. We know Zeno work (near forty arguments, including his famous paradoxes against the possibility of change [3, 71]) through his doxographers: Plato, Aristotle, Diogenes Laertius or Simplicius. The infinity in Zeno arguments is the actual infinity, although Zeno is obviously not doing infinitist mathematics, but logical reasoning in which infinite collections of points and instants appear. Zeno arguments only work properly if these collections are considered as complete infinite totalities (Zeno dichotomies are discussed in Chapter 20).

b) Aristotle (384-322 BC), one of the most influential thinkers of western culture. He introduced, in a broad sense, the notion of *one to one correspondence* just when he was trying to solve some of Zeno paradoxes [18, Books III-VII]. He also introduced the basic distinction between the potential and the actual infinity. A distinction that will be analyzed in the next section.

c) Georg Cantor (1845-1918), mathematician co-founder, together with R. Dedekind and G. Frege, of set theory at the end of the XIX century. His work on transfinite numbers [56] (cardinals and ordinals) lays the foundations of modern infinitist mathematics. He inaugurated the so called paradise of the actual infinity, where, according to D. Hilbert, infinitists will inhabit forever [157, p. 170]:

> Wherever there is the slightest prospect of fruitful concepts and conclusions, we will carefully track them, cultivate them, support them and make them usable. No one shall be able to drive us out of the paradise that Cantor has created for us.

From Zeno to Aristotle, the infinity discussed was usually the actual infinity, although this notion was far from being clearly established before Aristotle. From Aristotle to Cantor, there were defenders of both types of infinity (actual and potential), although with a certain hegemony of the potential infinity, especially since the 13th century, after Aristotle had been *christianized* by the medieval scholastics. In those pre-infinitist times, the same arguments could be used in support of one or the other infinity (for example, the arguments based on the correspondence between the points of a circle and the points of one of its diameters). But there is still no theory of mathematical infinity. The first mathematical theory of infinity appears at the end of the XIX century, Bolzano, Dedekind and especially Cantor being its most relevant founders. From Cantor until today, the hegemony of the actual infinity has been almost absolute and, moreover, free from serious criticism.

3.3 The actual and the potential infinity

(This section includes published texts by the author [212, p. 32-35])

In common parlance, the word infinite is used to refer to the quality of being immense, gigantic, unlimited, etc. C. F. Gauss (*Princeps Mathematicorum* [375, p. 1188]) said that infinity is a way of speaking (C.

F. Gauss, Letter to astronomer H.C. Schumacher, 12 July 1831 [127, Vol. II, p. 268]):

> I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a *façon de parler* [a way of speaking], the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction.

The consideration of an infinite magnitude (or an infinite sequence, for instance of numbers) as something completed is what we call *actual infinity* since Aristotle, who introduced the distinction between the potential infinity and the actual infinity [16, 17, Books III, VIII]. It is remarkable the fact that in the above quotation, Gauss implicitly includes the distinction between both infinities (see below in this section). I have the impression that most physicists think, like Gauss, in terms of the potential infinity, without worrying about the fact that they are building physics with the mathematics of the actual infinity.

As will be seen below, it is possible to give a precise definition of the concept infinity, albeit based on a primitive concept: the concept of set. However, the concept of set could be defined in operational, non-Platonic terms [212, p. 360]:

Definition 2 (of Set) *A set is the theoretical object that results from a mental grouping of different arbitrary objects previously defined.*

This definition has the advantage of avoiding the entanglements caused by self-reference, simply by requiring that the elements to be grouped be previously defined, which seems quite reasonable if we intend to know what we are grouping. It is convenient, on the other hand, to consider that some sets exist as complete totalities, i.e. as sets satisfying the following:

Definition 3 (of Complete Totality) *A complete totality is a set defined by comprehension in which every element that satisfies the corresponding membership definition of the set is in the set.*

In consequence, to a complete totality of a certain type of elements, it is not possible to add new elements of that type because it already contains *all of them*.

But returning to the concept of infinity, and apart from Gauss's opinion, the word "infinite" has also a precise meaning based on the primitive concept of set:

Definition 4 (of Infinite Set) *A set is said infinite if it can be put into a one to one correspondence with one of its proper subsets.*

which is the well known Dedekind's definition of infinite set [77, p. 115], an important element of the foundations of modern infinitist mathematics, which began its development at the end of the 19th century. As is well known, the controversial history of the (philosophical and) mathematical infinity has its roots in the pre-Socratic times, although here we are not interested in the details of that history (there is an abundant and excellent literature on the history of infinity, for instance: [376, 229, 318, 35, 300, 69, 223, 248, 251, 190, 191, 2, 252, 249, 66, 363, 22, 298]).

Now I will try to explain the distinction between the two infinities, the actual and the potential. The set of the natural numbers and *supertask theory* are two suitable instruments to evidence such a distinction. The set of the natural numbers needs no presentation. With respect to supertask theory it must be recalled that it is an infinitist theory based, as set theory, on the Axiom of Infinity (introduced in Section 3). It originated as a consequence of a seminal discussion about the possibility, or impossibility, of performing an infinite number of actions (tasks) in a finite time interval. [345, 38, 344, 364, 27].

Although the main objective of supertask theory was, and continue to be, the discussion on the actual infinity, its physical implications (including special relativity) have also been discussed in the last years [274, 284, 288, 312, 141, 143, 142, 284, 285, 286, 86, 287, 263, 7, 8, 289, 369, 160, 84, 85, 263, 83, 323]. In short:

> A supertask consists in performing an infinite sequence of actions $\langle a_i \rangle$ within a finite time interval $[t_a, t_b)$, each action a_i being performed at the precise instant t_i of a strictly increasing and convergent sequence of instants $\langle t_i \rangle$ within $[t_a, t_b)$, being t_b the mathematical limit of $\langle t_i \rangle$.

where the elements of $\langle a_i \rangle$ and $\langle t_i \rangle$ are ordered in the same way as the set of the natural numbers in their natural order of precedence: ω-order: 1, 2, 3,.... Notice in this ordering the set exist as a complete totality (Definition 3) and each element n has an immediate successor $n + 1$ (Peano's Axiom of the Successor, [272, p. 1]), where immediate successor is defined according to:

Definition 5 (of Immediate Successor) *All elements of an ordered set A succeeding (preceding) a given element n of A are successors (predecessors) of n in the considered order of A. An element n of an ordered set A is said the immediate successor (predecessor) of another element m of A if n succeeds (precedes) m in the considered ordering of A and no other element of A exists between m and n in that ordering.*

We are now in the appropriate position to analyze the difference between the actual and the potential infinity, Indeed, consider the list L_n of the natural numbers in their natural order of precedence:

$$L_n = 1, 2, 3, \ldots \tag{1}$$

The list L_n can be considered in two different ways:

a) As a complete totality, i.e. as a list in which every element that could be in the list, is in the list (actual infinity).

b) As an unlimited and uncompletable totality (potential infinity).

According to the Hypothesis of the Actual Infinity, the list L_n of the natural numbers in their natural order of precedence 1, 2, 3,... exists as a *complete totality*, i.e as a totality that contains, all at once, all natural numbers. The ellipsis (...) in $1, 2, 3, \ldots$ stands for *all* natural numbers. For all. The word "actual" in *actual infinity* means, therefore, that all elements of an infinite collection as L_n, exist all at once (in the *act*), as a complete totality. In consequence, the list L_n of the natural numbers in their natural order of precedence is considered as a complete totality despite the fact that no last number completes the list. To assume the Hypothesis of the Actual Infinity means, therefore, to assume that it is possible to complete the incompletable, as Aristotle would surely say [17, p. 291]. Or that the incompletable can exist as complete.

To emphasize this sense of completeness, let us consider the task of counting the successive elements of L_n, i.e. the successive natural numbers 1, 2, 3,... in their natural order of precedence. In agreement with the Hypothesis of the Actual Infinity we could count *all* natural numbers in a finite time, for example in an hour, or in a millisecond. The task of counting all natural numbers in a finite time interval, even in less than a second, is an example of supertask:

- *Count each of the successive natural numbers 1, 2, 3... at each of the successive instants t_1, t_2, t_3... of a strictly increasing sequence of instants $\langle t_i \rangle$ within the finite real interval (t_a, t_b), being t_a and t_b any two instants such that $t_a < t_b$, and t_b the mathematical limit of the sequence $\langle t_i \rangle$. For instance, the classical sequence defined by:*

$$t_n = t_a + (t_b - t_a)\,\frac{2^n - 1}{2^n} \tag{2}$$

As we will now prove, at t_b all natural numbers would have been counted. All. In effect, let each natural number n of the list L_n be counted at the precise instant t_n of $\langle t_i \rangle$. Being t_b the limit of $\langle t_i \rangle$, t_b is the first instant after all instants of $\langle t_i \rangle$, and all those instants do exist as a complete totality according to the Hypothesis of the Actual Infinity. So, the one to one correspondence f between L_n and $\langle t_i \rangle$ defined by:

$$f(n) = t_n, \ \forall n \in L_n \tag{3}$$

proves that at t_b all natural numbers of the list L_n has been counted. All. The reader can easily imagine why ellipsis and correspondences between sets are the key instruments for demonstrations in infinitist mathematics. Note, on the other hand, that the fact of pairing the elements of two infinite sequences (in our case the one of natural numbers and the other of instants) does not prove both sequences exist as complete totalities. They could also be potentially infinite with the same number of elements, a possibility usually ignored in modern infinitist mathematics.

The alternative to the Hypothesis of the Actual Infinity is the Hypothesis of the Potential Infinity, which rejects the existence of *complete* infinite totalities, and then the possibility to count all natural numbers. From this perspective, the natural numbers result from the *endless* process of counting: it is always possible to count a number greater than any given number (Peano's Axiom of the Successor [272, p. 1]). But it is impossible to complete the process of counting all of them, simply because there is not a last natural number to complete the process. So, the complete list of all natural numbers makes no sense, simply because it is incompletable.

The word "potential" in *potential infinity* means, therefore, that the elements of an infinite collection do not exist all at once, but potentially, as possible. The potential infinity is *the unlimited*, as the list L_n of the natural numbers in their natural order of precedence, but only finite collections can be considered as complete totalities, as large as wished but always finite. Similarly, only finite natural numbers can be considered, as large as wished but always finite. For the potential infinite there is not a last natural number (it is always possible to consider a number greater than any previously considered number), but neither is there the complete collection of *all* natural numbers. Contrarily to the actual infinity, the potential infinity assumes the incompletable cannot be completed, cannot exist as complete, precisely because it is not completable.

In short, the actual infinite hypothesis states that the infinite collections are complete totalities, even if no last element completes the collection, as in the case of the ordered list of the natural numbers. On the contrary, the hypothesis of the potential infinite proposes that the infinite collections do not exist as complete totalities, the only complete totalities are the finite totalities, though they can be unlimited in the number of their possible elements. All of which can be summarized in the following definition:

Definition 6 (of Actual and Potential Infinity) *An ordered collection of elements is infinite if there is no last (first) element that completes (initiates) it. The collection is actually infinite if it is considered a complete totality, and potentially infinite if it is not considered a complete totality.*

Where collection is by set, succession, sequence, list, etc. To be formally precise, the words *set*, *succession*, *sequence*, etc. should be replaced by the more general word *collection*. However, for the sake of brevity, it will not be necessary to do so. Therefore, in what follows all of them will be interchangeable with each other, unless otherwise specified.

The potential infinity (the 'improper' or 'non-genuine' infinity as Cantor called it [57, p. 70]) has never deserved the attention of contemporary mathematics. The infinity in Dedekind's Definition 4 of infinite set is the actual infinity (see next section). The infinitely many elements of an infinite set exist all at once, as a complete totality. Dedekind's Definition 4 is, therefore, based on the violation of the old Euclidean Axiom of the Whole and the Part (the whole is greater than the proper part) [106]. Set theory has been built on that violation.

The hegemony of the actual infinity in contemporary mathematics is absolute. As absolute as the submission of physics to infinitist mathematics. Some authors proceed as if the existence of complete infinite totalities had been formally demonstrated. Obviously, if that were the case we would not need the Axiom of Infinity to legitimize the existence of such infinite totalities. The Hypothesis of the Actual Infinity is just a hypothesis, not a proven fact. And physics should not be subject to infinitist mathematics. In fact, and in agreement with P. Dirac, it should not be subject to any kind of mathematics at all [81, p. VIII]:

> Mathematics is only a tool and one should learn to hold physical ideas in one's mind without reference to the mathematical form.

The three most important "proofs" of the existence of actual infinite totalities (by Bolzano, Dedekind and Cantor) are illustrative of what we could call *naive infinitism*. They also explain why modern infinitist mathematics had finally to establish the existence of actual infinite sets by an arbitrary law, i.e. by means of an arbitrary axiom (the Axiom of Infinity, which is introduced in the next section).

- Bolzano's proof goes as follow (taken from [249, p 112]):

> One truth is the proposition that Plato was Greek. Call this p_1. But then there is another truth p_2, namely the proposition that p_1 is true [But then there is another truth p_3, namely the proposition that p_2 is true]. And so *ad infinitum*. Thus the set of truths is infinite.

But the existence of an endless process (p_1 is true, then p_2 is true, then p_3 is true, then ...) does by no means prove the existence of a final result as a complete totality. At best it proves the existence of an endless (potentially infinite) process. But it does not prove the existence of an actual infinite totality.

• Dedekind's proof is similar (taken from [249, p 113]):

> Given some arbitrary thought s_1, there is a separate thought s_2, namely that s_1 can be object of thought [there is a separate thought s_3, namely that s_2 can be object of thought]. And so ad infinitum. Thus the set of thoughts is infinite.

The above comment on Bolzano proof also applies here. Dedekind gave another proof a little more detailed, albeit with the same formal defect, based on his definition of infinite set [77, p. 115].

• And finally, Cantor's proof: ([147, p 25], [249, p. 117]):

> Each potential infinite presupposes an actual infinity.

or ([55, p. 404] English translation [305, p. 3]):

> ... in truth the potential infinity has only a borrowed reality, insofar as a potentially infinite concept always points towards a logically prior actually infinite concept whose existence it depends on.

But this is an opinion, not a formal proof. It is now clear why the existence of an actual infinite set had to be finally established by law; that is, by means of an axiom.

Let us, finally, state a conventional use of the expressions actual infinite and potential infinity in this and the subsequent chapters of this book. From now on, and for sake of simplicity, the actual infinity will be referred to simply as infinity or actual infinity, while the potential infinity will always be referred to as potential infinity. Or put another way, the word "infinity" will always mean actual infinity, unless it is preceded by the word "potential", in which case it will obviously mean potential infinity. For the sames reasons of simplicity, the world "universe" will always denote the observable universe.

3.4 The infinity of the Axiom of Infinity

Nothing we have been able to observe and measure so far has been infinite. Nor has it been possible to divide anything into an infinite number of parts. On the other hand, and after more than twenty-seven centuries of arguments and discussions, it was not possible to prove (or disprove) the existence of the actual infinities. Infinitism had no choice but to accept that existence in axiomatic terms by means of the Axiom of Infinity. An axiom that simply states the existence of an infinite set:

Axiom 1 (of Infinity (ordinary language)):

There exists an infinite set.

Or in abstract, symbolic, terms:

Axiom 2 (of Infinity (abstract form)):

$$\exists A : \emptyset \in A \land \forall a \in A \ (a \cup \{a\} \in A \tag{4}$$

that reads: there exists a set A such that $\emptyset$ (the empty set) belongs to A and for every element a in A, the element $a \cup \{a\}$ also belongs to A. Although it is not explicitly declared the type of infinity involved in the set A, it can be easily proved that it is the actual infinity:

Theorem 1 (of the Actual Infinity) *The infinity in the Axiom of Infinity can only be the actual infinity.*

Proof: Since potentially infinite sets do not exist as complete totalities (Definitions 3 and 7), only two subsets with the same number of elements of the same potentially infinite set could be put into a one to one correspondence, and then Dedekind Definition 4 is not satisfied, because we would have a one to one correspondence between two proper subsets of a potentially infinite set, in the place of a one to one correspondence between a set and one of its proper subsets. In consequence, the infinity involved in the Axiom 1 of Infinity can only be the actual infinity. □

Obviously, an axiom is just an axiom, i.e. a statement that can be accepted or rejected. Some relevant authors as L.E.J. Brouwer, C. Hermite, S. Kleene, J. König, L. Kronecker, H. Poincaré, A. Robinson, L. Wittgenstein, or H. Weyl, among others, rejected the Axiom of Infinity, more or less explicitly. H. Poincaré went so far as to say that (quoted in [249, p. 121], [76, p. 1]):

> infinity is a perverse pathological illness that would one day be cured.

But the vast majority of contemporary mathematicians and physicists do not question the Axiom of Infinity. Indeed, in our days the criticism of the actual infinity is practically non-existent. And infinitism has become a current of thought absolutely hegemonic and quite intolerant of dissent, as if the existence of the actual infinite had been proven. And no, it has not been proven; it has been assumed. And one has the right and the duty to question that assumption, without being insulted and ostracized for it (as is currently the case).

3.5 A short proof of inconsistency

Over the last 30 years, and from different perspectives (set theory, supertask theory, transfinite cardinals, transfinite ordinals, transfinite arithmetics, geometry) I have developed more than forty formal proofs of the inconsistency of the Hypothesis of the Actual Infinity [212]. This section includes one of them, chosen for its brevity and simplicity: the next Theorem 5. First, however, it is necessary to consider the following formal elements:

Definition 7 (of the Types of Sets) *A set is finite if it has a definite and finite number of elements. A set of elements of a certain type is potentially infinite if it always contains a finite number of elements of that type and any finite numbers of new elements of that type can always be added to it, without the set ceasing to be finite and without it being necessary to change its name.*

Definition 8 (of the Types of Infinities) *The actual infinity is the infinity of the infinite sets. The potential infinity is the infinity of the potentially infinite sets.*

Definition 9 (of Inconsistent Set) *A set is inconsistent if a contradiction can be deduced from the number of its elements, or from the number of elements of at least one of its proper subsets.*

Corollary 1 (of Inconsistent Sets) *A set with the same number of elements as an inconsistent set, is also inconsistent.*

Proof: It is an immediate consequence of Definition 9. □

Definition 10 (of Denumerable Set) *A set is denumerable if its cardinal is the smallest infinite cardinal $\aleph_0$ of the infinite set of all natural numbers. An infinite set is non-denumerable if its cardinal is greater than the smallest infinite cardinal $\aleph_0$.*

Definition 11 (of ω-Ordered Sets) *A set is ω-ordered if being denumerable, it has a first element, each element has an immediate successor and an immediate predecessor, except the first one which has no predecessor.*

Theorem 2 (of Denumerable Sets) *It is always possible to define a one-to-one correspondence between any two denumerable sets.*

Proof: Let A and B be any two denumerable sets. They have the same number of elements: $\aleph_0$ elements (Definition 10). So their respective elements can be put into one-to-one correspondence, i.e. each of the different elements of A can be paired with a different and exclusive element of B, so that all elements of A and B result exclusively paired. □

Theorem 3 (of non-Denumerable Sets) *Every non-denumerable set has denumerable proper subsets.*

Proof: Let X be any non-denumerable set. Since its cardinal is greater than $\aleph_0$ (Definition 10), X contains proper subsets with only $\aleph_0$ elements, all of which are denumerable proper subsets of X (Definition 10). □

Theorem 4 (of Indexation) *The elements of a denumerable set can be reordered with the same order as the elements of any other denumerable set.*

Proof: Let $A = \{a, b, c, \dots\}$ and $B = \{\alpha, \beta, \dots\}$ be any two denumerable sets. There exists at least one bijection f between the elements of A and B (Theorem 2). Consequently, f pairs each element k of A with a unique and exclusive element, say δ, of B, which can be used to exclusively index that element k of A, so that element k can be rewritten as a_δ. Consequently, the elements of the set A can be reordered and rewritten to define the set $A' = \{a_\alpha, a_\beta, a_\gamma, \dots\}$ which has exactly the same elements as A, and ordered in the same way as the elements of B. □

The infinity of infinite sets is the actual infinity, not the potential infinity (Theorem 6 of the Axiom of Infinity). This implies the existence of certain infinite sets that are also complete totalities (Definition 3). For example the set $\mathbb{N}$ of ALL natural numbers in their natural order of precedence. It is not possible, then, to add new natural numbers to the set $\mathbb{N}$ of natural numbers because it already contains them all. And the same is true of many other numerical or non-numerical sets. For many authors, the existence of these ordered and complete totalities without a last element that completes them (or without a first element that initiates them) is a proven conclusion independent of the Axiom of Infinity. It is not. It is an existence assumed and legitimized by the Axiom of Infinity. Their existence is, therefore, as debatable as the Axiom of Infinity itself. So it is as legitimate to argue about that axiom as it is to argue about the existence of those complete totalities. This fully justifies the following:

Theorem 5 (of Denumerable Infinity) *All denumerable sets are inconsistent.*

Proof: Let A be any denumerable set. The set A allows us to define the set A' with the same elements as A but reordered as the set $\mathbb{N}$ of natural numbers in their natural order of precedence: $A' = \{a_1, a_2, a_3, \}$ (Theorem 4). The open interval of rational numbers $(0, 1)$ is densely ordered in the natural order of precedence (represented by the symbol $<$) defined by the natural values of the rational numbers. It is also a denumerable set, so there exists a bijection f between A' and $(0, 1)$

(Theorem 2). Consequently, $(0,1)$ can be reordered and rewritten as the set $\mathbb{Q}_{01} = \{q_{a_1}, q_{a_2}, q_{a_3}, \dots\}$, where $q_{a_i} = f(a_i), \forall a_i \in A'$, and the successive elements $q_{a_1}, q_{a_2}, q_{a_3}, \dots$ of $\mathbb{Q}_{01}$ are ordered by the successive natural numbers in their natural order of precedence, and not by their respective values as rational numbers. Let x now be a rational variable defined initially as q_{a_1}. And let the value of x be <-compared (i.e., compared according to the values of the rational numbers) with the successive elements of the set $\mathbb{Q}_{01}$, with x being redefined as the compared element q_{a_i} if, and only if, $q_{a_i} < x$.

For short, let us call comparison* this <-comparison and redefinition of x if, and only if, the value of the compared element is smaller than the current value of x. It is immediate to prove that for each natural number v it is possible to perform the first v comparisons* of x with the first v successive elements of $\mathbb{Q}_{01}$. Indeed, if it were not possible, there would be at least one natural number $n \leq v$ such that x could not be compared* with q_{a_n}, which is impossible because q_{a_n} is a rational number of $\mathbb{Q}_{01}$ that can be compared* with the current value of x, which is also a rational number. Once all possible comparisons* of x with the successive elements $q_{a_1}, q_{a_2}, q_{a_3}, \dots$ of $\mathbb{Q}_{01}$ have been made, the current value of x, whatever it may be, could only be the smallest rational number of that set. Indeed, if once performed all possible comparisons* of x with the successive elements of $\mathbb{Q}_{01}$ the current value of x were not the smallest rational number of $\mathbb{Q}_{01}$, there would be at least one element q_{a_n} in $\mathbb{Q}_{01}$ such that $q_{a_n} < x$. But that is impossible because n is a natural number; the first n comparisons* have been carried out; and therefore x was compared* with q_{a_n} and redefined as q_{a_n}; and in all subsequent comparisons*, x could only be redefined with values smaller than q_{a_n}. Therefore, it is impossible for $q_{a_n} < x$. But, on the other hand, it is also immediate to prove that once all possible comparisons* of x with the successive elements of $\mathbb{Q}_{01}$ have been made, the current value of x is not the smallest rational number of that set: every element of the infinite set $\{x/2, x/3, x/4 \dots\}$ is an element of $\mathbb{Q}_{01}$ smaller than x. This contradiction proves that the set A', defined exclusively with the elements of A, is inconsistent. Therefore A' and A are inconsistent (Definition 9). And A being any denumerable set, it must be concluded that all denumerable sets are inconsistent. □

Although the consistency of a mathematical proof of infinite steps is universally accepted without the need to perform all of its infinite steps, the theory of supertasks considers the possibility of performing them in finite time. In the case of the above successive comparisons* of x with each successive q_{ai} would be performed at each successive instant t_i of a strictly increasing and convergent sequence $\langle t_i \rangle$ of instants within the finite time interval (t_a, t_b), whose limit is t_b. The instant t_b is the first instant after all instants of $\langle t_i \rangle$, and therefore the first instant after having performed all possible comparisons* of x with the successive elements of Q_{01}. At the instant t_b the rational variable x will still be a rational variable with a certain value, whatever it is; and not, for example, an elephant (in which case anything could be proved). The problem is that the value of x at the instant t_b is and is not the least rational of Q_{01}.

Corollary 2 (of Inconsistent ω-Order) *ω-ordered sets are inconsistent.*

Proof: Since ω-ordered sets are also denumerable sets (Definition 11), they are inconsistent (Theorem 5). □

From the previous theorems and corollaries, we can immediately deduce, among many others, the following results:

3.6 The axiom of infinity is inconsistent

The above Theorem 5 proves the inconsistency of any denumerable set. It is then immediate to prove the following results:

Theorem 6 (of the Axiom of Infinity) *The Axiom of Infinity is inconsistent.*

Proof: Let us write the set A defined in Axiom 2:

$$\exists A : (\emptyset \in A \wedge \forall a \in A \ (a \cup \{a\} \in A)) \tag{5}$$

as:

$$A = \{a, s_1(a), s_2(a), s_3(a), \dots\} \tag{6}$$

where:

$$s_1(a) = a \cup \{a\} \tag{7}$$
$$s_2(a) = s_1(a) \cup \{s_1(a)\} \tag{8}$$
$$s_3(a) = s_2(a) \cup \{s_2(a)\}\} \tag{9}$$
$$s_4(a) = s_3(a) \cup \{s_3(a)\}\} \tag{10}$$
$$s_5(a) = s_4(a) \cup \{s_4(a)\}\} \tag{11}$$
$$\dots$$

Consider now the set $\mathbb{N}$ of the natural numbers, which is denumerable, and the set A defined by (6), which is the set whose existence claims the Axiom of Infinity. The one to one correspondence f between the denumerable set $\mathbb{N}$ and A defined according to:

$$f(n) = s_n(a), \ \forall n \in \mathbb{N} \tag{12}$$

proves that A is also an inconsistent set (Theorem 5 and Corollary 1). □

And from Theorems 1 and 6 it immediately follows the next three corollaries:

Corollary 3 (of the Inconsistent Infinity) *The actual infinity is inconsistent.*

Proof: It is an immediate consequence of Theorems 1 and 6. □

Corollary 4 (of the Actual Infinite Sets) *All actual infinite sets are inconsistent.*

Proof: It is an immediate consequence of Theorems 1 and 6. □

Corollary 5 (of Infinite Divisibility) *The actual infinite divisibility of any formal or physical object is inconsistent.*

Proof: From the actual infinite divisibility of any formal or physical object can only result an inconsistent infinite set of parts (Corollary 4). So that actual infinite divisibility is inconsistent. □

Corollary 6 (of Consistent Collections) *A set can be either a finite complete totality or a potentially infinite and uncompletable totality. Otherwise it is inconsistent.*

Proof: It is an immediate consequence of Definition 6 and Corollary 4. □

Let us now recall the following definition:

Definition 12 (of Densely Ordered Sets) *If no element of a strictly ordered set has an immediate predecessor nor an immediate successor, the set is said to be densely ordered or to define a continuum.*

We can now prove the following:

Theorem 7 (of Inconsistent Dense Order) *Densely ordered sets are inconsistent.*

Proof: Let X be a densely ordered set. Suppose X is finite. It will have a finite number of elements, say n. Let x_1 and x_2 be two elements of X such that x_2 is a successor of x_1. Since x_2 cannot be the immediate successor of x_1, there will exist between x_1 and x_2 at least one other successor x_3 of x_1. Since x_3 cannot be the immediate successor of x_1, there will exist between x_1 and x_3 at least one other successor x_4 of x_1. By repeating this argument $n-2$ times we will arrive at a successor x_{n-2} of x_1 that would have to be its immediate successor, which is impossible. Therefore, X cannot be finite. And being infinite it is inconsistent (Corollary 4). □

Corollary 7 (of the Inconsistent $\mathbb{Q}$ and $\mathbb{R}$) *When considered as complete infinite totalities, the set $\mathbb{Q}$ of the rational numbers and the set $\mathbb{R}$ of the real numbers are both inconsistent.*

Proof: It is an immediate consequence of Corollary 4, and also of Theorem 7, because they are densely ordered sets. □

Theorem 8 (of the Inconsistent Continuum) *The spacetime continuum is inconsistent.*

Proof: The spacetime continuum is the Cartesian product (cross product) of sets $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, each of whose factors is the set $\mathbb{R}$ of real numbers. Consequently it is an inconsistent set (Corollaries 7 and 1). □

The above results on the inconsistency of the infinite sets, including the inconsistency of the continuum and of densely ordered sets, will change everything. So deconstructing the arguments that follow here and in the subsequent articles of this series of articles, will involve proving the falsity of Theorem 5 (and the falsity of each of the more than 40 independent proofs included in [212]).

Let us now consider the following:

Definition 13 (of Discrete Sets) *A set is discrete if it has a first element, a last element and each of its elements (except the first one) has an immediate predecessor and (except the last one) an immediate successor.*

And then, let finally prove the following

Theorem 9 (of Discrete Sets) *All discrete sets are finite.*

Proof: Let A be any discrete set:

$$A = \{a, s_1(a), s_2(a), s_3(a) \dots \ s_v(a)\} \tag{13}$$

where $s_1(a)$ is the immediate successor of a; $s_2(a)$ the immediate successor of $s_1(a)$; $s_3(a)$ the immediate successor of $s_2(a)$; and so on. If an element $s_n(a)$ has a finite number n of predecessors, then its immediate successor $s_{n+1}(a)$ has also a finite number $n+1$ of predecessors: all n predecessors of $s_n(a)$ plus $s_n(a)$. And since the element $s_1(a)$ has a finite number of predecessors, just 1 predecessor, the element a, we can inductively conclude that all elements of A, including its last element, have a finite number of predecessors. Therefore, A has a finite number of elements. □

Theorem 10 (of the Strictly Ordered Sets) *Every strictly ordered set is discrete.*

Proof: Let a be any element of any strictly ordered set A, and suppose A has not a last element. Since a is not the last element of A, there exist successors of a in A. Let us consider one such successors and denote it by a_1. For the same reasons as in the case of a, we can consider and denote by a_2 any successor of a_1 in A. For the same reasons as in the case of a_1, we can consider and denote by a_3 any successor of a_2 in A. For the same reasons as in the case of a_2, we can consider and denote by a_4 any successor of a_3 in A. We would thus have a sequence of successors of a: a_1, a_2, a_3, $a_4 \ldots$ in which there is not a last element. The bijection f between A and the ω-ordered set $\mathbb{N}$ defined by $f(a_i) = i$ proves that A, like $\mathbb{N}$, would be infinite, and therefore inconsistent (Corollary 4). Consequently, A has a last element. Exactly the same argument now referring to the predecessors of a, proves also that A has a first element. Let a now be any element of A other than the last element of A. Suppose that a has not an immediate successor. Let a_1 be any successor of a. Since a_1 is not the immediate successor of a there will exist another successor a_2 of a between a and a_1. Since a_2 is not the immediate successor of a there will exist another successor a_3 of a between a and a_2. The same argument above shows that the sequence of successors a_1, a_2, $a_3 \ldots$ of a is inconsistent. Therefore a has an immediate successor. The same argument now referring to any element b different from the first element of A proves that b has an immediate predecessor. Consequently, A is discrete (Definition 13). □

Theorem 11 (of Discrete Sets) *Every set is either discrete or discretely orderable.*

Proof: Let A be any set. If it is strictly ordered, it is a discrete set (Theorem 10). If it is unordered and consistent, it will have a finite number n of elements. By a bijection f, each of its elements can be paired with a different natural number of the set $\mathbb{N}_n$ of the first n natural numbers in their natural order of precedence. The set A^* defined by f^{-1}:

$$A^* = \{f^{-1}(1), f^{-1}(2) \ldots f^{-1}(n)\} \tag{14}$$

is an ordered version of A, and therefore a discrete version of A (Theorem 10). □

As noted above, more than forty other different and independent arguments included in [212] reach the same conclusion about the inconsistency of the Hypothesis of the Actual Infinity subsumed in the Axiom of Infinity. This infinity is what Aristotle would surely call infinite by addition. In Chapter 16, it will be proved the inconsistency of the other Aristotelian infinitude: the infinite by division, which was the type of infinite involved in the formalized version of Zeno's Dichotomies I and II [52, 53, 358, 359, 312, 165, 363, 71, 236, 139, 140, 374, 141, 143, 142, 238, 237, 226, 227, 269, 6, 288, 312, 165, 323].

Physical models and theories work reasonably well (even very well) until the infinities appear. But physicists do not usually concern themselves with the formal consistency of the infinitist mathematics that they use in all their models and theories. Nor do they concern themselves with another problem essential to a consistent explanation of the physical world: the problem of the infinite regress (of proofs, definitions, and causes). As will be seen throughout this series of articles, it is possible to modify the infinitist models and theories used in physics by finitist and discrete versions in such a way that they remain compatible with all the accumulated empirical knowledge about the physical world. And, at the same time, they are much simpler, more physical and less extravagant than their infinitist counterparts.

3.7 Conclusion

If any one of the more than forty proofs of the inconsistent nature of the actual infinity given in [212] is right, then the Hypothesis of the Actual Infinity is inconsistent. One of those arguments has been reproduced here so that the reader can directly evaluate the possibility that, in fact, the Hypothesis of the Actual Infinity, and then the Axiom of Infinity were inconsistent. If so, we might draw our first two cosmological conclusions:

Corollary 8 (of the eternal universe) *A consistent universe cannot be eternal.*

Proof: In an eternal universe time would be infinite, with an actual infinite number of, for instance, seconds. Therefore, an eternal universe would contain inconsistent infinite sets of time units (Corollary 4), and then it would be inconsistent. □

Corollary 9 (of the Finite Number of Universes) *In a consistent reality only a finite number of universes could exist.*

Proof: It is an immediate consequence of Corollary 4. □

Corollary 10 (of the Finite Number of Cycles) *In a coherent reality there can only be a finite number of cycles of creation-destruction of universes.*

Proof: It is an immediate consequence of Corollary 4. □

Corollary 11 (of the Finite Universe) *A consistent universe cannot contains an actual infinite number of physical objects.*

Proof: It is an immediate consequence of Corollary 4. □

According to the Standard Model there exists a finite number of different elementary particles (six quarks, six leptons and five bosons), each with a different finite mass. Therefore, the following is also true:

Corollary 12 (of the Finite Mass-Energy) *The mass and the energy of the observable universe cannot be actually infinite.*

Proof: It is an immediate consequence of the Standard Model, Theorem 11 and the mass-energy relation. □

Theorem 12 (of the Finite Universe) *A consistent universe cannot contains an actual infinite number of physical objects.*

Proof: It is an immediate consequence of Corollary 4. □

According to the Standard Model there exists a finite number of different elementary particles (six quarks, six leptons and five bosons), each with a different finite mass. Therefore, the following is also true:

Corollary 13 (of the Finite Mass-Energy) *The mass and the energy of the observable universe cannot be actually infinite.*

Proof: It is an immediate consequence of the Standard Model, Theorem 12 and the mass-energy relation. □

4. Discrete Magnitudes and Functions

4.1 Introduction

One of the most ubiquitous and frequent concepts in the primary and secondary literature of the physical sciences is undoubtedly the concept of the spacetime continuum, considered as a 4-dimensional continuum of inextensive points and instants of zero duration. The theories of relativity, for example, are theories of the spacetime continuum. But even quantum mechanics, whose quantum surname alludes to the discrete, has been developed with the same infinitist mathematics of the spacetime continuum. The consideration of discrete spaces is still very much in the minority in contemporary physics, and that of discrete time is practically non-existent.

In this chapter, and after recalling the first (pre-Socratic) argument in favor of the discrete and the first modern consideration of energy as a discrete magnitude, we denounce a truly scandalous situation in the physics of our days that, as far as I know, no one has pointed out: the existence of a large number of mathematical functions whose outputs should be discrete values that are impossible because they involve continuous variables, such as space and time. And it is not a question of accuracy or approximation, but of representation of a discrete reality by means of an indiscrete language. If physicists forced themselves to express in a discrete language the discreteness of matter, electromagnetic energy and electricity, they would surely end up discovering the inconsistency of their infinitist mathematical language.

4.2 Democritus' argument

Many of the problems that arise today in physics were already raised in pre-Socratic Greece. Among them are those related to the finite or infinite divisibility of things. One such argument is that of Democritus concerning the divisibility of matter. Collected and recalled by Aristotle [18, A2, 316a], and slightly modified, Democritus' argument is as follows:

> Let us suppose that matter can be divided to infinity, and let us imagine that we do indeed successively divide a piece of matter as long as it is possible to do so. Could there remain extensive particles of that matter? The answer is no, because otherwise we would not have chopped up those remaining large pieces of matter. Therefore, we must continue chopping up those remaining large pieces of matter. But then we would get inextensive particles of that matter. And with inextensive particles of matter, it is impossible to reconstruct an extensive piece of matter. Therefore, we cannot think that matter is made of points without extension. Therefore, there must be minimally extended particles of matter that cannot be divided: the atoms.

Until the first years of the twentieth century some renowned scientists continued to deny the existence of atoms, such as the physicist E. Mach and the mathematician G. Cantor. From the latter are the following words [57, p. 78]:

> I cannot regards them [the atoms] as existent either in concept or in reality no matter how many useful thingshave up to a certain limit been accomplished by means of this fiction.

But, as is well known, the atomic theory ended up being universally accepted around that same time. Today no one doubts that, for example, the smallest possible amount of iron is exactly one atom of iron. This does not imply that this iron atom can be broken into the subatomic particles that form it, but these particles are no longer iron particles.

4.3 The ultraviolet catastrophe

In 1900 M. Planck published a paper [276] that can be considered as the first major step towards a new science in which the discrete takes center stage: quantum mechanics. Some 27 years later it was already the most successful science in the mathematical description of physical phenomena at the scale of the very small, specially at the atomic and subatomic scale. Another thing is the physical interpretation of its mathematical formalism, still to be solved and still with several alternative interpretations. In my opinion, it is important to highlight a fact that is almost never emphasized: the mathematical formalism of quantum mechanics is the formalism of the infinitist mathematics of the actual infinity. Or in other words: the science of the discontinuous built with the mathematics of the continuous.

Going back to the origins, to Planck's pioneering work of 1900, it is worth remembering that a very common method of solving physical problems by means of infinitist mathematics (differential and integral calculus, for instance) consists in trying first a discrete solution in order to make discreteness tends to zero and find there (in the continuum scenario) the correct solution. This was the method M. Planck was using to solve the so called ultraviolet catastrophe, an apparently unsolvable problems in those days, at the beginning of the XX century, just in 1900. Surprisingly enough, the correct solution appeared much more before discreteness vanishes in the infinitist scenario of the continuum. What we now call Planck constant gave the correct solution at the particular value of 6.626068×10^{-34} m^2 Kg s^{-1}. The key to the solution was to consider, as Planck did, rather as an artifice of calculation, that the electromagnetic energy was discrete, with indivisible minima (quanta), so that the electromagnetic energy E could be expressed by the simple equation:

$$E = h\nu \tag{1}$$

where h is Plack's constant and ν is the frequency of the electromagnetic radiation. Although Planck's discrete solution to the ultraviolet catastrophe was initially taken as provisional, it immediately led to the birth of quantum mechanics, the most successful science ever developed by man (as is often said about this discipline).

But as we have just indicated, quantum mechanics, the science of discreteness par excellence, the science where the indivisible and extensible minima play a fundamental role, is also made of infinitist mathematics: the mathematics of the continuum, where spacetime plays a capital role and where the indivisible and extensible minima of space and time are meaningless. This incompatibility is surely the cause of another apparently unsolvable problem: the incompatibility between quantum mechanics and the general theory of relativity. In S. Majid words [228, p 73]:

> The continuum assumption on space and time seems then to be the root of our problems in quantum gravity.

But physicists never question the formal consistency of the actual infinity, as if that consistence were a proved fact. Evidently that is not the case, otherwise the Axiom of Infinity would be unnecessary. The Hypothesis of the Actual Infinity, the belief that the infinite sets exist as complete totalities (Definition 3), is just a hypothesis. Brouwer, Poincaré or Wittgenstein, among others, rejected it.

4.4 Discrete magnitudes and functions

Although the concept of discrete magnitude needs no presentation, it is useful to give a formal definition of it in order to be able to use it in the discussions that follow in this and other chapters of the book:

Definition 14 (of Discrete Magnitudes) *A magnitude is discrete if any of its values is an integer multiple of an indivisible and invariant minimum.*

That in the year 2024 physics has not yet discovered preinertia, the most universal property of all physical objects, including photons, is described in Chapter 19 as a shame for contemporary physics. The reasons justifying that label is given there. But preinertia is not the only shameful issue for contemporary physics. Here we will see another one, and in this case the reader will be able to check it in any physics text available to him, or even on the Internet.

We already know that in physics there are discrete magnitudes with indivisible minima, for example the electromagnetic energy or the electric charge. Let M_d be one of these discrete magnitudes, and suppose that it is defined by a continuous function f of three variables x_1, x_2 and x_3, one of which, for example x_3, is a continuous variable such as time:

$$M_d = f(x_1, x_2, x_3) \tag{2}$$

Since f includes a continuous variable, the variable x_3, its output cannot be discrete but continuous, when it should be discrete because M_d is a discrete magnitude. For the output of f to always be a discrete output, f cannot contain continuous variables. Yet physics is full of continuous functions with continuous variables that should give discontinuous, discrete, outputs: integer multiples of indivisible minima. To give a very simple example:

$$q = 4\pi\epsilon_o Ur/Q \tag{3}$$

where q and Q are electric charges, π an irrational numerical constant, ϵ_o a physical constant (dielectric constant), U the electric potential energy, and r a continuous variable (distance). The output of equation 3 should be, but cannot be, discrete because of the assumed continuous nature of r (continuum spacetime).

There is in modern physics a multitude of cases similar to the above: functions whose output corresponds to a discrete magnitude involving continuous variables, such as space and time, which make the necessary discrete outputs impossible. And in none of these cases do physicists consider the incompatibility of the corresponding discrete outputs of the functions with the continuous nature of some of their variables. The problem is not a quantitative one, related to the accuracy of the results. It is a qualitative problem of representation; continuous functions are not the appropriate expression of physical phenomena that produce discrete magnitudes. To consider that they are, or that they are valid approximations, distorts the physical nature of the relationships between the physical magnitudes involved in physical phenomena. Or put in other words, it allows us to go and stay in the wrong direction in understanding the physical world.

The embarrassment noted above is the fact that for more than a century thousands of physicists have written tens of thousands (or hundreds of thousands) of functions of the type (2) and (3) just denounced here, without in any case having considered the inconsistencies between the continuous inputs of their variables and the necessary discrete outputs of their results. As if these inconsistencies were not of the slightest importance. But it is immediately clear that they are: confronting these inconsistencies would force us to look for new, more realistic ways of expressing the relationships between physical magnitudes, relationships that would surely reveal new aspects of the represented physical reality. It is this new search that is blocked by the mathematical routine, which comes to have the force of an intolerant religious creed. An inevitable consequence of the Pythagorean-Platonic extremism anchored in human science for more than twenty centuries.

Modern physics assumes the discrete nature of a certain number of essential physical magnitudes, such as electromagnetic energy, electric charge, or angular momentum of electrons, of atomic nuclei and atoms, each of them with one indivisible minimum unit (which generically could be called quantum). At the same time, and together with these discrete magnitudes, others are used which, like space and time, are considered continuous and densely ordered: between each two of their values there is another different value, which makes the existence of minimum indivisible units impossible in these continuous magnitudes. A significant problem arises here which, as far as I know, no one has considered.

Although the inevitable errors in these impossible mathematical relationships are negligible, there is a fundamental theoretical error when it is considered that these mathematical relationships between physical variables, such as the one given by the equation (3), describe in formal terms the reality of the physical world. In short, it is impossible to define in formal terms a discrete magnitude by means of indiscrete (continuous) variables. It is then immediate to prove the following results:

Theorem 13 (of the Discrete Physical Laws) *If the mathematical output of a physical law is a discrete magnitude, its definition cannot contain continuous variables.*

Proof: If the mathematical definition of a physical law contains any continuous variable, its output (result) will always be continuous, not discrete as required for a discrete magnitude. □

As is well known, ordinary matter is made up of atoms (118 in total, 92 of which are found in known nature), all of which are made up of the same three subatomic particles: electrons, protons and neutrons:

a) Electron (elementary particle):
   1.- Mass: $9.109383701 \times 10^{-31}$ kg.
   2.- Electric charge: $e^{-} = -1.602176634 \times 10^{-19}$ C.
b) Proton (2 quarks u y 1 quark d):
   1.- Mass: $1.672621923 \times 10^{-27}$ kg.
   2.- Electric charge: $p^{+} = +1.602176634 \times 10^{-19}$ C.
c) Neutron (2 quarks d y 2 quark u):
   1.- Mass: $1,674927498 \times 10^{-27}$ kg.
   2.- Electric charge: 0 C.

where C is the unit of electric charge of the International System of Units (the charge that for one second passes through the cross section of a conductor when the electric current is one ampere). So, each subatomic particle has a certain mass and a certain electric charge, both invariant and well defined. All electric charges are integer multiples of the same elementary charge: of the electron if it is negative, or of the proton if it is positive. Therefore the electric charge is a discrete quantity. (Definition 14).

The same is true for atoms: they all have an *electronic mass* (the sum of the masses of all its electrons), a *protonic mass* (the sum of the masses of all its protons) and a *neutronic mass* (the sum of the masses of all its neutrons), which in all three cases are discrete magnitudes (Definition 14). The same is true for molecules and macroscopic objects. There are in addition atomic, molecular and macroscopic masses, which are also discrete magnitudes (after correction for the effects of electromagnetic and nuclear strong and weak interactions). Since there is a mathematical relationship between the magnetic force ($\vec{F}$), the magnetic field ($\vec{B}$), the velocity ($\vec{v}$) and the electric charge q:

$$\vec{F} = q(\vec{v} \times \vec{B}) \tag{4}$$

all these four magnitudes must be discrete (Theorem 13). In short, electric charge, electromagnetic energy, electronic mass, protonic mass, neutronic mass, atomic, molecular and macroscopic masss, magnetic force, magnetic field and velocity, angular momentum of electrons, of atomic nuclei and of atoms, are discrete magnitudes. As it could not be otherwise, there is an enormous empirical evidence for the existence of discrete magnitudes, as in the cases just given. And taking into account all physical magnitudes that are mathematically directly or indirectly related to them should be as well discrete, it is inevitable to consider the following important formal consequence:

Theorem 14 (of Discrete Magnitudes) *All physical magnitudes are discrete, except those that are not mathematically related to discrete magnitudes, in which case they could be continuous.*

Proof: There are discrete magnitudes, such as electric charge or electromagnetic energy. Therefore, all physical magnitudes that are mathematically related to them must also be discrete (Theorem 13 of the Discrete Physical Laws). And the same is true for all physical quantities that are mathematically related to the latter. And the same is true for all physical quantities that are mathematically related to the latter. And so on. □

Comment: taking into account the discrete magnitudes already pointed out and those which must be discrete according to Theorem 14, one could state that all physical magnitudes are discrete. A conclusion that agrees with the finite results demonstrated in Chapter 3. But, as we will see below, other physico-mathematical problems remain to be solved.

Protons are formed by two *up quarks* and one *down quark*, while neutrons are formed by one *up quark* and two *down quarks*. Although in nature quarks always appear forming protons and neutrons (hadrons), we can speculate about their electric charges for the sole purpose of highlighting the problems that immediately follow from infinitist arithmetic. If we represent the electric charge of an up quark by q_u and that of a down quark by q_d, we can write:

$$\left.\begin{array}{rl} \text{neutron } n^o: & 2q_d + 1q_u = 0 \\ \text{proton } p^+: & 2q_u + 1q_d = e^+ \end{array}\right\} \tag{5}$$

System of equations whose immediate solution leads to the corresponding electric charges of up quark and down quark:

$$q_d = -\frac{1}{3}\, e^+ \tag{6}$$

$$q_u = \frac{2}{3}\, e^+ \tag{7}$$

But it happens that $1/3$ and $2/3$ are rational numbers with an infinite number of ω-ordered decimal places:

$$\frac{1}{3} = 0.33333\ldots \qquad \frac{2}{3} = 0.66666\ldots \tag{8}$$

and if one considers as a complete totality the infinite numbers of each of them, which is what it corresponds to do according to the Axiom of Infinity, then the consistency of those numbers is bound up with the consistency of that axiom. And we have already seen in this book that this axiom is inconsistent (Theorem 6 of the Inconsistent Axiom of Infinity, page 21). One would then have to decide and justify the number of digits to take in the above definitions of q_u and q_d. This and many other problems related to the existence of different quanta of different physical magnitudes will have to be posed and solved in the framework of a new finitist and discrete arithmetic. The last section of this chapter deals, albeit very briefly, with this possible new arithmetic.

4.5 Some discrete conclusions

The above Theoem 14of the Discrete Magnitudes is anything but irrelevant. Among many others, the following fundamental results are immediately drawn from it:

Theorem 15 (of Discrete Space and Time) *Space and time magnitudes are discrete, each with an indivisible and invariant minimum.*

Proof: Since space and time magnitudes, as distances or durations, are involved in the definition of discrete physical magnitudes, according to Theorem 14 they must be discrete entities, each with an indivisible and invariant minimum (Definition 14), otherwise the defined discrete magnitudes could not be discrete. □

In this book, and from now on, the minimum, indivisible and invariant unit of space will be called qusit (quantum space unit) and the minimum, indivisible and invariant unit of time will be called qutit (quantum time unit).

Theorem 16 (of Discrete the Threshold) *The laws of physics do not apply in spaces smaller than the minimum unit of space nor in times smaller than the minimum unit of time, both being of non-zero extension (duration).*

Proof: If the laws of physics could be applied to intervals of space smaller than the minimum unit of space, then that minimum unit it would be neither invariant nor indivisible, which is impossible (Theorem 15). The same argument holds for time. □

Although Theorem 16 of the Discrete Threshold has not been explicitly stated in contemporary physics, its statement has broad theoretical and empirical support. As will be seen throughout the pages of this book, it is a fundamental result for the construction of discrete models of the universe.

Corollary 14 (of the Physical Laws) *The laws of physics apply to all regions of space and time, provided they are not less than their respective indivisible minimum units.*

Proof: It is an immediate consequence of Theorem 16. □

Theorem 17 (of Adjacency) *No space exists between any two successive space minimum units, and no time elapses between two successive time minimum units.*

Proof: Let AB and CD be two successive space minimum units (simplified to a one dimensional version) and assume they are not adjacent, i.e assume that $0 < BC$. BC must be less than the space minimum unit, otherwise AB and CD would not be two successive space minimum units. Consequently, AD would not be an integer multiple of the space minimum units, which is impossible according to Definition 14 and Theorem 15. □

Other important consequences of Theorem 14 of the Discrete Magnitudes will be deduced in the following chapter.

4.6 Discrete arithmetic

Although related to the modern spacetime continuum, the problem of the continuous has a Pythagorean origin [231]. In my opinion, its importance in the history of science has not been sufficiently appreciated. The firsts Pythagorean believe in the existence of indivisible geometrical points with an extension δ greater than zero, consequently they believed that all lengths would have to be commensurable: the ratio between any two of these lengths, say L_1 and L_2, would be a ratio between two natural numbers [231, pp. 11-16]:

$$L_1 = n_1\delta;\ L_2 = n_2\delta \tag{9}$$

$$\frac{L_1}{L_2} = \frac{n_1\delta}{n_2\delta} = \frac{n_1}{n_2} \tag{10}$$

Somewhat later, the Pythagorean discovered the existence of non commensurable lengths: the length of the diagonal L_d of a square with the length of its side. For example, if the length of the side is 9δ, we would have:

$$L_d = \sqrt{9^2\delta^2 + 9^2\delta^2} \tag{11}$$

$$= 9\delta\sqrt{2} \tag{12}$$

$$\frac{L_d}{L_s} = \frac{9\delta\sqrt{2}}{9\delta} = \sqrt{2} \tag{13}$$

Unfortunately, they did not consider the possibility of a discrete arithmetic, for instance:

$$L_d = \lfloor\sqrt{9^2\delta^2 + 9^2\delta^2}\rfloor \tag{14}$$

$$L_d = \delta\lfloor\sqrt{9^2 + 9^2}\rfloor \tag{15}$$

$$= 9\delta\lfloor\sqrt{2}\rfloor \tag{16}$$

$$= 9\delta \tag{17}$$

$$\frac{L_d}{L_s} = \frac{9\delta}{9\delta} = 1 \tag{18}$$

where ⌊mathematical expression⌋ stands for the integer part of the mathematical expression. As we will see in Chapter 16 equations (14)-(18) represent the discrete version of Pythagoras theorem.

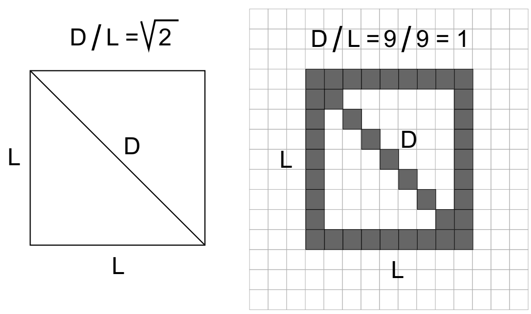


Figure 4.1 – Left: In continuous geometry the diagonal D and the side L of a square are not commensurable. Right: In discrete geometry the diagonal D and the side L of a square are commensurable.

On the other hand, although in the same direction of the discrete, the existence of different discrete magnitudes with their corresponding indivisible minima (quanta) raises new arithmetical problems that for the moment, and for the shameful reason given above, are completely ignored. But we will have to consider them and find the way to express those relations, and in the search for those relations it would not be strange that we would find new aspects or details of physical reality. If, for example, three discrete magnitudes M_1, M_2 and M_3, whose respective quanta are q_1, q_2 and q_3, are related in the form:

$$M_1 = M_2\, M_3 \tag{19}$$

It should be accomplished:

$$n_1 q_1 = n_2 q_2\, n_3 q_3;\ \ n_1, n_2, n_3 \in \mathbb{N} \tag{20}$$

$$q_1 = \frac{n_2 n_3}{n_1} q_2 q_3 \tag{21}$$

$$q_2 = \frac{n_1}{n_2 n_3} \frac{q_1}{q_3} \tag{22}$$

Consequently, the rational numbers $n_2 n_3 / n_1$ and $n_1 / n_2 n_3$ could only have a finite number of decimal places. This is just one example of the kind of problems that would have to be solved in a discrete arithmetic that correctly expresses the relationships between the different quanta of the different discrete physical magnitudes. In fact, and according to the inconsistency of the actual infinity, all physical magnitudes should be discrete if they are formally consistent. This type of discrete arithmetic is yet to be developed in formal and universal terms. The fact that its necessity has not even been raised is surely related to our perception of the physical world as essentially continuous, not discontinuous.

Indeed, it seems reasonable to assume that we model reality as a continuous system because we perceive it as a continuous system. The problem is that this perceived continuity is illusory. In fact, our brain takes a time greater than zero (≈13 ms [282]) to process each visual image (the base of the well known α, β, γ and δ movements, and of ϕ-phenomenon [102]), so that *a continuum* of visual images is physiologically impossible. The same illusory perception happens with motion when observed in a film. And in the same way a film is a discontinuous sequence of photograms, natural motion could also be a discontinuous sequence of changes in position, which is perceived as continuous by our brains and our physical instruments.

5. Infinite regress

5.1 Introduction

This chapter introduces an inductive principle (the Principle of Directional Evolution) from which several formal results about the consistent nature of the observable universe are proved, including the Theorem of the Consistent Universe, the Theorem of Formal Dependence and the Theorem of the First Element, all of which are fundamental to our own analysis of physical space to be developed from the next Chapter onward. The most important result demonstrated in this chapter is undoubtedly the existence of FIRST CAUSES that cannot be explained in terms of other causes deduced from our knowledge of the observable universe. Or in other words, the existence of first causes in all physical phenomena; causes that cannot be explained in terms of physical phenomena; causes that cannot be explained by human understanding. A logical result that goes far beyond the content of this book.

5.2 The Aristotelian infinite regress

The Aristotelian infinite regress of arguments says that since statements do not prove themselves, in order to prove a statement you will always need at least one other statement. And the same goes for that last statement(s). An inevitable indefinite regression of statements appears, an endless regress of statements that makes it impossible to complete the demonstration of the veracity of any initial statement. This is the reason why all sciences need basic statements whose truth is admitted without proof; these basic statements are the axioms, postulates and principles or fundamental laws. In some cases these primitive statements are obvious, in others they are arbitrary and not obvious at all, and in others they have more or less inductive confirmation.

The infinite regress of statements (arguments) extends to definitions and causes for the same reasons. Although rarely considered, the infinite regress of arguments, whether extended to definitions and causes or not, is a serious limitation of human knowledge, much more serious than Gödel's famous incompleteness theorems. But at the same time, and also considering the inconsistency of the actual infinity and infinite divisibility, it suggests the way we should go to explain the physical world, by clearly stating what can and cannot be explained in formal terms. In particular, it is proved here that every history either has an initial and unexplained instant or is inconsistent.

Although an infinite regress is any infinite sequence of elements that are recursively related (each element in the sequence is related in some, and always the same, way to its immediate predecessor in the sequence), we are interested here only in the particular cases of demonstrations, definitions, and causes. The case of demonstrations was treated by Aristotle [13, I.3], and it will be the case with which we begin our discussion, also making use of the famous Münchhausen Trilemma. The discussion will then be extended to definitions and causes.

Making use of the inconsistency of the actual infinity, the need for primitive concepts, axioms, fundamental laws, and inductive principles will be demonstrated. The Theorem of the First Element will also be proved. The common thread of all these demonstrations will be the Theorem 20 of Formal Dependence, directly deduced from the Principle 1 of the Directional Evolution. That theorem establishes that statements do not prove themselves; concepts do not define themselves; and objects and causes are not the cause of themselves. Obviously, the alternative would be an inevitable collection of nonsense incompatible with science.

As indicated above, the infinite regress of arguments was discovered more than twenty-two centuries ago. And for the reasons discussed below, it is a serious limitation of human knowledge. However, most contemporary scientists ignore it. For example, no science has established the list of its primitive concepts. The curious thing is that at the same time that modern science ignores these logically unavoidable limitations, it pays exaggerated attention to other, much less general limitations, almost always related to the contradictory self-reference [201].

In the particular case of physicists (experimental and theoretical), it is rare to see them concerned with these formal limitations, as if these limitations did not limit anything. This is not the best attitude, because until the problems posed by such limitations are resolved, physics cannot be built on adequate foundations. Experimental data drive and force the adjustments of physical theories, but (at least so far) do not determine all of their formal foundations.

Even more dramatic is the attitude of physics toward its most fundamental problem: the problem of change (physics is essentially the science of change). Raised by Parmenides and Zeno of Elea, no one has been able to explain how a simple change of position of a material object occurs. Physics has completely forgotten that it has this problem. Under these conditions, disagreements, even incompatibilities, between some physical theories are not strange.

5.3 The universe is consistent

The observable universe contains billions and billions of objects of the same type: galaxies, stars, planets, minerals, chemical elements... evenly distributed and of very different ages. This is only possible if the same things have always happened in the universe, and in such a way that the same consequences have always occurred under the same conditions. Something that has been suspected for at least a couple of centuries: the naturalists of the 19th century already assumed the Principle of Actualism-Uniformism, which suggests the same conclusion:

The laws of nature are the same in all places and times.

Even this general principle can be deduced in formal terms from another even more general principle. For, in effect, the observable universe has been producing the same type of objects throughout its history (over 13.8 billion years): there are planets, stars, galaxies etc. OF ALL AGES. And all these objects have been EVOLVING IN THE SAME WAY. In addition, there is a law of thermodynamics (though with numerous versions), the *Second Law* of thermodynamics, that also points in the same direction regarding the evolution of the heat-energy interconnections. All this, and taking into account that for most of its history the universe had no rational observers amply justifies proposing the following:

Principle 1 (of Directional Evolution) *The observable universe always evolves independently of its rational observers and in the same direction of increasing its global entropy.*

where entropy can be replaced with isotropy [202]. This primordial directionality has made possible other local directional evolutions that seem to go in the opposite direction: the creation and evolution of open systems that exchange matter and energy with their surroundings. In these systems there is a remarkable decrease in isotropy, but in return there is an even more remarkable increase in the isotropy of their surroundings, so that the final balance is a directional evolution of the universe in the sense of the above Principle of Directional Evolution. This is the case for crystalline minerals and for all self-organizing systems we call living beings [192, 193]. It could be said that the isotropic evolution of the universe produces strongly anisotropic residues, among which you, kind reader, find yourself (Figure 5.1).

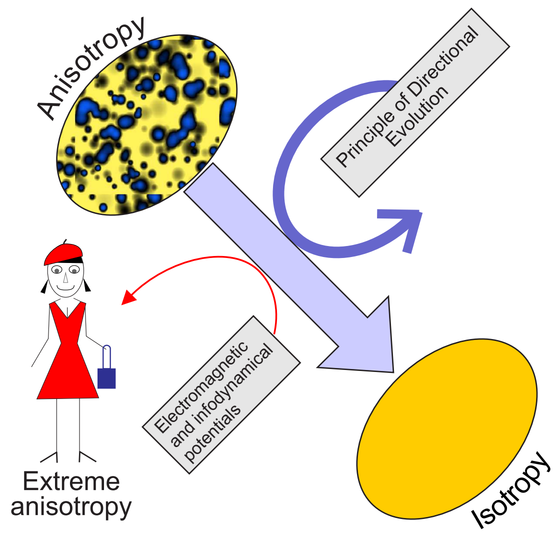


Figure 5.1 – Anisotropy as a residual product of the isotropic evolution of the universe.

As will be seen below, from the Principle of Directional Evolution we immediately deduce some fundamental results that detail the way in which the universe evolves. To begin with, the following definition is proposed:

Definition 15 (of the Consistent Set of Laws) *A set of physical laws is consistent if under the same conditions it always leads to the same results.*

It is now immediate to prove the following:

Theorem 18 (of the Consistent Universe) *The universe evolves under the control of a unique set of invariant and consistent physical laws.*

Proof: If the physical laws governing the evolution of the universe were not an invariable set of consistent laws, changes would occur with equal frequency in all directions, and no progress would be possible in any of them. Thus, directional evolution would not be possible, which violates the Principle 1 of Directional Evolution. Thus, the universe evolves under the control of a unique set of invariant and consistent physical laws. □

Note: The Theorem of the Consistent Universe could have been chosen as an inductive principle, and from it the Principle of Directional Evolution could be deduced as a theorem. There is overwhelming empirical evidence for both, and in fact their statements are mutually reinforcing inductively and formally. However, the alternative of the Consistent Universe Principle would have to be extended to the past (actually it would be the geological Principle of Actualism-Uniformitarianism) and for that we no longer have the same empirical evidence. On the contrary, we have been able to confirm on many occasions the existence in the observable (and observed) universe of the same objects with different ages, as well as very complex objects, such as the possible reader of this text, whose formation requires millions of years of directional evolution.

Corollary 15 (of Physical Laws) *The laws of physics apply to all regions of space and time.*

Proof: It is an immediate consequence of Theorem 18. □

Theorem 19 (of Identicality) *All particles of the same type have the same properties and behave the same way under the same conditions.*

Proof: It is an immediate consequence of Definition 15, Theorem 18 and Corollary 15. □

Comment: Evidently, this theorem goes against the Principle of the Identity of Indiscernibles, of G.W. Leibniz. Real identical objects exist and are distinguishable from each other because they do not occupy the same places in space, which requires (contrary to Leibniz's view) a real physical space.

Theorem 20 (of Formal Dependence) *No concept defines itself; no statement proves itself; no physical object is the cause of itself; and no cause is the cause of itself.*

Proof: If propositions proved themselves, then any proposition could prove itself, and consistent sets of laws would be impossible, which goes against Theorem 18. If concepts defined themselves, their meanings would be inaccessible to human knowledge, and they could not be used to establish the natural laws that we can only establish with those concepts, which also goes against Theorem 18. If physical objects, processes or causes were the cause of themselves, then anything could exist and anything could happen, so that the directional evolution of the universe would be impossible,which goes against the Principle of Directional Evolution. □

Then it is clear that in our observable universe, under the same conditions, the same results will occur. Another thing is that these results may seem strange or paradoxical to us. In that case, the corresponding laws will always produce the same strangeness or perplexity, as is the case with some aspects of quantum mechanics. Moreover, if the universe worked in a way similar to cellular automata, perhaps there could be two types of laws: the basic laws of the automaton (established before the automaton is started) and the laws that emerge from its evolution, laws that drive the relationships between the also emerging objects of the automaton. (see Chapter 18)

So far, we have not known any exception to the Principle of Directional Evolution of the Universe. Henceforth, and for the sake of simplicity, the directional evolution of the observable universe will simply be referred to as the evolution of the universe. This is the reason why the universe can be described by formal and computational languages. Whether the current infinitist mathematical language of physics is the most appropriate is another matter. And certainly it is not because it is based on an inconsistent axiom (the Axiom of Infinity). Unnecessary as it may seem, the consistent evolution of the universe is of interest beyond evolution itself. For example, it makes a Principle of Discrete Relativity unnecessary:

Theorem 21 (of the Reference Frames) *The laws of physics are the same in all discrete reference frames.*

Proof: Let RF_1 and RF_2 be any two discrete reference frames in relative motion with respect to each other. Their relative velocity will be a consequence of their different absolute velocities with respect to the absolute reference frame of the discrete space and time. Therefore, the observations made with respect to RF_1 and RF_2 can be modified in both cases by considering the absolute velocity of each system with respect to the absolute reference frame of the discrete space and time. By doing so, the same laws will be obtained in both systems, otherwise the Theorem 18 of the Consistent Universe Theorem would not be verified, and therefore neither would the Principle of Directional Evolution. □

Comment The problem is that for now, and due to preinertia (see Chapter 19), it is not possible to observe the absolute velocity of a discrete reference frame. Unless it is possible to do so by referring its motion to the isotropic reference frame of the Cosmic Microwave Background.

In reality, and as incredible as it may seem, the Theorem of the Consistent Universe is absolutely necessary. To understand this necessity, it is enough to remember a very famous phrase of A. Einstein [99, p. 315]:

> The eternal incomprehensible of the world is its comprehensibility

Popularized as:

> The fact that the universe is understandable is a miracle.

But, for the reasons given above, a universe that produces scientists (through a process that here on Earth has lasted several billion years) can only be a universe that evolves in a directional way and, consequently, under the control of a single set of invariant and formally consistent laws. That is, a universe understandable in terms of constant and consistent regularities. The miracle would be, then, that a universe with scientists were not understandable. Another thing is that the mathematical language used to explain the universe is appropriate. In this book I am showing that it is not. This inappropriateness would explain why it is so difficult to understand what should not be so difficult to understand.

5.4 Münchhausen Theorem

Also known as Agrippan Trilemma, the well-known Münchhausen Trilemma is an argument that tries to demonstrate the impossibility of proving the truth of any statement without making use of an arbitrary initial statement. According to this argument, a truth can only be proved by means of:

1. an infinite regress of proofs.
2. a first arbitrary statement.
3. a circular sequence of proofs.

the three of which are formally unsatisfactory. As we will see now, things do not improve with the Theorem 20 of Formal Dependence and the Theorem of the First Element 23. Arguably they get worse because they become a theorem whose consequences are the same as the Münchhausen Trilemma. Here, we will deal only with infinite regress of proofs, definitions, and causes, in relation to which it is immediate to demonstrate the following:

Theorem 22 (of Incompletable Regress) *Every recursive sequence S of proofs, definitions or causes in which there is a last element to be proved (defined, caused) and each element has an immediate predecessor that proves (defines or causes) it, is incompletable.*

Proof: If every element of the sequence has an immediate predecessor, then there is not a first element of the sequence, because this first element would have no immediate predecessor. Therefore, the sequence, if consistent, can only be potentially infinite and then incompletable (Corollary 6). □

In addition, the third option of the Münchhausen Trilemma would be inconsistent because there would be at least one element that proves (defines or is the cause) of itself: For example, if A is the cause of B, which is the cause of C which is the cause of A, the C is the cause of C, which goes against the Theorem of Formal Dependence 20. Therefore, and taking into account the above Theorem of the Incompletable Regress, only the second alternative of the Münchhausen Trilemma could be considered as part of the logical system for the explanation of the physical world. This is the subject of the next section.

5.5 Theorem of the First Element

As mentioned above, an infinite regress is a sequence of recursively related elements such that each element has the same type of relationship to its immediate predecessor. In our case, the elements of the sequence will always be formal elements: arguments, definitions, and causes. And their relations will always be the same formal relation: arguments in the infinite regress of arguments; definitions in the infinite regress of definitions; and causes in the infinite regress of causes.

We will now prove a general result that applies immediately to infinite regresses:

Theorem 23 (of the First Element) *A consistent sequence in which there is a last element and each element has an immediate predecessor is a complete totality only if it has a first arbitrary element without predecessors.*

Proof: Let $S = \ldots S_{3^*}, S_{2^*}, S_{1^*}$ be any sequence with a last element S_{1^*} and in which each element S_{n^*} has an immediate predecessor $S_{(n+1)^*}$, where n^* read last but $n-1$. If S is consistent it can only be finite or potentially infinite (Corollary 1). Therefore, if S is a complete totality it can only have a finite number n of elements (Definition 6). In these conditions, and taking into account that each element S_i of S has exactly one predecessor more than its immediate predecessor $S_(i+1)^*$, the element S_{1^*} has $n-1$, predecessors; the element S_{2^*} has $n-2$ predecessors; the element S_{3^*} has $n-3$ predecessors; etc. Consequently, the smallest number of predecessors that an element of S can have is $n-(n-1)=1$. That element will be $S_{(n-1)^*}$ whose predecessor can only be a first element S_{n^*} of the sequence that has no predecessor. So, S has a first element S_{n^*} with zero predecessors. □

5.6 Infinite regress of proofs

An immediate, and well known, consequence of Theorems 22 and 23 is that all formal sciences must be founded on a set of axioms, i.e. a set of statements whose veracity is assumed without proof (see next section for the role of definitions and primitive concepts in the foundations of formal sciences). Ideally, the number of axioms should be small and as self-evident as possible (otherwise, they would give rise to an excessively arbitrary and abstract science).

As is well known, classical Euclidean geometry was founded on five geometrical axioms, the fifth of which is the controversial axiom of parallels, whose statement is anything but self-evident. By contrast, Playfair, Hilbert and the author of this book founded their Euclidean geometries respectively on 30, 20 and 10 axioms (see Appendix C). As expected, the initial set of axioms will have important consequences on the resulting of science [196, 200, 279, 158]. For example, Euclidean and non-Euclidean geometries.

For its part, set theory is based on about ten axioms (depending on the version). One of them is the Axiom of Infinity. Assuming this axiom has enormous consequences, not only in mathematics but also, and above all, in physics, a science that pays little attention to the foundations of its mathematical language, and considers that this language is not an instrument for analyzing observations but a model to which observations must be adapted.

Since the beginning of the 20th century, physics has been built up with a type of mathematics (the infinitist mathematics of the space-time continuum) that assumes the existence of the complete list of the natural numbers in their natural order of precedence (which is a way of stating the Axiom of Infinity), even though there is no last natural number completing the list. They assume, as Aristotle would say, that the incompletable exists as complete.

The formal proofs of the inconsistency of theAxiom of Infinity has been available for more than twenty years, but it is taking too much effort to fight against the infinitist stream that assumes that axiom. An stream of thought absolutely dominant and hostile to dissidence. But in the end, well-constructed proofs will end up imposing their conclusions. The reader will be able to judge one of those proofs in Chapter 3 (Theorem 1) in this book, and forty others in [212, online link]. The consequences on physics will be enormous, and very positive. This book has a lot to do with that finitist and discrete future. Indeed, the inconsistency of the actual infinity will change all.

Experimental sciences have an additional element for their corresponding foundations: the inductive principles or laws. Statements whose veracity is accepted without formal proofs (such as axioms) but confirmed by experimentation and by observation of natural phenomena. This is the case, for example, of the Principle of Inertia, including preinertia. But here, too, it is important to be careful. In this sense, it is convenient to recall Russell's famous metaphor of the chicks [307, p. 31]): the innocent animals who lived happily on the farm in the care of their attentive farmer without suspecting the existence of fried chicken with potatoes. Fortunately, humans are, in general, smarter than chicks and we have discovered that it is wise to be cautious when drawing inductive conclusions about the physical world.

5.7 Infinite regress of definitions

Although the infinite regress has always been discussed for the case of arguments (demonstrations), its extension to other formal elements such as definitions and causes is immediate. And the consequences of Principle 20 and Theorem 23 on these new formal elements are also immediate.

Since concepts are not self-defining (Theorem 20), to define any concept it is necessary to use one or more different concepts; and the same goes for the latter. In this way a recursive sequence of definitions appears to which Theorem 23 can be applied, with the same consequences as in the case of axioms:

Corollary 16 (of Primitive Concepts) *Primitive concepts are inevitable in all sciences and languages.*

Proof: It is an immediate consequence of Theorems 20 and 23. □

Of course, the defined and undefined objects must be legitimized by the axioms or principles of the corresponding theories, or by formal proofs. This is usually the case in the formal sciences, but not always in the experimental sciences. Even in theoretical physics or theoretical biology, which are more formal and mathematical than their corresponding experimental branches. The problem is that when a science pays little attention to its fundamental basis, anything can happen.

The most basic concepts of science, such as set, number, point, force, mass, time, etc., are primitive concepts. Most of them are intuitive: We know what they are even though we have no formal definition of them. This is the case, for example, with set, number, mass, or force. In other cases we may have a false intuition. I think that's the case with point and instant. The intuition we have of point is confused with that of the mark on the paper or board that is trying to represent it. And the intuition of instant is confused with very short intervals of time. And neither is the case.

The concept of point is primitive and fundamental in physics: space would consist of points without extension, points that do not occupy places but define places; and there would be point masses, point charges, point particles, point trajectories, and so on. And as if that were not enough, there are as many points on a line one trillionth of a millimeter long as there are in the entire three-dimensional universe. Obviously, the same could be said of the concept of instant: the same number of them elapse in a microsecond as in the entire history of the universe. This is Cantorian Infinitism!

The effort to define objects, and to establish the axioms and proofs that justify them, may not always have been adequate. Of course, definitions must be formally productive: once legitimized by the axioms, they must be usable in subsequent proofs. A very notorious case is that of Euclidean geometry: it is possible to define a new foundational basis with 29 productive definitions and 10 axioms, in which it is possible to prove, like any other theorem, the statement of the Euclidean axiom of parallels [196, 200].

It is worth clarifying this issue further. The concepts of point, line, and straight line are primitive. So a straight line (a central concept in Euclidean geometry) is something for which we have no formal definition; a straight line belongs to a class of objects (lines) for which we also have no definition; and a line is made up of points for which we also have no definition. Perhaps too many *indefinitions*.

It is possible, however, to give a formal definition of a straight line (although the concepts of point and line remain primitive). It is a formally productive definition that, together with the rest of the new foundational elements of Euclidean geometry, allows us to demonstrate the Euclidean statement of parallels [196, 200, p. 40].

5.8 Infinite regress of causes

In accordance with Theorem 20 of Formal Dependence, physical (and formal) objects and natural phenomena are not self-causing, i.e. they cannot be the cause of themselves. In consequence Theorems 23 and 22 apply to them. We must, therefore, accept the following:

Corollary 17 (of the First Cause) *To explain any physical object or phenomenon, a first cause not explainable in terms of other causes deduced from our knowledge of the observable universe is necessary.*

Proof: It is an immediate consequence of Theorems 20 and 23. □

Taking into account the above results, and those demonstrated in the previous Chapter 4, the following results, which are anything but irrelevant, can be demonstrated.

Theorem 24 (of the Finite Universe) *The universe is finite in extent, age and number of components.*

Proof: The universe is consistent (Theorem 18 of the Consistent Universe). In consequence the number of its space discrete units (qusits) can only be finite. Therefore the spacial extension of the universe is finite. The same applies to the number of time discrete units (qutits) and to number of its physical components. Therefore, the age of the universe and the number of its physical components can only be finite. □

Theorem 25 (of the Origin of the Universe) *The universe must have had an origin whose cause is unknowable.*

Proof: The number of qutits of the observable universe is finite (Theorem 24), therefore, and taking into account that this number grows towards the future and decreases towards the past, there had to be a first qutit in the history of the universe. Therefore, the universe must have had an origin. Moreover, the universe as an object can only be explained by a first cause unknowable in terms of other causes deduced from our knowledge of the observable universe (Corollary 17 of the First Cause). □

Theorem 26 (of the Discrete Universe) *The universe has evolved through a finite and discrete sequence of qutits.*

Proof: The universe has evolved (Principle 1 of Directional Evolution) along a finite sequence of qutits in which there is a first qutit (Theorem 25 of the Origin of the Universe), a last qutit (the current qutit), and each qutit has an immediate predecessor, except the first one, and an immediate successor, except the last one. Therefore, that sequence is discrete (Definition 13) and finite (Theorem 9 of the Discrete Sets). □

As noted above, the consequences of the above theorems and corollaries on the universe are anything but irrelevant. In any case, do not forget that science should be free of prejudices, free even of religious and anti-religious prejudices. And that, on the other hand, without the necessary formal rigor, language cannot be scientific. Without the rigorous use of language, anything can be demonstrated. To affirm, for example, that the universe arose from a fluctuation of nothingness, implies that nothingness is not nothingness but something with the capacity to fluctuate universes. As T. Maudlin would surely say, the importance of the following conclusion of Corollary 17 of the First Cause cannot be exaggerated: The evolution of the universe, as such a natural process, must also have a first cause outside the evolution of the universe itself; a first cause that cannot be explained in terms of other causes deduced from our knowledge of the observable universe, i.e. a first cause that cannot be explained in physical or logical terms.

6. The Formal Scenario

6.1 Introduction

Although all the formal results obtained up to this chapter can be considered as elements of the formal setting for the discussions that follow in the following chapters, not all of them have the same relevance because some of them are intermediate results that are necessary in the proofs of the fundamental results that do need to be taken into account in those discussions. This short chapter is a simple listing of all the formal statements established or proved up to this chapter. Most of those results have to do with infinity and the geometry of space, and are dissenting with the dominant infinitist current in contemporary mathematics and physics. They are gathered here to facilitate their access to the reader.

6.2 Formal elements for a new theory of space and time

The 52 formal elements listed below (1 principle, 13 definitions, 25 theorems and 16 corollaries) were established and, where appropriate, proved in the previous three chapters. Their content can be summarized as follows:

> If one accepts the Principle of the Directional Evolution of the Universe, or put very informally: if one assumes that the gas released from a bottle of champagne when the bottle is uncorked will never spontaneously and naturally return to the uncorked bottle, then the universe is a consistent object that evolves according to a consistent set of laws. Furthermore, it had to have an origin whose cause cannot be established from knowledge drawn from within the universe itself. The age, extent, and number of its components, at any scale (from subatomic particles to galaxies), must be finite. None of its components, including space and time, can be divided into an infinite number of parts or units. Prac-

> tically all physical magnitudes have to be discrete, with indivisible minimal units. And the number of universes (if more than one exists), and the number of cycles of possible creations-destructions of the universe must also be finite. Consequently, the universe must be finite and discrete, and it must have had an origin whose cause is not cognizable by science constructed from within the universe itself. It could also be the case that the entire observable universe is a hoax, or a joke by something or someone, in which case there would be no point in continuing to do science.

It is a relate extracted from the following list of formal elements:

Definition 4 of Infinite Set: A set is said infinite if it can be put into a one to one correspondence with one of its proper subsets. (Page 12).

Definition 3 of Complete Totality: A complete totality is a set defined by comprehension in which every element that should be in the set, is in the set. (Page 11).

Definition 5 of Successors and Predecessors: In strictly ordered sets, all elements that, in the ordering of the set, follow (precede) a given element of the set, are its successors (predecessors). If between the given element and one of its successors (predecessors) there is no other element, then this successor (predecessor) is the immediate successor (predecessor) of the given element. (Page 12).

Definition 6 of Actual and Potential Infinity: An ordered collection of elements is infinite if there is no last (first) element that completes (initiates) it. The collection is actual infinite if it is considered a complete totality, and potential infinity if it is not considered a complete totality. (Page 15).

Axiom 1 of Infinity: There exists at least one infinite set. (Page 17).

Definition 7 of the Types of Sets: A set is finite if it has a definite and finite number of elements. A set of elements of a certain type is potentially infinite if it always contains a finite number of elements of that type and any finite numbers of new elements of that type can always be added to it, without the set ceasing to be finite and without it being necessary to change its name. (Page 18).

Definition 8 of the Types of Infinities: The actual infinity is the infinity of the infinite sets. The potential infinity is the infinity of the potentially infinite sets. (Page 18).

Definition 9 of Inconsistent Set: A set is inconsistent if a contradiction can be deduced from the number of its elements, or from the number of elements of at least one of its proper subsets. (Page 18).

Corollary 1 of Inconsistent Sets: A set with the same number of elements as an inconsistent set, is also inconsistent. (Page 18).

Definition 10 of Denumerable Set: A set is denumerable if its cardinal is the smallest infinite cardinal $\aleph_0$ of the infinite set of all natural numbers. An infinite set is non-denumerable if its cardinal is greater than the smallest infinite cardinal $\aleph_0$. (Page 18).

Definition 11 of ω-Ordered Sets: A set is ω-ordered if being denumerable, it has a first element, each element has an immediate successor and an immediate predecessor, except the first one which has no predecessor. (Page 18).

Theorem 6 of the Axiom of Infinity: The infinity subsumed in the Axiom of Infinity can only be the actual infinity. (Page 21).

Theorem 2 of Denumerable Sets: It is always possible to define a one-to-one correspondence between any two denumerable sets. (Page 18).

Theorem 3 of Non-Denumerable Sets: Every non-denumerable set has denumerable proper subsets. (Page 19).

Theorem 4 of Indexation: The elements of a denumerable set can be reordered with the same order as the elements of any other denumerable set. (Page 19).

Theorem 5 of the Denumerable Infinity: All denumerable sets are inconsistent. (Page 19).

Corollary 2 of ω-Ordered: All ω-ordered sets are inconsistent. (Page 21).

Corollary 4 of the Inconsistent Infinite Sets: All actual infinite sets are inconsistent. (Page 22).

Corollary 6 of the Inconsistent Axiom of Infinity: The axiom of infinity is inconsistent. (Page 21).

Theorem 3 of the Actual Infinity: The actual infinity is inconsistent. (Page 22).

Corollary 5 of Infinite Divisibility: The actual infinite divisibility of any formal or physical object is inconsistent. (Page 22).

Corollary 6 of Consistent Collections: A set can be either a finite complete totality or a potentially infinite and incompletable totality. Otherwise it is inconsistent. (Page 22).

Definition 12 of Densely Ordered: If no element of a strictly ordered set has an immediate predecessor nor an immediate successor, the set is said to be densely ordered or to define a continuum. (Page 22).

Theorem 7 of the Inconsistent Dense Order: Densely ordered sets are inconsistent. (Page 22).

Corollary 7 of the inconsistency of Q and R: When considered as complete infinite totalities, the set $\mathbb{Q}$ of the rational numbers and the set $\mathbb{R}$ of the real numbers are both inconsistent. (Page 23).

Theorem 8 of the Inconsistent Continuum: The spacetime continuum is inconsistent. (Page 23).

Definition 13 of Discrete Sets: A set is discrete if it has a first element, a last element and each of its elements (except the first one) has an immediate predecessor and (except the last one) an immediate successor. (Page 23).

Theorem 9 of the Discrete Sets: All discrete sets are finite. (Page 23).

Theorem 10 of the Strictly Ordered Sets: Every strictly ordered set is discrete. (Page 23).

Theorem 11 of Discrete Sets: Every set is either discrete or discretely orderable. (Page 24).

Corollary 8 of the Eternal Universe: A consistent universe cannot be eternal. (Page 25).

Corollary 9 of the Finite Number of Universes: In a consistent reality only a finite number of universes could exist. (Page 25).

Corollary 10 of the Finite Number of Cycles: In a consistent reality there can only be a finite number of cycles of creation-destruction of universes. (Page 25).

Corollary 12 of the Finite Universe: A consistent universe cannot contains an actual infinite number of physical objects. (Page 26).

Corollary 13 of the Finite Mass-Energy: The mass and the energy of the observable universe cannot be actually infinite. (Page 26).

Definition 14 of Discrete Magnitudes: A magnitude is discrete if any of its values is an integer multiple of an indivisible and invariant minimum. (Page 30).

Theorem 13 of the Discrete Physical Laws: If the mathematical output of a physical law is a discrete magnitude, its definition cannot contain continuous variables. (Page 31).

Theorem 14 of the Discrete Magnitudes: All physical magnitudes are discrete, except those that are not mathematically related to discrete magnitudes, in which case they could be continuous. (Page 33).

Theorem 15 of the Discrete Space and Time: Space and time magnitudes are discrete, each with an indivisible and invariant minimum. (Page 34).

Theorem 16 of Discrete the Threshold: The laws of physics do not apply in spaces smaller than the minimum unit of space nor in times smaller than the minimum unit of time, both being of non-zero extension (duration). (Page 34).

Corollary 14 of the Physical Laws: The laws of physics apply to all regions of space and time, provided they are not less than their respective indivisible minimum units. (Page 35).

Theorem 17 of Adjacency: No space exists between any two successive space minimum units, and no time elapses between two successive time minimum units. (Page 35).

Principle 1 of Directional Evolution: The observable universe always evolves independently of its rational observers and in the same direction of increasing its global entropy. (Page 41).

Definition 15 of the Consistent Set of Laws: A set of physical laws is consistent if under the same conditions it always leads to the same results. (Page 42).

Theorem 18 of the Consistent Universe: The universe evolves under the control of a unique set of invariant and consistent physical laws. (Page 42).

Corollary 15 of the Physical Laws: The laws of physics apply to all regions of space and time. (Page 43).

Theorem 19 of Identicality: All particles of the same type have the same properties and behave the same way under the same conditions. (Page 43).

Theorem 20 of Formal Dependence: No concept defines itself; no statement proves itself; no physical object is the cause of itself; and no cause is the cause of itself. (Page 43).

Theorem 21 of the Reference Frames: The laws of physics are the same in all discrete reference frames. (Page 44).

Theorem 22 of Incompletable Regress: Every recursive sequence S of proofs, definitions or causes in which there is a last element to be proved (defined, caused) and each element has an immediate predecessor that proves (defines or causes) it, is incompletable. (Page 45).

Theorem 23 of the First Element: A consistent sequence in which there is a last element and each element has an immediate predecessor is a complete totality only if it has a first arbitrary element without predecessors. (Page 46).

Corollary 16 of Primitive Concepts: Primitive concepts are inevitable in all sciences and languages. (Page 48).

Corollary 17 of the First Cause: To explain any physical object or phenomenon, a first cause not explainable in terms of other causes deduced from our knowledge of the observable universe is necessary.

(Page 49).

Theorem 24 of the Finite Universe: The universe is finite in extent, age and number of components. (Page 49).

Theorem 25 of the Origin of the Universe: The universe must have had an origin an origin whose cause is unknowable. (Page 49).

Theorem 26 of the Discrete Universe: The universe has evolved through a finite and discrete sequence of qutits. (Page 50).

7. Space in ancient Greece

7.1 Introduction

Prior to early Greece, usually referred to as pre-Socratic Greece, there is little documentation on the use of space-related concepts. In the so-called river Mesopotamian cultures ([308, 33, 317, 266, 329]) units of measurement related to length, extent and volume were introduced, all linked to the interest of their practical use. The only abstraction here would be a practical abstraction, i.e. the conventional use of those practical units of measurement [176]. We have no documented evidence that a philosophical interest in the notion of space existed in these cultures.

The philosophical interest in the concept of space, as in almost all concepts, appears in the early Greece of the first pre-Socratics, especially in the first cosmologists, the Pythagoreans and the atomists. Even then, certain problems related to space were posed, which are still posed in the 21st century, without being solved, except for the final solution given by an important part of contemporary physics: space does not exist, it is not real, and therefore poses no problem. In these first abstract reflections on space, on extension, concepts related to space are already used, such as the concept of emptiness and the concept of place, the place occupied by physical bodies.

At this early stage, some problems related to the limits of space, to the finite/infinite nature of its extension, and to its divisibility are already discussed. In fact, the problems posed by the actual infinity appear already from the first abstractions on space, and continue to be posed, not solved, until our days. As in the case of physicists with space, mathematicians proposed at the beginning of the last century a solution that claims to be definitive, final, although in this case in the opposite direction to that given by physicists to space: now it will be the dogmatic acceptance of the existence of the actual infinity (Axiom of Infinity). A decision that could be wrong for reasons that have already been given in [212], and that will continue to be given in this book.

It is interesting to note at least two aspects of the first conceptions of the universe. One is the proposal of models in which there is a single principle of creation (and destruction) of all things. Although their relationship to Cellular Automata Like Models (CALMs) is currently unclear, they all belong to the same basic model of explaining the world. On the other hand, and also related to CALMs (with their intimate spatial structure), we have to consider the fact that one of the first theoretical conceptions of space was that of a discrete space defined by extended points, points of non-zero extension. Although in the later development of the theory, the irrational numbers (all of them with an actually infinite number of decimal places) made their appearance and there ended the first discrete possibility for space. Plato and his (rebellious) student Aristotle, certainly the two most important figures of classical Greek culture, also devoted part of their thinking to space, much more the latter than the former. Although in both cases, the former more than the latter, their reflections on space were more ontological and metaphysical than physical.

Some of Aristotle's spatial arguments have the historical interest of having been an inexhaustible source of contradictions for a good part of his Arab followers from the tenth century AD. These contradictions motivated the criticism of the great Greek thinker, who despite the criticism in this matter, continues to be the great thinker of reference in the Arab world and in the European medieval scholastic world, nourished by the translations that the Arabs made of the main works of Aristotle, including those in which space is discussed more extensively, for example his *Physics* [16], or his *On Heaven* [14].

Although he is also a Greek classic, like Plato or Aristotle, the importance of Euclid in the content of this book is so outstanding that he deserves a chapter exclusively devoted to his geometry, as is done in Chapter 8 of this book. As will be seen there, Euclid is more mathematician and physicist than metaphysician, and his influence on modern mathematical and physical sciences is the most important of all Greek authors, pre-Socratic and classical. Euclidean three-dimensional space is considered since Newton as the geometrical structure of physical space, real and absolute for Newton, not so real and not so absolute for most contemporary physicists. This question, the reality of physical space, will be a key issue in the content of this book.

7.2 The first cosmologists

As already indicated, it is in early Greece (8th-7th century BC) that space is included as an object of study proper to philosophy. What existed before that, according to historical documentation, was a pragmatic use of this concept: the knowledge of the spatial directions in which things are located, a kind of system of reference of practical and social interest, and the definition of measurements related to the extension of objects (length, area and volume) for exclusively practical purposes, almost always related to surveying and construction.

Cosmogony, the origin of all things (the origin of the universe in modern terms), appears at the beginning of practically all known philosophies, including those developed in greater or lesser detail by the first Greek authors in what we now generally know as the pre-Socratic philosophers, which include mainly the first cosmologists (Hesiod, Thales, Anaximander, Anaxagoras, Heraclitus) and the Eleatic, Pythagorean and materialist (atomist) philosophies.

The first idea of space to appear in Greek literature is found in Hesiod (8th-7th century BC): the CHASM, a kind of extension in which things can originate and exist [314, p. 18-20]. The chasm, then, was the first thing to exist. (Note that this idea of Hesiod's is not far from the idea of space as the generator and container of all physical objects, an idea not alien to the concept of CALM (Cellular Automata Like Model), which will emerge throughout the book as a consistent model for beginning to explain the physical world. The second thing that must have existed, according to Hesiod, was our Gaia, which gave birth to Ouranos (above) and Tartarus (below). And thus appear the three basic directions of space (six, if one considers the two senses of each of them, as was usual at that time). But Hesiod did not go into details, he did not develop his spatial concepts, which, moreover, are not clearly distinguished from the temporal concepts.

A little later, in the school of Miletus, another idea appears that is also related to the concept of physical space that will be introduced here, although at the moment this relationship may seem somewhat obscure. This idea is that of a basic generating principle (the ARCHÉ) of all things, of the whole universe. The first proposal of such a principle was that of Thales of Miletus (624-546 BC): that generating principle of all that exists in the universe would be *water*. According to Thales, the other elements are derived from this generating principle.

Anaximander of Miletus (610-545 BC) rejected this proposal, considering that water being one of the fundamental elements, characterized by the cold and the wet, could not originate other fundamental elements such as fire, characterized by the opposites hot and dry. Instead he proposed a principle, the APEIRON (the unlimited and indeterminate), because being itself indeterminate and unlimited it could originate the fundamental elements and a multitude of material, spatial and temporal processes [314, p. 23]:

1. The apeiron is that from which the heavens and the worlds arise. Therefore, it appears to be a matter from which everything can

arise. The generation of everything from it is possible, since it itself is not determined or limited as is a piece of matter or a material thing.

2. The apeiron is eternal and ageless and, therefore, temporally infinite. Infinite duration is a necessary condition for the unceasing processes of the phenomenal world.

3. The apeiron surrounds all worlds and is therefore the most encompassing space. It allows all movements and changes to take place in it.

As is also the case of CALMs, Anaximander's apeiron has spatial, temporal and material qualities. Everything is born and dies in it as a consequence of the struggle between opposites (interactions) [33, p. 48-58],[268, p.33-38]. Thus:

> Things perish in the very thing that gave them being, according to necessity [268, p. 36].

But the birth of a thing in the bosom of the apeiron would be an act against the uniqueness of being, the uniqueness of the apeiron. It would be a kind of sin that time makes pay with the disappearance, with the death of that thing. The universe would have, as such an object, an allotted time of existence, at the end of which it is destroyed and returns to the apeiron. This cyclical vision of the universe and of time is not rare in Greek authors.

Anaximenes of Miletus (590-525 BC) did not agree with his teacher Anaximander that the indeterminate, the apeiron, could originate the determinate, the material objects of the physical world, the physis. He then proposed AIR as the generating principle, although it would be an element different from the material air of our days. That of Anaximenes is more complex and not only material: it would contain the generating principle of life. And by different degrees of rarefaction and condensation it would originate other material objects, such as fire, water or earth. Anaximenes therefore proposes a transforming cause that originates the different things of the world from the original principle of all of them: rarefaction/condensation.

Heraclitus of Ephesus (6th-5th centuries BC) also considered a single generating-regulating principle of all things: that without which nothing could be explained. His principle is more relational, more legal and logical than material. It is the LOGOS, the law that binds and holds together the things that constitute physis. It would be present in all things, although it cannot be visualized in any of them, except in their becoming. The physis itself would be giving us (logical) signs of its existence, although humans, according to Heraclitus, tend to ignore those signs led by habit and lack of deep reflection. Only that deep reflection can lead to a thought that reflects the true becoming of physis. A truth that is not to be sought in mysticism but in the simplest objects of the physical world.

Heraclitus also defended what could be called the Principle of Anti-Identicality, as opposed (at least in some respects) to the Theorem of Identicality which will be proved later, in Chapter 5 of this book. According to this Heraclitean principle all things share the same attribute: *being different from one another, at least in the space and time occupied by each thing*. In his deep and obscure style he tells us (quoted in [268, p. 69]):

> Listening not to me, but to the logos, it is wise to recognize that all things are one.

Naturally, both the Anti-Identicality Principle and the Identicality Theorem also apply in the world of CALMs, and in practically all scientific models that try to explain the physical world.

7.3 Parmenides and Zeno of Elea

If Heraclitus is the philosopher of becoming, of continuous change, Parmenides (530(515)-? BC) is the philosopher of the permanent being: being is incompatible with non-being; being cannot not be; being is the opposite of nothingness, and nothingness is not even something that can be considered or conceived, (quoted in [268, p. 112]):

> ... so that you could never cut so that being does not follow with being.
>
> It must be what can be said and conceived. Because there is being, but nothing, there is not.
>
> And it is that such a thing will never be violated, so that something, without being, is.

Parmenides' philosophy contains, then, the two great principles that will end up founding logic: the Principle of Identity (a thing is what it is, and it is not what it is not), and the Principle of Non-Contradiction (it is not possible to be and not to be at the same time). It would not be out of place to consider Parmenides as one of the fathers of Classical Logic.

The philosophy of Parmenides, surely one of the most closed, challenging and difficult in the history of human thought, can also be opened and explained by admitting, as will be seen later in this book, that, indeed, that which is, can only be; but both as what it was, or as something else into which it has been transformed. It is the complete being, it is the being that includes both what it is and what it can be transformed into. We shall also see in this book that for the same reasons that a first cause is necessary to explain being (Corollary of the First Cause, Chapter 5), a final cause is also necessary to explain complete non-being (Chapter 5). The universe is closed, everything is transformed but nothing disappears from the universe, it cannot disappear without the final inexplicable cause, in that Parmenides is right. And it is an important detail that all cosmology should take into account.

Like his teacher Parmenides, Zeno of Elea (490-430 BC) was more interested in ontology than in physics, but he developed part of his famous arguments by making use of a mathematical property of space. A property that, both in his time and in ours, is still a hypothetical property of space: its supposed infinite divisibility (actual infinity) and its consequent modern dense order: between any two points of a simple straight line of 1mm length there exists a non-numerable infinity ($2^{\aleph_0}$) of distinct points; and between any two points of those infinite points there exists another non-numerable infinity of distinct points; and between any two points of those infinite points there exists another non-numerable infinity of distinct points; and so on and on. In Zeno's words [33, p. 177]:

> If there are many beings, beings are infinite, for there are always others in the midst of beings, and in turn others in the midst of these, and thus beings are infinite.
>
> What moves, does not move where it is or where it is not.
>
> If everything there is is in a space, it is evident that there will be a space of space, and that will go on to infinity.

By making use of the supposed infinite divisibility of space, Zeno proves, for example, the impossibility of motion:

> To go from A_1 to A_2, it is first necessary to go from A to the middle A_3 of A_1A_2.
>
> To go from A_1 to A_3, it is first necessary to go from A to the middle A_4 of A_1A_3.
>
> To go from A_1 to A_4, it is first necessary to go from A to the middle A_5 of A_1A_4.
>
> To go from A_1 to A_5, it is first necessary to go from A to the middle A_6 of A_1A_5.
>
> and so on to infinity.
>
> Therefore, the motion from A_1 to A_2 cannot be started.

Already here appears the fascination of humans with the strange and bizarre, as if the strange and bizarre added scientific value to theories. We like to prove things like the impossibility of motion despite its overwhelming evidence. It seems clear in this case that, in the face of such overwhelming existence, Zeno would have to have considered the possibility of some flaw in his argument. And the flaw, as we shall see, is the Hypothesis of the Actual Infinity which, briefly stated, considers that the ordered list of natural numbers exists as a complete totality, even though there is not a last natural number completing the list. As if the incompletable could exist as completed, as Aristotle would say (recall that a complete totality is a set defined by comprehension in which every element that should be in the set, is in the set). The infinite will accompany us throughout the book, meanwhile the reader can analyze some demonstrations of its inconsistency in [204, 210, 211] and especially in [212] and [213, Link].

Indeed, in a discrete physical space, with minimal indivisible units (qusits) there exists immediate successiveness, so that between a qusit and its immediate successor there is no other qusit. Under these conditions it will be proved that between any two qusits there always exists a first qusit, a last qusit and between them a finite number of qusits. All Zeno paradoxes are immediately dissolved in this finite and discrete scenario.

7.4 The Pythagoreans

The abstract notion of extension and place appears in the first Pythagoreans. It is a spatiality linked to the natural numbers, which for them were more real than sensible reality itself. Indeed, according to Aristotle [16, 213b, p. 230] the Pythagoreans considered the natural numbers with a certain spatiality, necessary to guarantee their discrete character: between two successive natural numbers no other natural number can exist. The void delimits the natural numbers. They also considered the void as a kind of division or separation between objects. This primitive space (pneuma apeiron) had no physical implications, except that of separating things [176, p. 9]. Some pre-Socratics identified it with the limitless, with the void, with air, and even with night [47, p. 433-434]. It is only the beginning of the abstract conception of space.

Already in this epoch, the earliest of abstract thought, the first idea of a discrete space appears. Indeed, the early Pythagoreans believed in the existence of a geometric space formed by indivisible points with a length δ greater than zero. Consequently, all lengths would have to be commensurable: the ratio between any two of them, say L_1 and L_2, would be the ratio between two natural numbers, i.e. a rational number [231, pp. 11-16]l:

$$L_1 = n_1\delta;\ L_2 = n_2\delta \tag{1}$$

$$\frac{L_1}{L_2} = \frac{n_1\delta}{n_2\delta} = \frac{n_1}{n_2} \tag{2}$$

But, as is well known, they themselves discovered the incommensurability of two well-defined lengths: the length L of the side of any square and the length L_d of its diagonal. For example, for a square whose side has length 6δ we would have:

$$L_d = \sqrt{6^2\delta^2 + 6^2\delta^2} \tag{3}$$

$$= 6\delta\sqrt{2} \tag{4}$$

$$\frac{L_d}{L_s} = \frac{6\delta\sqrt{2}}{6\delta} = \sqrt{2} \tag{5}$$

where $\sqrt{2}$ is an irrational number: a number with an infinite, non-periodic number of decimal places. We will deal with numbers with infinite decimal places later in this book. For the moment we regret that the Pythagoreans did not discover discrete (integer) division, the only division compatible with discreteness, for example:

$$L_d = \lfloor\sqrt{6^2\delta^2 + 6^2\delta^2}\rfloor \tag{6}$$

$$L_d = \delta\lfloor\sqrt{6^2 + 6^2}\rfloor \tag{7}$$

$$= 6\delta\lfloor\sqrt{2}\rfloor \tag{8}$$

$$= 6\delta \tag{9}$$

$$\frac{L_d}{L_s} = \frac{6\delta}{6\delta} = 1 \tag{10}$$

where $\lfloor x \rfloor$ stands for the integer part of x. As we will see later in this book, equations (6)-(10) represent the discrete version of Pythagoras theorem, which we will also deduce later on. And naturally, the Pythagorean metric, based on the Pythagorean theorem, will be key in the geometries and theories of space to come.

The newly discovered incommensurability between certain lengths led the Pythagoreans toward the notion of continuous space [231], a precedent of the relativistic spacetime continuum. Perhaps due to the enormous influence of our sensory perception of the physical world as a continuous space-time scenario, discrete (discontinuous) arithmetic was not developed in Greek culture, and is yet to be developed in ours in formal and universal terms.

In any case, one of the consequences of the Pythagorean discovery of incommensurable lengths was the abandonment of extensive points in favor of non-extensive points, which are the same ones we still use today in all continuous geometries, Euclidean and non-Euclidean, practically the only geometries in contemporary physics. But the story is not over, as we will see throughout this book.

The Pythagorean Archytas (435/410-360/350 BC) seems to have written a book on space, although only a few fragments have survived. He is one of the first authors to consider the problem of the limit of space. He did so with a well-known and recurrent argument (his discussion is repeated at least until 1690, when J. Locke uses it in his famous text *Essay concerning human understanding* [217, C. XIII, 21, p 102]): Archytas wonders whether placed a man at the boundary of space he may, or may not, extend his arm beyond that boundary.

The same question we can ask ourselves today: what will we find if we travel from the center of the Universe in a straight line 46 billion light years and stand at its boundary? Does that boundary exist? Does the outside of the Universe exist? What could happen if we emit a visible laser beam from that supposed boundary in the direction of the supposed outside of the Universe? And if the outside of the universe does not exist, does it not exist because it is infinite, or does it have some kind of physical limit? If the universe is consistent, and we will prove that it is, and infinity is inconsistent, and we will prove that it is, then it must be finite, and therefore could have a physical limit. But, in addition to asking questions similar to those we have just asked about the limits of the universe, Archytas reflected on other aspects of space, especially in relation to the objects contained in space:

1. He distinguishes between space and matter, and considers space to be independent of matter.
2. Every object occupies a place and the object cannot exist if the place does not exist first. The place must exist before all things.
3. A salient feature of space is that it contains all things, but space is not contained in anything else.
4. Space determines the volume of all bodies: it exerts a kind of pressure on them, preventing them from reaching an infinite size.
5. Space is thus a kind of primitive atmosphere with pressure and tension.
6. Beyond space lies the infinite void.
7. Since there is nothing after all things, there is no outside, and therefore space is without end and without limit. And no matter in which region it is situated, it will have the same infinitude in all directions.

7.5 The atomists

As it is well known, and its name indicates, for the atomists Leucippus (460 - 370 BC) and his disciple Democritus (460 BC - 370 BC) matter was discrete, it was formed by indivisible units: the atoms. With non-zero shape and size, the atoms existed in an infinite number (actual infinity) and therefore would occupy an infinite space, an empty extension without influence on the motion of the atoms, incessant motion due to their continuous collisions. The existence of empty space had been rejected by other pre-Socratic thinkers such as Melissus (quoted in [176, p. 11]):

> Nor is there anything empty, for the empty is nothing and that which is nothing cannot exist

But for the atomists it was a logical necessity, according to their atomic theory of matter. The disciples of Leucippus and Democritus added weight to the atoms as the cause of their upward and downward motion, which added to space a preferred directionality: the vertical. Space was then homogeneous but anisotropic. There remains the doubt as to whether the unlimited space was for the atomists something that penetrated all bodies and was penetrated by all bodies, or was only the sum of all the gaps between all atoms and between all bodies.

Lucretius, a late atomist (99 BC - 55 BC) expressed the foundations of atomic materialism in a long and famous poem that was lost for several centuries until it was found in 1418: The Nature of Things. With respect to space we can read [221, p. 108]

> Let us return to our reasoning:
> all nature, then, is based
> on two principles: bodies and void (420)
> in which they swim and move:
> that there are bodies, common sense
> proves it; an irresistible principle
> without which reason, abandoned
> from error to error would be lost.
> If there were not, therefore, that space
> which we call emptiness, there would not be
> bodies would not be seated, nor would they move
> could, as I have just told you

And also: [221, p. 129]

> If, in addition, space is limited

and someone stands at the end
and shoots a flying arrow,
do you want it to be shot with great strength
it flies lightly to reach the target,
or do you think that impeded by some hindrance
its flight does not let it go forward?
One or the other you must confess. (1220)
Whichever one you choose, you must forcibly
you must remove the limits to the whole:
For it may well be an obstacle that hinders
and hinders the arrow from reaching the target,
or else it passes it, here there is no end:
where you set limits, I will at once
I will ask what has become of the arrow:
you will never find the end like this;
its immensity always leaves a space
for the fugitive arrow to cross.

Space becomes according to Lucretius an infinite receptacle of all things (bodies). On this infinity, Lucretius gives a new argument, invoking, like other disciples of Leucippus and Democritus, the weight of atoms and a directional preference in space (above and below). [221, p. 129]:

Moreover, if nature (1231)
had set limits to the whole,
already the matter with its own weight
would be gathered in the deepest places;
beneath the vault of heaven
nothing would be produced.

The atomists, the first materialists in history, were also the first to admit that something immaterial like the unlimited void would have an existence as real as that of material objects. Real but different in their essence, as also stated by Gorgias (460-380 BC), who also gave one of the first proofs of the finiteness of space (reconstructed in [176, p. 14]):

> The first clear idea of space and matter as belonging to different categories is to be found in Gorgias. Gorgias first proves that space cannot be infinite. For if the existent were infinite, it would be nowhere. For were it anywhere, that wherein it would be, would be different from it, and therefore the

> existent, encompassed by something, ceases to be infinite; for the encompassing is larger than the encompassed, and nothing can be larger than the infinite; therefore the infinite is not anywhere. Nor on the other hand, can it be encompassed by itself. For in that case, that wherein it is found would be identical with that which is found therein, and the existent would become two things at a time, space and matter; but this is impossible. The impossibility of the existence of the infinite excludes the possibility of infinite space.

7.6 Space according to Plato

According to Aristotle, Plato was not very satisfied with the explanations given by his predecessors about the existence of space, so he tried to explain it [16, 209b]. And he did so in his Timaeus, a dialogue between Socrates, Critias and Timaeus of Lycritus (an old Pythagorean of dubious historicity). The dialogue is sometimes obscure, but it is undoubtedly one of the most influential works in the history of philosophy and science. The Timaeus is a cosmogony that includes reflections on matter and on living beings [278]. For the reasons that will be given below, Plato's text will be very significant for the physical discussion of physical space proposed in this book. In the Timaeus we can read [278, Pos. 520-536] (the texts in straight brackets are mine):

> The same reasoning also applies to the nature that receives all bodies. We must say that it is always identical with itself, for it does not change its properties at all. Indeed, it always receives everything without adopting in the least any form similar to anything that enters it, since by nature it underlies everything as a mass which, because it is changed and shaped by what enters, appears diverse at various times; and both what enters and what leaves are always imitations of beings, imprinted from them in a difficult to conceive and admirable manner which we shall investigate later. Certainly, now we need to conceptually differentiate three genres:
>
> (i) That which becomes, [the objects that are formed in the imperfect material world],
>
> (ii) that [the medium] in which it becomes, [the receptacle in which material objects are formed]
>
> (iii) and that through the imitation of which that which becomes is born. [The Ideas or Perfect Forms]
>
> And one can also liken the vessel to the mother, that which is imitated to the father, and the intermediate nature to the

> son, and think that, similarly, when a relief is to be of a great variety, the material on which the engraving is to be made would be well prepared only if it lacked all those forms which it is to receive from somewhere. If it were similar to anything of what goes into it, by receiving the opposite or what is in no way related to that, it would imitate it badly because it would manifest, in addition, its own appearance. It is therefore necessary that that which is to take all species in itself should be exempt from all forms. As happens in the first instance with artificially perfumed oils, the liquids that are to receive the perfumes are made to be as odorless as possible. Those who attempt to print figures on some soft material do not allow any figure at all, but flatten it first and leave it completely smooth. It likewise corresponds that what is to receive often and well to its full extent imitations of eternal beings should by nature lack all form. Therefore, let us conclude that the mother and receptacle of the visible become and completely sensible is neither earth, nor air, nor fire, nor water, nor whatever is born of these, nor that from which they are born. If we affirm, on the contrary, that it is a certain invisible, amorphous species, which admits everything and which participates in the most paradoxical and difficult to understand way of the intelligible, we will not be mistaken.

Consequently, and according to Plato, in addition to that which becomes (i), and that through whose imitation that which becomes (iii) is born, it is also necessary to consider that in which it becomes: the receptacle (ii). Note that this Platonic receptacle is material, but of a different matter from that which forms the material objects of the physical world, and while these are in continuous change, the same is not true of the receptacle, which remains unalterable. Plato will also end up calling the receptacle space. Plato then explains how the universe was set in motion, starting from a receptacle that already included traces of the four fundamental elements (fire, air, water and earth) in initial chaotic movement, but that story is already alien to our objectives.

Before leaving the Platonic receptacle, let us think about the matrix of cells of a CALM (Cellular Automata Like Model) and the functioning of the CALM. It is in this matrix that CALM objects are formed and evolve, resulting from the CALM laws and the state of the cells. CALM objects, as such, are different from the cells and can move through the CALM while the cells remain immobile. It seems appropriate, then, that from now on we also refer to the array of discrete elements (cells) of a CALM as a receptacle.

7.7 The Aristotelian space

Aristotle's texts have the reputation of being unfinished, of being drafts for future manuscripts that, although mentioned by his first commentators, do not really exist [164, p. 72]. This situation is conducive to different interpretations of some Aristotelian texts, which is what happens with those devoted to space, as we will see here. The situation is complicated because space could be a primitive concept (indefinable in terms of other more basic concepts) that Aristotle tries to define using twisted dialectical means. Without success, of course. As indicated in the presentation of this book, until now no one has succeeded. Neither the authors we have already remembered nor those we will remember. In such situations I always remember Newton's words, which are almost a joke. I repeat them again here [259, p. 77]:

> I do not define time, space, place and motion, as being well known to all.

In his Categories [15, Part 6, p. 15], Aristotle distinguishes between discrete and continuous magnitudes according to whether there are discontinuities or jumps between their successive values (discrete magnitudes), or not (continuous magnitudes). Between any two values of a continuous magnitude there are always intermediate values (dense order), which does not occur with discrete magnitudes (discrete order). For Aristotle, the quantities of space and time are of continuous type, since there are no discontinuities between their respective parts. The intervals of space and time are infinitely divisible, although the infinite parts do not all exist in act (actual infinity) but in potency (potential infinity).

To reconcile the indivisibles with the divisibles ad infinitum, Aristotle defined three types of relations [16, Book V, 228a]: successiveness, contiguity, and continuity, establishing the conditions to be met by continuous, contiguous, and successive elements. Between elements of one type there could be elements of the other types. Successiveness implied neither contiguity nor continuity, but if two successive elements touched, then they were contiguous (juxtaposed), as occurs, for example, when air touches the surface of a glass of water. If adjacent elements had coincident boundaries, then they would be called continuous and would be a single reality (half of a stick is continuous with the other half). Physical contiguity is mathematical continuity when all contiguous elements are of the same type, homogeneous. In the case of different physical objects, there can be contiguity but not continuity, because they are qualitatively different elements.

It is mainly in his Physics that Aristotle analyzes the concepts of space and place, although he hardly uses the word space, probably trying to avoid confrontation with a predictably primitive, indefinable term. In fact, the word "*space*" appears only 10 times in Aristotle's Physics, 9 of which, according to other authors, refer to space without giving a definition of it. Only once is the word place used in the sense of space (chôra). On the contrary, the word "*place*" appears 383 times because it gives a definition of place, which, as we will see here, is a pseudo-definition. Recall that the universe, according to Aristotle, is a finite sphere composed of several concentric layers with a center at the center of the Earth. The outermost shell, which limits the size of the universe, is formed by the fifth Aristotelian element (quintessence) called ETHER, which is distinct from the other four elements (earth, water, air, and fire) that form the inner sphere or sublunar world.

The outer layer or envelope, the heavens in Aristotelian terms, would be in continuous rotation, which leads to certain contradictions with Aristotelian mechanics that were discovered by the Arabian scholars of the ninth to twelfth centuries. We will analyze them in Chapter 9. For its part, and in an ideal state, the internal sphere would be formed by four concentric layers that from the exterior to the interior would be: fire, air, water and earth. But in its real state the four elements are mixed, although conserving, as one of their essential properties, their tendency to move towards their corresponding natural places, that is to say towards each of these four layers ordered in the way that has been indicated. These movements would be natural, not forced.

The natural motion is, therefore, teleological: the fundamental elements would move with the purpose of occupying their natural places in their corresponding layers. In addition to these natural movements towards natural places, there would also be forced movements of objects towards other places. Aristotle then tries to characterize the concept of place in order to then try to deduce what the place of a thing must be [16, Book 4]. Among these characteristics of place he highlights:

1. Places have a real existence although not independent of the things that occupy those places.
2. The place of a thing is that which embraces it.
3. The place of a thing is not part of that thing.
4. The place of a thing is neither larger nor smaller than the thing.
5. The place and the thing it contains can be separated.
6. Each location implies absolute directionality up and down.
7. The place has no place.
8. The place is different from its changing content, then it is real.

Aristotle then proposes four possible definitions of place, to prove that three of them are impossible:

(i). The place is the form.

(ii). The place is the matter.

(iii). The place is a certain extension between the extremes of that which contains the thing.

(iv). The place is the end that contain the thing.

The first two alternatives are impossible because neither form nor matter are separable from the thing, while the place of the thing is separable from the thing. The proof that the third alternative is also impossible is much more obscure [16, 211b, p. 224-225]. Therefore, the only possible definition left is the fourth (iv) of the above alternatives. [16, 212a, p. 225]:

> The place of a thing is the limit of the containing body that is in contact with the contained body.

And as the reader will have guessed, the above definition does not define anything unless it is indicated what the containing body is, which is nowhere to be found in the Aristotelian text, which at most refers to vessels and liquids contained in the vessels, insisting that the containing body must be in contact with the contained body, leaving no gaps between the two. So, after all and as expected, the place of a thing is the boundary of something that is not known what it is and that is in contact with the contained body. The place of an object is a kind of shell that envelops the object, but a shell of an unknown nature. That is the Aristotelian (pseudo) definition of place, because the definition of space does not exist. Only in [16, 208b, p. 213] one can read:

> Hence, it may appear that the place or space [chôra] to which or from which the bodies have changed is distinct from them.

where Aristotle equates space with place. But he does not give an explicit definition of space. Some authors consider that Aristotelian space would be the set of all the places of all objects [176, p. 20], that is to say the set of all the external shells of all material objects. These external shells would be surfaces without thickness, with the same morphology as the external morphology of the enveloped body [164, p. 77].

Some authors give a definition of Aristotelian place different from the one given above. For example, in [314, p. 42] the following definition of Aristotelian place is given (which appears written in both Greek and English):

> The first/immediate unmoved limit of that which surrounds-that is topos.

The place would be, in addition to the immediate boundary or first, that which surrounds and is immovable. For example, the place of a boat sailing down a river would not be the outer shell of the boat, but the immovable boundary of the riverbank, since the container object must be immovable. It would be something like a material (physical) reference frame in which it is possible to describe motion (in this case of the boat), because a boundary of the contained body of the same size as the contained body would not allow motion, and motion is one of the fundamental concepts of Aristotle's physics.

As M. Jammer points out, an interesting aspect of the Aristotelian concept of place is its similarity to a field of forces [16, 208b, p. 213]:

> Moreover, the displacements of simple natural bodies, such as fire, earth, and the like, not only show us that place is something, but also that it exerts a certain power. For each of these bodies, if nothing prevents it, is carried toward its proper place, some upward and some downward.

In the same sense, other authors [314, 233] emphasize the dynamic role of the "container body" as a transmitter of forces and as a cause of the maintenance of the directions of motion of the "contained bodies", reminiscent respectively of preinertia (which will be discussed in Chapter 19 of this book) and Newtonian inertia.

But the fundamental problem of a definition of space remains unresolved. And as will be seen throughout this book, it remains so today. Consequently, the main questions raised by the physical nature of space remain open:

- Does space have substance?
- If it does, what kind of substance is it? because it must be different from ordinary matter.
- How does it interact with ordinary matter?
- Does it penetrate all objects?
- Does it allow itself to be penetrated by all objects?
- What is space the cause of? or is it the cause of nothing?

Although in our days, all of them are solved by the expeditious way of the negation of space: space does not exist, it is not real. However, space can expand, deform, vibrate and be the transmitting medium of its own vibrations and of other vibrations such as electromagnetic waves. And one wonders how something that is not real can do all that? How can the vibrations of an object that does not exist be recorded experimentally? We will return to these questions and propose some answers at the end of the book.

7.8 Space in post-Aristotelian Classical Greece

Although Plato and Aristotle were the fundamental pillars of metaphysical and physical thought until at least the Scientific Revolution of the 16th-17th centuries [182, 103, 320], their approaches to space were already disputed by Aristotle's own disciples and later by his Arabic translators [352, 357]. Thus, his disciple and successor in the direction of the Lyceum, Theophrastus (371 BC-287 BC), came to the conclusion that space is not an entity of its own but a system of relations between bodies that determines their relative positions. Here appears already a theory of the group of relational theories as a counterpoint to the substantival theories. Naturally, Euclidean space should also be included in this section, but, as already indicated, given its relevance to the central theme of this book, chapter 8 will be devoted to it.

The Stoics modified the Aristotelian definition of place: instead of the bounding surface of the first immobile container body, they used the alternative of the interior volume defined by that bounding surface, a concept closer to the intuitive notion of space occupied by an object. The purely geometrical continuity in Aristotle becomes a physical principle in the Stoics, allowing the propagation of physical phenomena and physical interactions between objects throughout the universe, even beyond the sublunar world: for example, Posidonius (135 B.C. -51 B.C.) discovers the influence of the Moon on terrestrial tides.

The universe according to the Stoics would be formed by the set of material objects physically interrelated through the physical space in which they are included, all surrounded by an undifferentiated vacuum, externally unlimited, infinite, and without influence on the underlying material world. According to the Peripatetics, the material universe of the Stoics would eventually have to dissipate into the infinite void, but the Stoics argued that such dissipation into the external void would not occur because of the tension and interactions between the material parts of their universe [176].

Strato of Lampsacus (335-268 B.C.) was a Greek peripatetic philosopher who succeeded Theophrastus in the direction of the Lyceum founded by Aristotle. He was especially devoted to the study of nature, including new natural elements in the explanation of the world, always seeking an agreement with daily experience, to the point that the intervention of the gods in the creation of the universe was unnecessary. Strato defined space as the container of all things. A container that would exist even if there were no things inside it. So, for Strato, the vacuum was not entirely impossible, it could exist in the interstices of material particles. The Alexandrian engineer Heron (10?-70? B.C.) used the penetration of material rays of light and heat in water as proof of the existence of such interstices.

The first indication of a connection between space and God appears in Palestinian Judaism of the first century. In Greek philosophy the use of the word place as a reference to God does not occur. In Sextus Empiricus (160 - 210 B.C.) an empiricist Greek physician and philosopher, we can read a hint of this usage (quote taken from [176, p. 29]):

> And so far as regards these statements of the Peripatetics, it seems likely that the First God is the place of all things. For according to Aristotle the First God is the limit of Heaven. Either, then, God is something other than the Heaven's limit, or God is just that limit. And if He is other than the Heaven's limit, something else will exist outside Heaven, and its limit will be the place of Heaven, and thus the Aristotelian will be granting that Heaven is contained in place; but this they will not tolerate [...] And if God is identical with Heaven's limit, since Heaven's limit is the place of all things within Heaven, God -according to Aristotle- will be the place of all things; and this, too, is itself a thing contrary to sense.

Whereas in the Jewish theology of the time, and probably earlier, they wrote things like (quoted in Latin in [176, p. 29]):

> The Hebrews do not doubt God, because no one contains Him, but He Himself, by His immense power, contains all things, having to be called "makom" or place, as is often done in the booklet of the Paschal rites published by Rittangelius.

8. Euclidean space

(Content partially taken from [200])

8.1 Introduction

This chapter is entirely devoted to Euclid's Elements, a fundamental work in the history of science, written more than 2300 years ago by Euclid (~325 BC - ~265 BC), and still valid today, at least in its most basic aspects. Euclid's Elements have the added interest of providing a possible geometric model for physical space, which is the central theme of this book.

As we shall see, and despite the fact that physical space is not real for most contemporary physicists, physical space could be real and discretely Euclidean[1], although massive objects could locally deform it and transform its geometry from Euclidean to non-Euclidean. In Chapter 25 of this book we will see that there are Euclidean alternatives to explain these non-Euclidean deformations.

After a brief presentation of Euclid and his Elements, one of the formal shortcomings of the Euclidean text is analyzed: the absence of a functional definition of a straight line, i.e., the absence of a definition that, realized in terms of the properties of lines, characterizes straight lines exclusively and that can be used explicitly in formal proofs. It is also explained why this deficiency is important, although the matter is treated in detail in Appendix C of this book. Finally, the physical reasons why it is still believed that physical space is Euclidean are discussed.

8.2 Euclid

Euclid is the name of a Greek mathematician, the author of the Elements, a book that laid the foundation for a mathematical discipline now known as Euclidean geometry. As with many other great thinkers

[1]As we will see later in this book, physical space cannot be continuous, as in Euclidean geometry, but discrete.

of ancient Greece, almost nothing is known about the man. In fact, all we know about the author of the Elements comes from two texts, one by Proclus Diadochus (412-485 AD) and the other by Pappus of Alexandria (290-350 AD). From Proclus' text [283] we infer that Euclid lived in the time of Ptolemy I Soler (367-283 BC), and that he was "younger than the pupils of Plato, but older than Eratosthenes and Archimedes". It seems reasonable to conclude that he flourished around 300 BC and that he received his mathematical education in Athens, from the students of Plato.

In the same passage of Proclus's text we can read the well known anecdote on Euclid (quoted from [150, p. 1]):

> . . . Ptolemy once asked him [Euclid] if there was in geometry any shorter way than that of the Elements, and he answered that was no royal road to geometry.

From Pappus' text [265] it can be inferred that Euclid *taught and founded a school in Alexandria* because Pappus wrote about Apollonius of Perga (262-190 BC) that *he spent a very long time with the pupils of Euclid in Alexandria, and it was thus that he acquired such a scientific habit of thought.* In the same text, Pappus wrote a favorable comment on Euclid as a response to the less favorable opinion of Apollonius on Euclid's work on conics.

From 1332 to 1493 Euclid was believed to be the philosopher Euclid of Megara who lived about 400 BC. In 1493, Constantinus Lascaris resolved definitively the error. Other misunderstandings and questionable anecdotes related to Euclid and his Elements come from the Arabian authors, some of which defended the theory that it was Apollonius, not Euclid, the author of the Elements. Euclid not only wrote the Elements, at least half a dozen of other scientific works were surely authored by Euclid. Among them:

- The Data: an introduction to higher analysis.
- The Phenomena: on theoretical astronomy.
- The Optics: on the (rectilinear) propagation of light.
- Elements of Music: on harmony and Pythagorean theory of music.
- The Porisms: three lost books of very controversial content (probably advanced mathematics).

8.3 Euclid's Elements

In my opinion, the 13 books that make up Euclid's Elements are the first great scientific work in the history of science. It is certainly the most edited work and the most widely used source of scientific knowledge by people of all times and places. The word "Elements" in the title of a scientific work (Elements of Chemistry, Elements of Geology, etc.) usually indicates that it is an introductory text to a discipline, the aim of the work being to provide the reader with the basics to get started in a science. Generally, these works contain the definitions, principles, and axioms upon which the science in question is built. This is a tradition that dates back to ancient Greece. Such is the case with Euclid's Elements, although here there is something more. In fact, Euclid's Elements represent:

a) A model of how to proceed in the development of a mathematical theory.
b) A model of mathematical reasoning.
c) A prototype of the axiomatic method (scientific method of the formal sciences).
d) A compendium of the main geometric results known in Euclid's time.
e) The creation of a new science.

But Euclid's Elements were neither the first nor the last Elements of Geometry. Among other authors of this type of works we can mention Hippocrates of Chios (not to be confused with physician Hippocrates of Kos), Leon, Theudius of Magnesia, Amyclas of Heraclea, Cyzicenus of Athens, Philippus of Mende or Aristaeus. The success of Euclid's Elements, perhaps the most read and studied book ever, made practically disappear the other *Elements*. According to T.L. Heath [150, p. vii], Euclid's work is "*one of the noblest monuments of antiquity*". I fully agree.

Naturally, Euclid's text was built on the basis of the geometry known at the time. Among the authors on whose experience and achievements Euclid built his Elements we must mention the followings [136, p. 9]:

a) Pythagoreans: Books I, II, III, IV, VII and IX.
b) Archytas: Book VIII.
c) Eudoxus: Books V, VI, and XIII.
d) Theaetetus: Books X and XII.

The thirteen books include 5 general axioms, 5 geometric axioms, 131 definitions and 465 propositions. The propositions proved in one of the books can be used to prove other propositions in the same or in other subsequent books, so that between them there exists a complex network of formal relations that are now being analyzed with the aid of graph theory and computer programming [316]. These types of analyses allow to calculate the number of formal connections between any two propositions as well as the number of logical paths connecting two propositions. The Book I, which has been always considered as the most perfect of the thirteen books, is the richest regarding the number of formal connections between its propositions. For instance, between Proposition 1 and Proposition 45 there is a formal path composed of 20 different propositions, and 558 different logical paths connecting them [316, p. 25]. It is also the book that poses Euclid's enigma we will examine in the next section.

8.4 The enigma of the parallel straight lines

To avoid the infinite regress of arguments and circular arguments, all sciences, whether formal or experimental, must be built on assertions whose veracity must be accepted without proof. In the formal sciences these assertions are known as axioms. Ideally they should be short in number and highly self-evident. If we construct a science on an excessive number of axioms the output could result excessively speculative. If the axioms are not self-evident the output would be excessively abstract. For these reasons the set of axioms selected to found a formal science should be carefully examined. In the case of the experimental sciences, biology, geology, physics and chemistry, it is the inductive knowledge (that of Russell's chicks (see Chapter 5) which guides the choice of axioms, which are usually called principles or fundamental laws.

Euclid's Element are based on five general axioms (that apply to all sciences) and five geometric axioms (Euclid's Postulates). It is this group of axioms, or postulates, which poses the problem of Euclid's enigma, also known as the parallel enigma. A simple reading of these five axioms suffices to understand from were the problem arises.

Let the following be postulated [150, p. 154-155]:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

The first four postulates are short and self-evident assertions. The fifth one is neither short nor self-evident. It has rather the aspect of a typical Euclidean proposition or theorem. For both reasons it was put into question, as such a postulate, from the very beginning of the history of Euclidean geometry. For centuries, the same questions have been being asked: can this Fifth Postulate be derived from the other four? can Euclidean geometry be built without the Fifth Postulate? what differences would there be between a geometry with the Fifth Postulate and one without the Fifth Postulate? These questions summarize the enigma of the Fifth Postulate. In 1868 E. Beltrami proved the Fifth Postulate cannot be deduced from the other four [26], so it is not necessary to include it in the foundational bases of other types of geometries (see next section).

The first known attempt to resolve the enigma of the Fifth Postulate dates from the 2th century AD. And the attempts continued until the end of the 19th century, even after the birth of non-Euclidean geometries. So, the accumulated literature on the Fifth Postulate is enormous (see [294]). Among the main authors that tried to solve the problem of parallels we found: C. Ptolemy (2nd century), Proclus (5th century), al-Gauhary (9th century), Omar Khayyam (11th century), Nasir ad-Din at-Tusi (13th century), John Wallis (1616-1703), Gerolamo Saccheri (1667-1733), J. H. Lambert (1728-1777), J. L. Lagrange (17-36-1813), or A. M. Legendre (1752-1833).

Euclidean geometry is intuitive because it is closely and unequivocally related to our interactions and experiences with the physical world, in which we perceive space and objects arranged in that space. For that reason, Euclidean geometry is easy to understand. Although, for that very reason, it is not uncommon to take for granted what cannot be taken for granted. Or in other words, it is very easy to assume hypotheses implicitly, without realizing that one is assuming an implicit hypothesis, i.e. an hypothesis that is not included in the initial basis of assumed hypotheses (axioms) that should be the only hypotheses used in demonstrations. This type of error has always been present in the history of Euclidean geometry, particularly in the history of the Fifth Postulate. A matter on which, after centuries of discussions, the only thing that could be found were alternative statements for Euclid's Fifth Postulate. Most of them are in themselves problems of great geometric interest. These include the following ones (taken from [150, p. 220] and [257]):

1) Through a given point only one parallel can be drawn to a given straight line (Proclus and Playfair).
2) If a straight line intersects one of two parallels, it will intersect the other also (Proclus).
3) Straight lines parallel to the same straight line are parallel to one another (Proclus).

4) Parallels remains, throughout their length, at the same finite distance from one another (Proclus).
5) There exist straight lines everywhere equidistant from one another (Posidonius and Geminus).
6) Non-equidistant straight lines converge in a direction and diverge in the other (Thabit ibn Qurra).
7) If in a quadrilateral figure three angles are right angles, the fourth angle is also a right angle (Clairaut).
8) Two perpendiculars of the same length to the same straight line defines a rectangle (Farkas Bolyai).
9) There exists a triangle in which the sum of the three angles is equal to two right angles (Legendre).
10) A straight line perpendicular to a side of an acute angle cuts also the other side (Legendre).
11) Through any point within an angle less than two-thirds of a right angle, a straight line can be drawn which meets both sides of the angle (Legendre).
12) There exists no triangle in which every angle is as small as we please (Worpitzky).
13) Given any figure, there exists a figure similar to it of any size we please (Wallis, Carnot and Laplace).
14) There exist two unequal triangles with equal angle (Saccheri).
15) A rectilineal triangle is possible whose area is greater than any given area (Gauss).

The positive side of all this work is that, thought Euclid's enigma could not be resolved, Euclidean geometry was enriched and extended with an increasing collections of new and exciting problems. But the question that interests us here is: Can the Fifth Postulate be proved on a foundational basis DIFFERENT from Euclid's? Ockham's razor suggests an affirmative answer. And Ockham's razor is not usually wrong.

8.5 The definition of straight line

The infinite regress of definitions makes the use of primitive (indefinable) concepts inevitable in all sciences, which is not always explicitly recognized. Euclid's Elements is not an exception. Book I includes 23 definitions, some of which are rather confusing. Among the first definitions are those of point, line and straight line:

1. A point is that of which there is no part.
2. A line is a length without breadth.

3. A straight-line is a line which lies evenly with the points on itself.

The definition of point contains a generic, unspecified "*that*" which raises more questions than answers: does it have no parts because it has no size or because it is homogeneous (there are many different homogeneous objects)? Can something exist that has no size? Can extended Euclidean objects be composed of non-extended Euclidean objects? In the case of the definition of line it is clear that a line is not a length. And the definition of straight line does not specify what it is to lie evenly, unless it means to extend in a straight line, in which case the definition is circular. As expected, Euclid did not use any of these definitions in the formal proofs of his propositions. And the same is true in later Euclidean geometries, in which the three concepts are often used as if they were primitive concepts.

The Euclidean definitions of point, line and straight line are thus formally non-productive: a productive definition of an object establishes properties of the object which, once the object is axiomatically legitimated, are explicitly used in demonstrations. Surely the concepts of point and line can only be primitive concepts; if they were not, they would have to be defined by other more basic concepts, which would be the new primitive concepts. There is no escape from this potentially infinite regress. Another thing is the definition of a straight line. In this case it is possible to give a productive definition. Indeed, let us consider the following definitions of straight line that followed one after the other in post-Euclidean time:

- Definition by Heron of Alexandria (10-70 DC) [150, p. 168]: [a line such that] all its parts fit on all (other parts) in all ways.
- Definition by Proclus (412-485 DC) [150, p. 168]: that line which with one another of the same species cannot complete a figure.
- Definition by J. Playfair (1748-1819 DC) [279, p. 8]: if two lines are such that they cannot coincide in any two points without coinciding altogether, each of them is called a straight line.
- Definiton by E. Beltrami (1835-1900 DC) [26, p. 2]: [a line whose] specific character is to be completely determined by only two of its points, because two [straight] lines cannot pass through two given points of space without coinciding in all their extension.

These definitions point to a unique property of straight lines: if two straight lines have two common points, all points between those common points are also common points. The New Elements of Euclidean Geometry [200] is built on a new foundational basis that includes 29 definitions, 10 axioms and 45 corollaries. In that new foundation straight lines are defined as follows:

1. Points and segments that do not belong to the same line are said

non-collinear. Non-collinear lines with at least one common segment are said locally collinear.

2. Lines whose segments have the same definition as the whole line are said uniform. Two or more uniform lines are said mutually uniform iff any segment of any of them has the same definition as any segment of any of the others.
3. To extend a given line by a given length is to define a line, said extension of the given line, that is adjacent to the given line, has the given length, and the extension and the extended line are lines of the same class as the given line. Lines that can be extended from each endpoint and by any given length are called extensible lines.
4. **Definition 16 (of Straight Lines)** *Straight lines: Extensible and mutually uniform lines that can neither be locally collinear nor have non-common points between common points.*

Therefore, it is possible to give an exclusive definition of straight line not based on metric concepts alien to the nature of lines[2] but on concepts proper to the topological nature of lines. It is also a functional, productive, definition. By the way, the new foundational base of Euclidean geometry allows to prove as a theorem the Euclidean postulate of the parallel straight lines, see Appendix C for a summary and [200] for a full discussion

8.6 Physical space is Euclidean

Until the beginning of the 19th century, the axioms of Euclidean geometry had enough evidence to consider that the geometry of physical space was Euclidean. Recall Gauss's famous fictitious experiment that proved the physical reality of that geometry: the angles of a triangle were 180º when the sides of that triangle were three rays of light properly emitted from the tops of three mountain peaks. Although the experiment is only a fiction, it illustrates well the conviction that the geometry of physical space was (and still is) truly Euclidean. What is not a fiction, as will be seen below, is the estimate of the critical density of the universe. However, we must first remember the birth of non-Euclidean geometries, around the same time as Gauss's fictitious experiment.

In the early years of the 19th century, all attempts to deduce Euclid's Fifth Postulate from the other four had failed. For that reason, the parallel's problem was called at that time the shame of mathe-

[2] As is the case of the non-Euclidean definition of straight line as the line that minimizes the distance between any two given points.

matics [264, p. 9]. Frustration with parallel lines led to the birth of non-Euclidean geometries, which occurred in the first half of that century [297, 136, 264, 246, 135, 311, 257, 134, 275, 348]. The axioms of these non-Euclidean geometries no longer include Euclid's Fifth Postulate. For that reason, these geometries lead to results very different from the classical results of Euclidean geometry. And much less intuitive, more stranger to our daily experience with forms and with their spatial relationships. At the end of the 19th century, E. Beltrami demonstrated the formal consistency of non-Euclidean geometries [25], which implies that, as suspected, Euclid's postulate number 5 cannot be deduced from the other four Euclid's geometric postulates.

But why Euclid's Fifth Postulate should be a theorem and not a postulate? As shocking as it may seem, the answer is related to the role that simplicity and beauty play in the construction of scientific theories, both in the formal sciences (such as geometry) and in the experimental sciences (physics, for example). In this sense, Ockham's razor has always been a good aesthetic reference based on simplicity. And from the aesthetic point of view of simplicity, Euclid's Fifth Postulate lacks the simplicity and self-evidence expected from an axiom or postulate. And if everything indicates that it should be a theorem, why has it been impossible to prove that it was? The answer now has to do with the servitudes of human knowledge. The contrast between Euclidean and non-Euclidean geometries is quite clear:

The Hyperbolic Axiom reads:

> There exists a line l and a point P not in l such that at least two distinct coplanar lines parallel to l pass through P.

The Elliptic Axiom states:

> Through a point exterior to a given line, there is no line parallel to the given line.

While Playfair's Axiom (a variant of Euclid's Fifth Postulate) reads:

> Through a given point one, and only one, parallel can be drawn to a given straight line.

Apart from the non existence of parallels, another notable difference between Euclidean geometry and Riemann elliptic geometry is that in the latter there are infinitely many different straight lines passing through the same two points, which contradicts the strong version of Euclid's First Postulate, according to which there is only one straight line between any two points. Euclid's original statement (weak version of the First Postulate) establishes the existence of AT LEAST one straight line between two points. Hence, his statement is compatible with the existence of more than one straight line between two points. Although it does not seem probable that this was Euclid's belief, nor that of the majority of the subsequent Euclidean authors. In Appendix C, it will be proved that any two points can be the endpoints of one, and only one, straight line (according to the Definition 16 of straight line proposed above). Other abuses of language in non-Euclidean geometries, all of them related to the definition of a straight line, will be discussed in Chapter 14.

As is well known, the general theory of relativity states that, depending on its energy density at the time of its formation, the universe could be closed (elliptical geometry with curvature greater than zero), open (hyperbolic geometry of curvature less than zero) and plane (Euclidean geometry of zero curvature). In the first two cases, the al-Tutsi-Legendre version of Euclid's 5th Postulate [200] is not verified (in the elliptic case the internal angles of a triangle add up to more than 180º and in the hyperbolic case less than 180º). Also in the case of the closed universe, the universe would collapse gravitationally, while in the case of the open universe the expansion would be forever [340, 113, 233, 291, 41, 95, 91, 93, 88, 90, 367, 224, 149, 131, 290, 334, 341, 189, 40].

There is a single value for the initial energy density of the universe that separates closed universes from open universes, the value that corresponds to the flat universe. This unique density is called the critical density ρ_{crit}. The energy density of the present universe has been calculated from astronomical observations in three independent calculations: the energy density due to ordinary matter ($\rho_{om} = 0.049\rho_{crit}$), the energy density due to dark matter ($\rho_{dm} = 0.268\rho_{crit}$) and the energy density of dark energy ($\rho_{de} = 0.683\rho_{crit}$). As can be seen, the sum of the three independent measurements is precisely the critical density ρ_{crit}, which corresponds to that of the flat universe:

$$\rho_{om} + \rho_{dm} + \rho_{de} = \rho_{crit} \tag{1}$$

It should be noted that in the case of a closed universe and in the case of an open universe, the formation of structures such as galaxies would be compromised, at least in the long and very long term. And it should also be noted that the initial energy of the universe had to be such that its initial energy density could not differ from the critical density ρ_{crit} by a factor greater than 10^{-62}, an extremely small factor. Explaining this coincidence is one of the biggest problems facing cosmology today. It also has a great interest in the discussion about the origin of the universe. Obviously, equation (1) can be considered, or not, as a mere random coincidence. But in any case it would be proving that the geometry of physical space is Euclidean.

It is also well known the relativistic explanation of gravity by the general relativity: instead of a force, gravitational attraction would be caused by the local deformation of spacetime, in turn caused by the local presence of massive objects. The geometry of that deformed space would no longer be Euclidean but Riemannian. As will also be seen in Chapter 25 of this book, other physical explanations based on preinertia are possible that do not need to deform neither space nor time.

9. Space light and Gold

9.1 Introduction

This chapter examines some of the ideas about physical space that developed between the first century and the Scientific Revolution of the 16th-17th centuries. During this period the Aristotelian theory of space and place is discussed, discussions in which theology played an important role. Indeed, the words "God" and "place" were used in Alexandrian Jewish theology as practically synonymous words. An equivalence that will be introduced also in Christian theology, which became an important factor in the development of theories about space from the time of Philo (20 B.C.-50 A.D.) to Newton (1642-1727), or even later; that is, at least from the 1st century to the 18th century. With the development of these scientific theories, their corresponding authors also tried to prove in formal terms the existence of God.

9.2 The Neoplatonics

By 146 BC, Greece was already a Roman province, as was Egypt a little later. The two great centers of Greek culture, Athens and Alexandria, came under the political control of Rome. The Roman Empire (29 BC - 476 AD) came to occupy a large part of Europe, from Spain to the Rhine and from North Africa to Persia. More focused on the efficient administration of its vast empire, Rome devoted most of its efforts to the development of law, public administration, and great civil works. Neoplatonism is part of the late Hellenism that developed in Athens and Alexandria during the Roman period. It is a synthesis of Platonic elements, enriched with contributions from other great Greek authors such as Zeno and Aristotle, and Eastern mystical influences from both Hinduism and Judaism of the time. Neoplatonism, in turn, had a great influence on medieval Christian mysticism.

Plotinus (204-270), disciple of Ammonius Sachas (175-242), is one of the most significant authors of Neoplatonism. According to him, there would be a First Principle as the cause of all being, which is why that First Principle, the One, cannot be described as being; it must be understood as beyond being, as something completely indeterminate. This inevitable indeterminacy of the One is the consequence of what we will here call the potentially infinite regress of causes and the logical necessity of a first cause not explainable in terms of other causes, contained in the Theorem of the First Element, deduced from the Principle of Directional Evolution (Chapter 5). For Plotinus, light has the highest degree of existence. It is the medium that maintains the universal order and permeates the space of the whole universe. In its purest reality, light is God. According to the Cabala, the light of the Holy and Infinite One originally occupied the entire universe and then was withdrawn and concentrated into its own substance, creating empty space.

But the first unequivocal redefinition of neoplatonic space is found in the work of the Greek neoplatonic philosopher Proclus (412-485). Indeed, in Proclus we find a definition of space similar to that of the Stoics:

Space is the interval between the limits of bodies.

Consequently, it must be a magnitude commensurable with corporeal objects, although it must also be immaterial and immobile. Space contains the whole material world but is not contained by the material world, being therefore coextensive with the domain of light. This neoplatonic metaphysics of light and space will spread through Jewish philosophy and mysticism and will exert a great influence on most of the natural philosophers of the Renaissance.

The neoplatonic Damascius (458-538?) conducted a profound investigation on the nature of space. A key concept of that research was the position or location of an object. For Damascius, position is an inseparable attribute of every object, and has a double meaning: on the one hand it denotes the relative location of the different parts of the object, and on the other it signifies the position of the whole object in the universe as a whole. If position is a quality of each object, space makes it possible to determine that quality in quantitative terms. In this sense, space is the numerical measure of position.

Space is different from position in the same sense that time is different from motion. Position is an inseparable quality of the object, even when it is in motion. Position is not transferable from one object to another, although always changing it never becomes the position of another object, it simply ceases to exist when the object acquires a new position. One could say that for Damascius space functions as a kind of absolute reference frame.

Damascius also discussed another important issue related to space: the first relativistic question of whether all motion requires the existence of a body at rest. Euclid had already weighed in, saying that an object can appear to be at rest to an observer walking toward it, or it can appear to be in motion to the observer if the observer considers himself to be at rest. But in general, Aristotelian scientists and philosophers thought that motion required the existence of an immovable object (which might imply the immobility of the Earth).

For Damascius, on the contrary, motion does not presuppose the existence of an immobile object, only our perception of motion requires that existence. As Galileo would later say, humans do not have sensors to perceive uniform rectilinear motion [126, p. 529] (as we do, for example, to perceive temperature). Although Damascius still maintains the traditional doctrine of natural places, which remain fixed and motionless, i.e., independent of the actual motion of the concrete parts of the universe. The natural place is, for Damascius, the directing force towards perfection.

J. Philoponus, also called John the Grammarian (490-566), discovered an inconsistency in Aristotle's theory of space: what is the (Aristotelian) place of the sublunar world? According to Aristotle it is the concave surface of the first celestial sphere, which is the orbit of the moon. That sphere is in rotation, but the rotation of the sphere was not considered motion because the sphere, as such a sphere, remains always in the same place. Philoponus disagreed: the parts of the sphere move because they occupy different places over time. Therefore, if the place of an object has to be the first immobile envelope of that object, the place of the sublunar world could not be the concave sphere of the first celestial sphere. Moreover, there is the problem of the place or space in which the outermost celestial sphere moves, because there is no space outside.

For Philoponus, therefore, a new definition of place and space was necessary: space would be a three-dimensional incorporeal volume, different from the objects contained in it. Space and emptiness would be identical. Any region of space could successively receive different material bodies, but space would not intervene in the movement of objects. If objects move toward their natural places it is not because of the intervention of space but because of their tendencies to reach the places that the Demiurge has assigned to them. Changing the Demiurge for the laws of physics, the story of Philoponus acquires a certain relevance. At this point it seems appropriate to recall the words of Copernicus (quoted in Latin in [176, p. 57]):

> In fact, I believe that gravity is nothing other than a certain natural desire given by divine providence to the parts of the

> universe to come together in unity and integrity in the form of a group.

9.3 Arab and Judeo-Christian ideas about space

This section briefly analyzes the path followed by classical Greek science, together with Indian and Arabic science, towards the Christian world. An essential first step for the birth of universal modern science from the so-called Scientific Revolution of the sixteenth and seventeenth centuries.

Science in the Roman Empire

As noted above, by 146 BC, the Roman Empire (29 BC - 476 AD) came to occupy a large part of Europe (from Spain to the Rhine and Persia) and North Africa. More focused on the efficient administration of its huge empire, Rome devoted most of its efforts to the development of law, public administration and great civil works. In 320 AD it recognized the Christian religion, which gradually became the majority religion.

The pressure of the "barbarbarian" peoples of the north eventually led to the fall of the western part of the Roman Empire in 476. The eastern part of the Empire held out until 1453, when the Turks conquered Constantinople. But the barbarian peoples settled in the former western part of the Empire quickly became Romanized, and also Christianized (even before they became Romanized). The common language of the Empire, Latin, remained the common cultured language throughout western Europe until the 18th century, which facilitated the spread of culture throughout the different European regions.

Philosophy aroused less interest in Rome than in Greece. Few Greek authors were translated into Latin in the Western Roman Empire: Plato's Timaeus, Nicomachean Arithmetic and some of Euclid's books. Scientific knowledge was mainly oriented towards practical applications related to surveying and major public works. In this regard, Cicero writes (quoted in [352, p. 68]:

> The Greeks gave the geometrician the highest honor; according to this nothing had a more brilliant progress than mathematics. But we have set as the limit of this art its usefulness for measuring and counting.

Among the authors of scientific-practical works and encyclopedias of the Roman period, the following stand out:

1. Titus Lucretius Carus (198-55 BC)): *De rerum natura*, a work written in verse on Epicurean atomic theory. The work begins with a

very significant principle for contemporary science[1] [221, p. 98-101]:

> We will begin with a principle of his:
> no thing is born from nothing;
> the divine essence cannot do it:
> though it represses all mortals
> fear so that they are inclined
> to believe produced by the gods
> many things of heaven and earth,
> because they do not understand their causes.
> [. . .]
> To this is added the fact that nature
> annihilates nothing, but reduces
> everything to its primitive bodies;

Note the second verse and compare it with the last statement of the following theorem which will be formally proved later (Chapter 5):

Theorem of Formal Dependence: No concept defines itself; no statement proves itself; no physical object is the cause of itself; and no cause is the cause of itself.

Among many other things, Lucretius' work denies action at a distance and infinite division; gives an explanation of colors, sounds and atmospheric phenomena; and expounds a corpuscular theory of light and heat.

2. Marcus Vitruvius Pollio (s. I A.C.): *De architectura*, in which a great variety of problems related to architecture and some strictly scientific problems collected from Greek authors, for example the theory of sound due to waves through the air, are dealt with.
3. Marcus Terentius Varro (116-27 BC): *Disciplinae*, first encyclopedia written in Latin.
4. Marcus Tullius Cicero (106-43 BC): *Somnium Scipionis*, which includes a description of Greek geocentric cosmology.
5. Sextus Iulius Frontinus (40-103 AC): *De aquis urbis Romae*, on aqueducts and water conduction, including some general laws of hydraulics of Greek origin.
6. Lucius Annaeus Seneca (4 BC-65 AC): *Questions*, 7 books on natural phenomena taken from Greek books on meteorology.
7. Gaius Plinius Secundus (Pliny the Elder) (23-74 AC): *Historia naturalis*, 37 books covering most of the knowledge available at the time.

[1] Theorem of the First Element, Corollary of the First Cause and Theorem of the Arrow of Time [207].

8. Martianus Minneus Felix Capella (365-440 AC): *De nuptiis Mercurii et Philologiae*, includes the theory of Heraclides according to which Venus and Mercury revolve around the Sun.

Indian science in the 5th-13th centuries

Between 3000 and 2000 BC, the Indo-Aryan peoples settled in India, and with them began its cultural development. The oldest books (the Vedas) date back to 1500 BC. It was not an isolated culture but maintained cultural contacts with Babylon, Persia, Greece and the Roman Empire. Its cyclical conception of time is well known, linked to its religion, in which there was no real separation between the divinity (Brahma) and the physical world. In contrast to Greece, India paid considerable attention to arithmetic and algebra, focusing more on calculation than on proving, which was undoubtedly due to its excellent numbering system, which would eventually become universal (albeit with Arabic symbols).

The positional (sexagesimal) number system originated in Babylon by the Sumerians, although it was eventually lost. The Indian decimal numbering system was also positional and probably originated in the 7th century. The Arabs copied it, incorporated their own symbols for the first ten numbers (Arabic numerals: 0, 1, 2, 3,. . . 9), and exported it to the territories of their empire. Over time it spread to the rest of the world. Today it is a universal numbering system, with some variations (binary, octal, decimal and hexadecimal). As is well known, in the decimal numbering system the value of each digit in the expression of a number depends on its position in that expression, the value of each position being 10 times greater than that of the previous position, to the left of its writing. The decimal numbering system allows numbers of any size to be represented. Although the latter is just a figure of speech. Indeed, one can define natural numbers (the counting numbers, all of which are finite) so enormous that there is not enough matter in the Universe to represent them graphically at a standard scale of for example 5mm/digit (see Chapter 16).

In addition to the decimal numbering system, including the use of zero, Indian mathematicians of the time also developed algorithms for the four basic arithmetic operations, both with positive and negative numbers, as well as basic notions of trigonometry and methods of solving first-degree and second-degree equations and systems of equations. They also knew some basic theorems of plane geometry. All this Indian mathematical knowledge would pass to the West through Arab authors, including their own contributions.

Arab science in the 7th-13th centuries

Islam as a religious doctrine was established in the Koran around the year 640. From that date, the Umayyads spread throughout North Africa and entered southern Europe, invading part of the Iberian Peninsula. In Cordoba (Spain), an independent caliphate is created. Baghdad and Cordoba become the most important cultural centers of the time. Astronomy and mathematics are the most developed disciplines in both centers. From the year 800 onwards, translations of the great Greek and Indian authors began, which, in addition to being translated into Arabic, were widely commented on. This work was completed in approximately 100 years. From the 9th to the 12th centuries, a period of great splendor developed, especially in mathematics and astronomy. The incursions of the Turks in the 14th century caused the beginning of the decline of this period of splendor in Arab scientific culture, not sufficiently recognized in our arrogant western world.

Among the most noteworthy aspects of this period of Arab scientific splendor, the following can be highlighted:

1. Introduction and use of the Indian decimal numbering system, with its own symbols, including zero.
2. Calculation procedures for square and cube roots.
3. Modern use of fractions.
4. Development of the algebra of first and second degree equations.
5. Method for calculating areas and volumes precursor of integrals.
6. Development of trigonometry applied to astronomy.
7. Important development of astronomy with the construction of astronomical observatories and the perfection of instruments such as the astrolabe.
8. Thabit ibn Kurra (836-901 AC) raises a very significant numerical paradox, antecedent of Galileo's infinitist paradox [318, 78, 212].
9. Development of applied mathematics, especially in optics and mechanics.
10. Interpretation of light as particle rays with finite velocity and different for each transparent medium.
11. Establishment of the laws of the reflection of light.
12. Study of the refraction of light, establishing some of its laws, although not the law of sines.
13. Interpretation of the rainbow as a phenomenon caused by the interaction of light with water droplets in the air.
14. Translation and commentary of all the great philosophical and

mathematical works of the Greeks. In this regard, it is appropriate to recall the following words of Alhazen (965-1040), quoted in [352, p. 79]:

> The seeker of truth is not one who studies the writings of the ancients and following their natural disposition puts his trust in them, but, rather, one who suspects his faith in them and questions what they present [...] Thus the obligation of one who investigates the writings of the philosophers, if he seeks to learn the truth, is to make himself an enemy of all he reads, and applying his mind to their contents, to attack them from all their angles[2].

15. Critique of Aristotle's theory of motion, proposing ideas similar to the theory of momentum (linear momentum).
16. Studies of static equilibrium and analysis of the centers of gravity.
17. Averroes (1126-1198 AC), the great commentator of Aristotle, maintains that there are two ways to reach the truth: reason and the revelation of the Koran.
18. According to Averroes himself, nothing in the world is born or destroyed, but is transformed, an idea that underlies the later statements of the Principle of Conservation of Matter.

Latin translations of the Greek authors

After its reconquest in 1085 by Alfonso VI of Castile, the city of Toledo (Spain) became the center of contact between Christian, Arab and Jewish cultures. At the same time, translations into Latin of Greek works, previously translated into Arabic, and of Arabic works began. At the end of the 11th century, Sicily and southern Italy also became centers of translation of Greek works. In both cases, the translated works were almost exclusively scientific. It is not possible to understand the subsequent history of European science, since the Scientific Revolution, without these translations. They occupy, therefore, a relevant place in the history of physics, including our debate on the reality of physical space.

With the Greek works were also translated the commentaries of their Arabic translators, and some scientific works of the Arabs themselves, including their Arabic numerals, the decimal system of numeration, and Indian arithmetic and algebra, as they are still used today throughout the world. This, together with the translation of Aristotle's Logic, Euclid's geometry and optics, Archimedes' mechanics and the Arabic works on optics and mechanics (by Alhazen and Al Farisi (1260-1320 AC), for example), led to a true scientific revolution in the twelfth and

[2]A strategy that should be applied today by all future scientists.

thirteenth centuries. The religious and anti-religious intolerances of our time should look to this wonderful light of knowledge instead of their present dark prejudices.

From the same period are the monastic schools and the European cathedral schools, which from the 12th century onwards will become universities, as is the case of Paris, Oxford, Salamanca, etc. Their role in the development of science will be fundamental up to the present day, including what in my opinion is still a very negative aspect for scientific progress: the intolerant nature of the main currents of thought, which emerged, and continue to emerge, in university centers all over the world. It is worth recalling at this point the words of Adelard of Bath (1080-1152 AC) (quoted in [352, p. 93]):

> One thing is what I have learned from the Arabian masters, under the guidance of reason, and another thing is what you, seduced by the mask of authority, are tied to like a yoke. For what other name but yoke is there to give to authority? You allow yourselves to be led by authority like animals that do not know where they are being led or why.

Fortunately, there are no longer prison sentences (or worse) for intellectual dissidence, but there are sentences of insult and ostracism. I attest to this.

Atomic theory of space in Kalam

Once Greek philosophy became familiar in the Arab world, particularly in the case of the Umayyads, the authority of Aristotle prevailed in almost all physical and metaphysical matters. One of the few exceptions was the Aristotelian theory of space, against which a discrete (atomic) alternative was constructed within the philosophical and theological current known as Kalam (9th-10th century), perhaps comparable to medieval Christian scholastic mysticism. Here, too, the dialectical method was used as a support for theological speculations. The atomic theory of Kalam did not originate in religious speculations, although it was from the religious background that it drew its emotional force of conviction. Kalam was defined as the science of the fundamentals of faith and intellectual proofs in support of theological truths.

According to Kalam, matter is formed by indivisible particles, atoms, equal to each other and devoid of spatial extension. The extension arises, in the three spatial directions, from the establishment of relations between different atoms. The existence of these atoms would be transient, of a very short duration, which requires the continuous divine intervention to maintain the coherence and continuity of the universe. Another important characteristic of the atomic doctrine of

Kalam is the necessary existence of empty space, which, like matter, must also be formed by indivisible units, and the same must apply to time.

Consequently, motion must also be discrete, in jumps; it must consist of a discontinuous succession of jumps in each of which successive positions in space will be occupied. The successive jumps would be separated by one or more discrete units of time in which the corresponding atoms remain at rest, in the same units of space. Slower objects separate their jumps by a greater number of units of time in which they remain occupying the same atoms of space. Interestingly, the theory of physical space proposed at the end of this book, although deduced exclusively in physical and mathematical terms, is reminiscent in many respects of the theory of Kalam.

Judeo-Christian ideas about space

There is sufficient evidence that the Judeo-Christian religious tradition exerted a remarkable influence on the development of physical theories of space from the first to the eighteenth century. Space was nothing but an attribute of God, even the same thing as God. In Palestinian Jewish culture the word "place" was frequently used to denote God. And the divine omnipresence, God occupies all space, is the consequence of a long process of theological thought (which did not occur in the polytheistic religions). As we shall see later in this section, for H. More space is the divine extension. And remember that for Newton space is the divine sensorium.

Another trend in the history of theories of space, very similar to its mystical-theological character and the association of God with space, was the identification of space with light. Light is the medium in which God becomes visible to man: *Ego sum lux mundi.* As noted above, the Infinite Sacred Oneness, whose light originally occupied the entire universe, withdrew its light and concentrated on its own substance, thereby creating empty space. This apotheosis of light became a fundamental feature of late Neo-Platonism and medieval mysticism.

The Franciscan R. Grosseteste (1175-1253) was one of the first scholastics who defended the neoplatonic metaphysics of light: he assumed that light was the first corporeal form and the first principle of motion. The creation of space in the universe was the self-diffusion of light, which according to Grosseteste propagates instantaneously, as can be proved, according to Grosseteste himself, with visible light, which is the basis of spatial extension. Hence the importance of the study of optical geometry and the great interest in mathematics and optics in the 13th century. Light was the means by which universal order was maintained. In its pure reality light was God. In the words of St.

Bonaventure (1221-1274), God was the spiritual light that is actualized in all the senses. The Polish friar Witelo (1230-1314), clearly influenced by Grosseteste, is another Neoplatonic advocate of the identity of light and space: light is the source of all existence, the all-pervading power. Space and light are one.

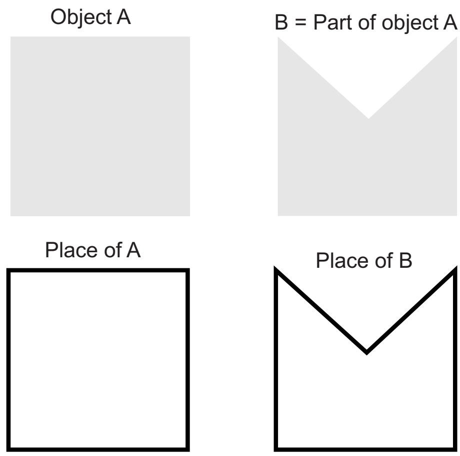


Figure 9.1 – Crescas' Paradox: The aristotelian place of the whole (Left) is less than the aristotelian place of one if its parts (Right).

The Jewish philosopher and jurist Hasdai Crescas (1340-1412) discovered some inconsistencies in the Aristotelian theory of place, among them the so-called Crescas Paradox: the Aristotelian place of the whole can be less than the place of one of its proper parts (Figure 9.1). Consequently, he proposed to change the Aristotelian definition so that the place occupied by any object would always be equal to the sum of the places of its parts, whatever the division of the object into parts.

The theologian and scientist Nicholas of Cusa (1401-1464) offered another solution to the Aristotelian problem of the definition of place: The circumference and the center of the universe could only be God, but from the physical point of view it is absurd to be at the same time the circumference and its center. There is, then, neither the circumference nor the center of the universe. Consequently, the Earth is not the center of the universe or of space, nor is a body at rest necessary for motion to exist, which eliminates the possibility of absolute motion. It would therefore be a relativistic theory of position.

For Tomaso Campanella (1568-1639) space was an absolute spiritual entity characterized by divine attributes. Space was homogeneous and undifferentiated, immobile and incorporeal, penetrated by matter and penetrating matter, destined for the placement of mobile entities. Campanella states that space is in God, but God is not limited by space, space is His "divine creature". It is important to note that P. Gassendi (1592-1655) was in contact with Campanella, and Gassendi was in contact with Newton (1642-1727) and Leibniz (1646-1716).

R. Descartes (1596-1650) identified matter with extension, its key quality. There would be three kinds of matter: ether, the most subtle, identified with space itself; luminous matter, which forms the Sun and the stars; and denser matter, which forms the Earth and the planets. *Empty space* does not exist: a vessel from which all air is extracted must collapse. Space is thus matter, not a separate entity containing matter. Euclidean geometry correctly describes space, and was used by Descartes (and Leibniz) to introduce coordinate systems. As for motion, Descartes explains it on the basis of three fundamental laws [79, p. 421-422]:

1. Each thing remains in the state in which it is as long as nothing modifies that state.
2. Every body that moves tends to continue its movement in a straight line.
3. If a moving body collides with another body stronger than itself, it does not lose any of its movement, but if it meets another weaker body that can move, it loses as much movement as it communicates to the other.

Naturally, the Cartesian theory of space has to overcome the Aristotelian objection that the place occupied by matter and matter itself must be separable. Descartes proposes two ways of conceiving the problem of the extension occupied by a moving object:

1. The extension of a moving object moves with the object: particular non-separable extension.
2. The moving body moves with respect to a fixed extent defined by its position relative to other objects: generic separable extent.

But although it can be thought of as separable and non-separable, in reality this is not possible at the same time. When an object moves, does it remain made up of the same matter, or does the matter remain for the properties to move and attach to successive pieces of matter? The solution Descartes seems to give is that although the matter and place of an object are made of the same substance at any given time, its place is to be identified by its form and relative location.

An outstanding example in which a strong religious bias can be seen in the conception of space is the theory of Henry More (1614-1687). More considers it necessary to complete Descartes' science with cabalistic and Platonic concepts. As for his theory of space, More refers to the cabalistic doctrine as explained by Cornelius Agrippa (1486-1535) in his *De occulta philosophia*, in which space is specified as one of the attributes of God (quoted in [176, p. 42]):

> And so through that same door through which Cartesian philosophy seems to want to exclude God from the world, I, on the contrary, enter again and strive to introduce him.

The great motive behind More's preoccupation with the problem of space, like that of his whole philosophy, is to find a convincing demonstration of the indubitable reality of God, spirit and soul. He rejects then the Cartesian identification of matter with extension. To demonstrate the existence of spirit it is enough to show that extension is spiritual, provided that extension is real. On the basis of this reasoning, More's treatment of space can be divided into three parts:

1. Extension is not the distinguishing attribute of matter. Matter has the property of impenetrability; impenetrability is the distinguishing criterion between matter and extension. Space is the common ground between the world of matter and the world of spirit. Extension is not an attribute of matter alone but of matter and spirit.
2. Space is real, with real attributes. *Empty space* does not exist for More. But if space is empty of matter it will be filled with spirit. The existence of space is guaranteed by its own measurability. Since there is no accident (measurement) without cause, its measurement proves its existence. Space is incorporeal because it is penetrable, which proves that it is different from matter.
3. Space is of divine character. The necessary existence of space, even without matter, leads More to the final identification of space with God. Space and God both have the property of necessary existence, they are thus the same thing.

More had a great influence on J. Locke (1632-1704), I. Newton (1643-1727) and S. Clarke (1675-1729) and on the eighteenth-century philosophy.

Some Arabic translators and commentators of Aristotle's work had updated the old contradiction between the Aristotelian definitions of place and motion: the ultimate celestial sphere would be in motion without being able to be in motion, since there is nothing beyond the ultimate sphere itself, that ultimate sphere would have to move without a place in which to move. It was necessary to change at least one of the two definitions. William of Ockham (1287-1347) proposed to use the notion of distance from an object to another object of reference to define the object's position. The immobility of a given place was reduced to the constancy of its distance to a given reference body (quoted in [176, p. 72]):

> ... If you are at rest, and even if all the air around you, or any body which surrounds you, is moving, your are always at the same place; for you are always at the same distance

> from the center of the poles of the universe. With regard to these the place is therefore called immobile.

On the contrary, N. Copernicus (473-1543) was in favor of eliminating the rotational motion of the outer celestial sphere [72, Book 1, Chapter 5, P. 23]:

> Moreover, since the heavens, which enclose and provide the setting for everything, constitute the space common to all things, it is not at first blush clear why motion should not be attributed rather to the enclosed than to the enclosing, to the thing located in space rather than to the framework of space. This opinion was indeed maintained by Heraclides and Ecphantus, the Pythagoreans, and by Eficetas of Syracuse, according to Cicero. They rotated the earth in the middle of the universe, for they ascribed the setting of the stars to the earth's interposition, and their rising to its withdrawal.

The consequences of these ideas of Copernicus are well known, and will be developed in the rest of the book.

10. Newton absolute space

10.1 Introduction

Newton's Mathematical Principles of Natural Philosophy (Principia) [260] is rightly considered one of the essential works in the history of human thought. First published in 1687, it is still being published and its study (direct or indirect) is still mandatory to know the foundations of classical mechanics, and even of modern mechanics. Newton's Principia represent for mechanics what Euclid's Elements represent for geometry: they contain the first hypothetico-deductive model for rational mechanics. The model of space used in the Principia is, moreover, the Euclidean three-dimensional model.

As is well known, Newton defended the absolute nature of space, time and motion. Or in other words: for Newton, space, time and motion were real, not fictitious or relative. This chapter is devoted to these three absolute (real) notions of Newton, which, moreover, are also closely related to those proposed in this book, although here the general perspective of the continuous will be exchanged for the discrete and finite perspective. By way of dialectical contrast, the chapter also includes the relational view of space defended by Leibniz during Newton's own time, Although Newton's absolute space prevailed over Leibniz's relational space, the latter will be the only one considered by modern physics, which, as everyone knows, is essentially relativistic. A relational view that will eventually prevail until it becomes the practically unique view of space, time and motion in contemporary physics. Dissent also exists, but in a very marginal way, without echo in the "orthodox" scientific community of our days (2024).

10.2 The formal language of Newton's Principia

As the first hypothetico-deductive construction of mechanics, the Principia are compatible with more than one interpretation of the physical world, which among other attributes could be infinite or finite, continuous or discrete, although those matters are not discussed in Newton's text. The Principia consist of three books:

Books I y II: On the motion of bodies.

Book III: On the world system.

The formal content of the three books is richer than that of Euclid's Elements, and includes: definitions, laws, lemmas, propositions, theorems, corollaries, problems, additional hypotheses, additional definitions, rules for philosophizing, phenomena, scholia and examples; distributed as follows:

1. Book I: 8 definitions; 3 laws with 6 corollaries; 29 lemmas; 98 propositions, of which 50 are theorems; 48 problems.

2. Book II: 1 additional definition (of fluid); 1 additional hypothesis; 7 lemmas; 53 propositions, of which 41 are theorems, and 12 problems.

3. Book III: 4 rules for philosophizing; 2 additional hypotheses; 6 phenomena; 11 Lemmas; 42 propositions of which 20 are theorems and 22 problems.

   Some of the lemmas, propositions and theorems also include corollaries and scholia (commentaries).

Geometrical representations and proves are very abundant in the Principia, where geometry (and mathematics in general) have a very realistic, or naturalistic, use: they are not a simple set of logically related formal elements but in a certain way a part of mechanics. The laws (axioms) and hypotheses of the Principia are not arbitrary statements disconnected with the real world; for Newton they are an inevitable consequences of the immediate experience with the physical world. Although he makes a distinction between science and metaphysics (quoted in [176, p. 97]):

> I consider here mathematical quantities, not as composed of the smallest possible parts, but as described by continuous motion. These genes have a real place in nature and are seen in the motion of bodies every day.

Newton sought to separate science from the religious and transcendental, with the sole exception of space, one of the four fundamental concepts for the study of nature. The other three are time, mass, and force, which are as real as space. Although, as is always the case with the most basic concepts, their definitions are imprecise or circular. Even Newton's definitions. This inevitable limitation of human knowledge is hardly ever acknowledged. Not even in the Principia. Here we will devote a good part of Chapter 5 to it.

Unlike the Cartesian model, not only space ceases to be relative to be absolute, i.e. real, but also extension ceases to be the (Cartesian) fundamental attribute of matter, which is now its mass. Although the definition given by Newton is circular. The same happens with the definition of impressed force and centripetal force [260, p. 121-123]:

> Definition I. *The quantity of matter is the measure of the same, arising from its density and bulk conjunctly.*
>
> Definition IV. *An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of moving uniformly forward in a right line.*
>
> Definition V. *A centripetal force is that by which bodies are drawn or impelled, or any way tend, towards a point as to a centre.*
>
> (The innate force (inertia) of Newton's Definition III is discussed in Chapter 19 of this book).

Newton immediately calls the quantity of matter of Definition I *mass* or *body* [260, p. 121]. The primitive nature of the concepts of space and time are discussed beginning in the next section of this chapter and in other chapters of this book.

10.3 Space, time and motion in the Principia

In the scholium to his first eight Initial Definitions, Newton tells us that he thought it appropriate to explain the lesser known terms and the sense in which they are to be taken in the future [260, p. 127]. These are the terms included in his first eight Initial Definitions. As for time, space, place and motion, Newton tells us in that same scholium:

> I do not define time, space, place and motion, as being well known to all.

It is Newton's (not too honest) way of getting around the inevitable problem of primitive concepts. Or perhaps he did not realize that this was indeed the problem he faced. And this was done by, in my opinion, the most important scientist in the history of science, to whom I once again express my deepest admiration.

Although there is no formal definition of space (if there were, it would have to be in terms of at least another more basic concepts that would also have to be defined, or declared as new primitive concepts), in the scholium that follows his first eight Initial Definitions, Newton explains the differences between absolute space, time and motion and their corresponding relative versions, as well as how absolute motion could be detected experimentally [260, p. 127-134]:

> Hitherto I have laid down the definitions of such words as

are less known, and explained the sense in which I would have them to be under stood in the following discourse. I do not define time, space, place and motion, as being well known to all. Only I must observe, that the vulgar conceive those quantities under no other notions but from the relation they bear to sensible objects. And thence arise certain prejudices, for the removing of which, it will be convenient to distinguish them into absolute and relative, true and apparent, mathematical and common.

I. Absolute, true, and mathematical time, of itself, and from its own nature flows equably without regard to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year.

II. Absolute space, in its own nature, without regard to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies; and which is vulgarly taken for immovable space; such is the dimension of a subterraneous, an aereal, or celestial space, determined by its position in respect of the Earth. Absolute and relative space, are the same in figure and magnitude; but they do not remain always numerically the same. For if the Earth, for instance, moves, a space of our air, which relatively and in respect of the Earth remains always the same, will at one time be one part of the absolute space into which the air passes; at another time it will be another part of the same, and so, from the absolute point of view, it will be perpetually mutable.

III. Place is a part of space which a body takes up, and is according to the space, either absolute or relative. I say, a part of space; not the situation, nor the external surface of the body. For the places of equal solids are always equal; but their surfaces, by reason of their dissimilar figures, are often unequal. Positions properly have no quantity, nor are they so much the places themselves, as the properties of places. The motion of the whole is the same thing with the sum of the motions of the parts; that is, the translation of the whole, out of its place, is the same thing with the sum of the translations of the parts out of their places; and therefore the place of the whole is the same thing with the sum of the places of the parts, and for that reason, it is internal, and in the whole body.

IV. Absolute motion is the translation of a body from one absolute place into another; and relative motion, the translation from one relative place into another. Thus in a ship under sail, the relative place of a body is that part of the ship which the body possesses; or that part of its cavity which the body fills, and which therefore moves together with the ship: and relative rest is the continuance of the body in the same part of the ship, or of its cavity. But real, absolute rest, is the continuance of the body in the same part of that immovable space, in which the ship itself, its cavity, and all that it contains, is moved. Wherefore, if the Earth is really at rest, the body, which relatively rests in the ship, will really and absolutely move with the same velocity which the ship has on the Earth. But if the Earth also moves, the true and absolute motion of the body will arise, partly from the true motion of the Earth, in immovable space; partly from the relative motion of the ship on the Earth; and if the body moves also relatively in the ship; its true motion will arise, partly from the true motion of the Earth, in immovable space, and partly from the relative motions as well of the ship on the Earth, as of the body in the ship; and from these relative motions will arise the relative motion of the body on the Earth. As if that part of the Earth, where the ship is, was truly moved toward the east, with a velocity of 10010 parts; while the ship itself, with a fresh gale, and full sails, is carried towards the west, with a velocity expressed by 10 of those parts; but a sailor walks in the ship towards the east, with 1 part of the said velocity; then the sailor will be moved truly in immovable space towards the east, with a velocity of 10001 parts, and relatively on the Earth towards the west, with a velocity of 9 of those parts.

Absolute time, in astronomy, is distinguished from relative, by the equation or correction of the vulgar time. For the natural days are truly unequal, though they are commonly considered as equal, and used for a measure of time; astronomers correct this inequality for their more accurate deducing of the celestial motions. It may be, that there is no such thing as an equable motion, whereby time may be accurately measured. All motions may be accelerated and retarded; but the true, or equable, progress of absolute time is liable to no change. The duration or perseverance of the existence of things remains the same, whether the motions are swift or slow, or none at all: and therefore it ought to be distinguished from what are only sensible measures thereof; and out of which we collect it, by means of the astronomical equation. The necessity of which equation, for determining the times of a phenomenon, is evinced as well from the experiments of the pendulum clock, as by eclipses of the satellites of Jupiter.

As the order of the parts of time is immutable, so also is the order of the parts of space. Suppose those parts to be moved out of their places, and they will be moved (if the expression may be allowed) out of themselves. For times and spaces are, as it were, the places as well of themselves as of all other things. All things are placed in time as to order of succession; and in space as to order of situation. It is from their essence or nature that they are places; and that the primary places of things should be moveable, is absurd. These are therefore the absolute places; and translations out of those places, are the only absolute motions.

But because the parts of space cannot be seen, or distinguished from one another by our senses, therefore in their stead we use sensible measures of them. For from the positions and distances of things from any body considered as immovable, we define all places; and then with respect to such places, we estimate all motions, considering bodies as transferred from some of those places into others. And so, instead of absolute places and motions, we use relative ones; and that without any inconvenience in common affairs; but in philosophical disquisitions, we ought to abstract from our senses, and consider things themselves, distinct from what are only sensible measures of them. For it may be that there is no body really at rest, to which the places and motions of others may be referred.

But we may distinguish rest and motion, absolute and relative, one from the other by their properties, causes and effects. It is a property of rest, that bodies really at rest do rest in respect to one another. And therefore as it is possible, that in the remote regions of the fixed stars, or perhaps far beyond them, there may be some body absolutely at rest; but impossible to know, from the position of bodies to one another in our regions whether any of these do keep the same position to that remote body; it follows that absolute rest cannot be determined from the position of bodies in our regions.

It is a property of motion, that the parts, which retain given positions to their wholes, do partake of the motions of those wholes. For all the parts of revolving bodies endeavour to recede from the axis of motion; and the impetus of bodies moving forward, arises from the joint impetus of all the parts. Therefore, if surrounding bodies are moved, those that are relatively at rest within them, will partake of their motion. Upon which account, the true and absolute motion of a body cannot be determined by the translation of it from those which only seem to rest; for the external bodies ought not only to appear at rest, but to be really at rest. For otherwise, all included bodies, beside their translation from near the surrounding ones, partake likewise of their true motions; and though that translation were not made they would not be really at rest, but only seem to be so. For the surrounding bodies stand in the like relation to the surrounded as the exterior part of a whole does to the interior, or as the shell does to the kernel; but, if the shell moves, the kernel will also move, as being part of the whole, without translation of the shell vicinity.

A property, near akin to the preceding, is this, that if a place is moved, whatever is placed therein moves along with it; and therefore a body,which is moved from a place in motion, partakes also of the motion of its place. Upon which account, all motions, from places in motion, are no other than parts of entire and absolute motions; and every entire motion is composed of the motion of the body out of its first place, and the motion of this place out of its place; and so on, until we come to some immovable place, as in the before-mentioned example of the sailor. Wherefore, entire and absolute motions can be no otherwise determined than by immovable places: and for that reason I did before refer those absolute motions to immovable places, but relative ones to movable places. Now no other places are immovable but those that, from infinity to infinity, do all retain the same given position one to another; and upon this account must ever remain unmoved; and do thereby constitute immovable space.

The causes by which true and relative motions are distinguished, one from the other, are the forces impressed upon bodies to generate motion.True motion is neither generated nor altered, but by some force impressed upon the body moved: but relative motion may be generated or altered without any force impressed upon the body. For it is sufficient only to impress some force on other bodies with which the former is compared, that by their giving way, that relation may be changed, in which the relative rest or motion of this other body did consist. Again, true motion suffers always some change from any force impressed upon the moving body; but relative motion does not necessarily undergo any change by such forces. For if the same forces are likewise impressed on those other bodies, with which the comparison is made, that the relative position may be preserved, then that condition will be preserved in which the relative motion consists. And therefore any relative motion may be changed when the true motion remains unaltered, and the relative may be preserved when the true suffers some change. Upon which accounts; true motion does by no means consist in such relations.

The effects which distinguish absolute from relative motion are, the forces of receding from the axis of circular motion. For there are no such forces in a circular motion purely relative, but in a true and absolute circular motion, they are greater or less, according to the quantity of the motion. If a vessel, hung by a long cord, is so often turned about that the cord is strongly twisted, then filled with water, and held at rest together with the water; after, by the sudden action of another force, it is whirled about the contrary way, and while the cord is untwisting itself, the vessel continues for some time in this motion; the surface of the water will at first be plain, as before the vessel began to move: but the vessel, by gradually communicating its motion to the water, will make it begin sensibly to revolve, and recede by little and little from the middle, and ascend to the sides of the vessel, forming itself into a concave figure (as I have experienced), and the swifter the motion becomes, the higher will the water rise, till at last, performing its revolutions in the same times with the vessel, it becomes relatively at rest in it. This ascent of the water shows its endeavour to recede from the axis of its motion; and the true and absolute circular motion of the water, which is here directly contrary to the relative, discovers itself, and may be measured by this endeavour. At first, when the relative motion of the water in the vessel was greatest, it produced no endeavour to recede from the axis; the water showed no tendency to the circumference, nor any ascent towards the sides of the vessel, but remained of a plain surface, and therefore its true circular motion had not yet begun. But afterwards, when the relative motion of the water had decreased, the ascent thereof towards the sides of the vessel proved its endeavour to recede from the axis; and this endeavour showed the real circular motion of the water perpetually increasing, till it had acquired its greatest quantity, when the water rested relatively in the vessel. And therefore this endeavour does not depend upon any translation of the water in respect of the ambient bodies, nor can true circular motion be defined by such translation. There is only one real circular motion of any one revolving body, corresponding to only one power of endeavouring to recede from its axis of motion, as its proper and adequate effect; but relative motions, in one and the same body, are innumerable, according to the various relations it bears to external bodies, and like other relations, are altogether destitute of any real effect, any otherwise than they may perhaps partake of that one only true motion. And therefore in their system who suppose that our heavens, revolving below the sphere of the fixed stars, carry the planets along with them; the several parts of those heavens, and the planets, which are indeed relatively at rest in their heavens, do yet really move. For they change their position one to another (which never happens to bodies truly at rest), and being carried together with their heavens, partake of their motions, and as parts of revolving wholes, endeavour to recede from the axis of their motions.

Wherefore relative quantities are not the quantities themselves, whose names they bear, but those sensible measures of them (either accurate or inaccurate), which are commonly used instead of the measured quantities themselves. And if the meaning of words is to he determined by their use, then by the names time, space, place and motion, their measures are properly to be understood; and the expression will be unusual, and purely mathematical, if the measured quantities themselves are meant. Upon which account, they do strain the sacred writings, who there interpret those words for the measured quantities. Nor do those less defile the purity of mathematical and philosophical truths, who confound real quantities themselves with their relations and vulgar measures.

It is indeed a matter of great difficulty to discover, and effectually to distinguish, the true motions of particular bodies from the apparent; be cause the parts of that immovable space, in which those motions are per formed, do by no means come under the observation of our senses. Yet the thing is not altogether desperate: for we have some argu-

> ments to guide us, partly from the apparent motions, which are the differences of the true motions; partly from the forces, which are the causes and effects of the true motions. For instance, if two globes, kept at a given distance one from the other by means of a cord that connects them, were revolved about their common centre of gravity, we might, from the tension of the cord, discover the endeavour of the globes to recede from the axis of their motion, and from thence we might compute the quantity of their circular motions. And then if any equal forces should be impressed at once on the alternate faces of the globes to augment or diminish their circular motions, from the increase or decrease of the tension of the cord, we might infer the increment or decrement of their motions: and thence would be found on what faces those forces ought to be impressed, that the motions of the globes might be most augmented; that is, we might discover their hinder most faces, or those which, in the circular motion, do follow. But the faces which follow being known, and consequently the opposite ones that precede, we should likewise know the determination of their motions. And thus we might find both the quantity and the determination of this circular motion, even in an immense vacuum, where there was nothing external or sensible with which the globes could be compared. But now, if in that space some remote bodies were placed that kept always a given position one to another, as the fixed stars do in our regions, we could not indeed determine from the relative translation of the globes among those bodies, whether the motion did belong to the globes or to the bodies. But if we observed the cord, and found that its tension was that very tension which the motions of the globes required, we might conclude the motion to be in the globes, and the bodies to be at rest; and then, lastly, from the translation of the globes among the bodies, we should find the determination oi their motions. But how we are to collect the true motions from their causes, effects, and apparent differences; and, vice versa, how from the motions, either true or apparent, we may come to the knowledge of their causes and effects, shall be explained more at large in the following tract. For to this end it was that I composed it.

For Newton, his First Law of Mechanics is a fact of immediate experience that requires absolute space. Thus, space becomes a logical and ontological necessity. But Newton's mechanics is invariant with respect to Galileo's Transformation, therefore his reference frame (the reference frame of the world) is not determined in a single way. And so he admits it in Corollary V, which anticipates the modern Principle of Relativity [260, p. 144]:

> The motions of bodies included in a given space are the same among themselves, whether that space is at rest, or moves uniformly forwards in a right line without any circular motion.

In order to solve this problem, Newton introduces Hypothesis I in Book III of the Principia [260, p. 641]:

> The center of the world system is at rest.

A hypothesis that cannot be experimentally tested, but which he uses to prove Theorem XI of Book III [260, p. 641]:

> The common center of gravity of the Earth, the Sun and all planets is at rest.

Note that Newton does not take into account the stars, whose relative motions were not known to Newton. But space was not only a logical and ontological necessity for Newton, in the general scholium of Book III, and in line with the thought of H. More, Newton identifies space and time with attributes of God (italics mine) [260, p. 783]:

> And from his true dominion it follows that the true God is a living, intelligent, and powerful Being; and, from his other perfections, that he is supreme, or most perfect. He is eternal and infinite, omnipotent and omniscient; that is, his duration reaches from eternity to eternity; his presence from infinity to infinity; he governs all things, and knows all things that are or can be done. He is not eternity or infinity, but eternal and infinite; he is not duration or space, but he endures and is present. He endures for ever, and is every where present; and by existing always and every where, *he constitutes duration and space.*

It is therefore surprising that a 21st century quantum physicist writes in the foreword of a book (about physics and God) [319, p. 7]:

> In the 17th century, Isaac Newton gave birth to a mathematical science that almost eliminated the idea of God's intervention in the material world of physics and chemistry.

The authors of this elimination were Newton's critics, all of them convinced relationists and relativists. As the reader of this book will see in Chapter 5, it is possible to develop a formal demonstration that the observable universe had to have a first cause exterior to the universe itself.

10.4 Critique of Newton's absolute space

The real or illusory nature of Newton's absolute space does not seem an irrelevant scientific matter. It is therefore striking that some relativists deny the real existence of space simply because that existence is not necessary to explain motion. For that same reason they accused Newton of being contradictory, since his First Rule for Philosophizing says [260, p. 615]:

> No more causes of natural things should be admitted than those that are true and sufficient to explain their phenomena.

As if the real or illusory nature of space were not itself a major scientific issue in describing the universe. In any case we would have two ways of explaining motion, which does not detract from the importance of the problem of the physical reality of space. Naturally, one can argue for or against the reality of space, as G. Leibniz (1646-1716) did, for example, in this case against that physical reality and in favor of a purely relational interpretation of space (quoted in [176, p. 117]):

> I will here show, how Men come to form to themselves the Notion of Space. They consider that many things exist at once, and they observe in them a certain Order of Co-existence, according to which the relation of one thing to another is more or less simple. This Order is their Situation or Distance. When it happens that one of those Co-existent Things changes its Relation to a Multitude of others, which do not change their Relation among themselves; and that another Thing, newly come, acquires the same Relation to the others, as the former had; we then say it is come into the Place of the former; and this Change we call Motion in That Body, wherein it is the immediate Cause of Change. And though Many, or even All the Co-existing Things, should change according to certain known Rules of Direction and Swiftness; yet one may always determine the Relation of Situation, which every Co-existent acquires with respect to every other Co-existent; and even That Relation, which any other Co-existent would have to this, or which this would have to any other, if it had not changed or if it had changed any otherwise. And supposing, or feigning, that among those Co-existents, there is a sufficient Number of them, which have undergone no Change; then we may say, that Those which have such a Relation to those fixed Existents, as Others had to them before, have now the same Place which those others had. And That which comprehends all those Places, is called Space.

So, for Leibniz, space is relational, a simple order of co-existence; a situation of bodies among themselves. He also claim that all empirical knowledge can be derive from a simple axiom: the Principle of Sufficient Reason, that reads:

> There ought to be some sufficient reason why things should be so, and not otherwise.

from which one could prove the existence of God, but not that God is space, for if such were the case God, like space, would be infinitely divisible, which is absurd. And although Newton did not say that space was an organ of God, Leibniz used this assertion in his criticism of Newtonian space (quoted in [176, p. 114]):

> Sir Isaac Newton says, that Space is an Organ, which God makes use of to perceive Things by. But if God stands in need of any Organ to perceive Things by, it will follow, that they do not depend altogether upon him, nor were produced by him.

Statement answered by S. Clarke with the following words ([176, p. 114]):

> Sir Isaac Newton doth not say, that Space is the Organ which God makes use of to perceive Things by; nor that he has need of any Medium at all, whereby to perceive Things; But on the contrary, that he, being omnipresent, perceives all Things by his immediate Presence to them, in all Space whereever they are, without the Intervention or Assistance of any Organ or Medium whatsoever.

Leibniz developed other arguments against Newton's absolute space, two of the best known being the static shift argument and the kinematic shift argument (taken from [164, p. 162-163]):

> **The static shift**: Imagine a second universe just like ours except that all the matter is located in (i.e. shifted to) another place in absolute space, without any change in the relations of one object to another. Since space is a Euclidean plane, the two places are exactly alike, and so no differences will be seen.
>
> **The Kinematic shift**: Imagine a second universe just like ours except that the absolute velocity of every piece of matter differs by (i.e. is shifted by) a fixed, constant amount, without any change in the relations of one object to another. Since the two velocities differ only by a constant amount, no differences will be seen.

With respect to Newton's bucket experiment, Leibniz had to admit (quoted in [176, p. 119]):

> However, I grant there is a difference between an absolute true motion of a Body, and a mere relative Change of its Situation with respect to another Body. For when the immediate Cause of the Change is in the body, That Body is truly in Motion; and then the Situation of other Bodies, with respect to it, will be changed consequently, though the Cause of that Change be not in Them.

Leibniz seems to have been trapped in an awkward situation: on the one hand assuming kinematic relativism, and on the other hand the phenomenon of circular motion claiming the existence of an absolute space.

For G. Berkeley (1685-1753), space is a theoretical concept formed by the perception and abstraction of extension. He argues that it is impossible to imagine the motion of a body without imagining it moving with respect to another object (quoted in [164, p. 171]:

> ... no motion can be understood without some determination or direction, which in turn cannot be understood unless besides the body in motion our own body also, or some other body, be understood to exist at the same time.

In addition, Berkeley rejects that Newton's bucket experiment proves the reality of absolute space, and also rejects that the motion of water in the bucket is circular if the rotational and translational motions of the Earth around the Sun are taken into account. The argument of the tension in the cord joining the two rotating globes is not valid for Berkeley either, because without a material reference it is not possible to conceive the motion of the globes and therefore no inertial effect can be attributed to the motion of the two globes. According to Berkeley, the idea of absolute space and motion is a mere fiction without empirical foundation, and he relates all such motions (such as those of water in Newton's bucket) to the reference frame of the fixed stars (quoted in [176, p. 109]):

> If we suppose the other bodies were annihilated and, for example, a globe were to exist alone, no motion could be considered in it; so necessary is it that another body should be given by whose situation the motion should be understood to be determined.

Though different, Berkeley's statement resembles Mach's Principle: The inertia of a body is determined by the masses of the universe and their distribution.

Another critique of Newton's famous bucket experiment is that of C. Huygens (1629-1695), this time from a different relativistic perspective (quoted in [176, p. 125]):

> For a long time I had thought that rotational motion by means of centrifugal forces contains a criterion for true motion. Indeed, with regard to other phenomena it is the same whether a circular disk or a wheel rotates near me, or whether I circle round the stationary disk. However, if a stone is put on the circumference this will be projected only if the disk rotates, and therefore I formerly thought that circular motion is not relative to any other body. Still, this phenomenon showed only than the parts of the wheel, owing to the pressure acting on the circumference, are driven in relative motion among themselves in different directions. Rotational motion is therefore only a relative motion of the parts, which are driven to different sides, but held together by a rope or other connection.

We are then before an argument of pure relativistic dynamics that anticipates a good part of contemporary physics.

In any case, and in spite of its criticisms, Newton's absolute space eventually prevailed in science, philosophy and theology, at least until the first decades of the eighteenth century. As we will see in the following chapters, the eighteenth, nineteenth and twentieth centuries are especially significant for our discussion on the nature of space.

11. Questioning Leibniz's Principle of Sufficient Reason

Abstract.-In the formal framework defined by the Principle of Directional Evolution of the Universe (towards its maximum entropy) and by the Theorem of the Inconsistent Infinity, this chapter proves the incompleteness of Leibniz's Principle of Sufficient Reason, which renders inconclusive Leibniz's critique of Newton's absolute space, and also makes inevitable the existence of first causes that cannot be explained in terms of other causes deduced from our present knowledge of the observable universe.

11.1 Two Leibniz's Principles

As is well known, after the publication of Newton's Principia [260] a famous epistolary debate took place between S. Clarke and G.W. Leibniz, the former defending Newton's absolute space and the latter denying it, and both cases considering its consequences on the very existence of God. Without going into the details of the debate (for which the reader may consult, for instance [233, 234, 176, 291, 164, 352]), Leibniz introduced in it two of his famous principles:

> **Principle of Sufficient Reason**:*There must be a sufficient reason for things to be one way and not another.*

According to Leibniz, the above principle would turn metaphysics into a deductive science.

> **Principle of Identity of the Indiscernibles**: *There cannot exist two different things that are indistinguishable from each other.*

Since (according to Leibniz) two different and indiscernible things cannot exist, and since in Newton's absolute space things could be located in several different and indiscernible ways, Leibniz argued that God would have had to choose one of these indistinguishable ways, without any reason to choose one of them to the detriment of the others, which for Leibniz is not proper to God. Therefore, absolute space cannot exist. Clarke argued in the opposite sense, not defending the possibility of contingent events, but making God's will intervene as the only reason why things were one way and not another.

The Principle of Identity of Indiscernibles is no longer accepted by contemporary science, but the Principle of Sufficient Reason (PSR) is at least partially accepted. So here we respond to this principle. The following answer could be given without the advantages of the knowledge accumulated from Leibniz's time to ours: it would differ very little from the one given in this article. Since the important thing is the PSR answer, whether or not Leibniz is present to answer it, this advantageous knowledge will be used here, including that which has been published but not yet sufficiently accepted in contemporary science, as is the case with the inconsistency of the actual infinity. A key inconsistency for the future of mathematics and especially for the future of physics and the logical understanding of the physical world.

11.2 The formal setting of the discussion

The PSR will be discussed here within a formal scenario whose two fundamental pillars will be the Principle of Directional Evolution of the Universe and the inconsistency of the actual infinity, the latter not as a principle but as a formally demonstrable theorem. From both of them we can deduce the rest of the formal elements that constitute the formal scenario in which Leibniz's famous principle will be contested. All of these formal elements are briefly demonstrated in the appendices to this chapter. As usual, I invite the reader to jump to the proof of the inconsistency of the actual infinity (Theorem 6 of the Axiom of Infinity). I could have chosen any of the more than forty demonstrations contained in [212]. The one included in the first appendix is a very simplified variant of one of those demonstrations, which was also one of the first I was able to develop. It contains less than 300 words that can be read in less than 3 minutes, and if the reader does not find it a correct argument, he/she can stop reading the rest of the article right there.

The formal elements to be used in the PSR discussion, which are formally proved in chapters 3-5, are the following:

1. **Principle of Directional Evolution**:*The observable universe always evolves independently of its rational observers and in the same direction of increasing its global entropy.*
2. **Theorem of the Inconsistent Infinity**: *The actual infinity subsumed in the Axiom of Infinity is inconsistent.*
3. **Theorem of the Consistent Universe**: *The universe evolves under the control of a unique set of invariant and consistent physical laws.*
4. **Theorem of Identicality**: *All particles of the same type have the same properties and behave the same way under the same conditions.*

5. **Theorem of Formal Dependence**: *No concept defines itself; no statement proves itself; no physical object is the cause of itself; and no cause is the cause of itself.*
6. **Theorem of the First Element**: *A consistent sequence in which there is a last element and each element has an immediate predecessor is a complete totality only if it has a first element without predecessors.*
7. **Corollary!of the First Cause**: *No physical object or process can be fully explained without a first cause that cannot be explained in terms of other causes.*

The PSR could also be stated in terms of logical causes: *there is always a logical cause which explains why things are as they are and not otherwise*. In the following, both forms, Leibniz's original and the latter, will be used interchangeably.

11.3 The Principle of Sufficient Reason

The infinite regress of arguments was already considered by Aristotle [13, I.3]:

> We, on the other hand, hold that not every form of knowledge is demonstrative, but that the knowledge of ultimate principles is indemonstrable. The necessity of this fact is obvious, for if one must needs know the antecedent principles and those on which the demonstration rests, and if in this process we at last reach ultimates, these ultimates must necessarily be indemonstrable.

This, of course, is why we have always needed, and will always need, axioms and inductive laws in the foundations of all sciences. In our case, this need is demonstrated by the Theorem of Formal Dependence, a consequence of the Principle of Directional Evolution of the Universe, which is inductively based on overwhelming empirical evidence.

In fact, no one expects the shards of broken glass to spontaneously reassemble into the exact original shape of the broken glass; or that the gas released from a bottle of champagne spontaneously returns to the champagne in the bottle. These typical examples are often used to illustrate the Second Law of Thermodynamics, which is immediately incorporated into the Principle of the Directional Evolution of the Universe. This principle, moreover, permits the formal deduction of results that extend the Aristotelian infinite regress of arguments to definitions, and causes of objects and natural phenomena. As indicated elsewhere in this book, the case of first causes certainly goes far beyond the content of this book. And the reader can easily see why.

As noted above, contemporary science still allows the PSR to be applied, with the exception of contingent events. But both in contemporary science and in Leibniz's arguments, applying the PSR implies applying the principle of infinite regress of causes. And this is the key fact that was initially absent in Leibniz's arguments and is still absent in contemporary physics, although in Leibniz's case he came to admit a first cause of why things are as they are and not otherwise:

Because the universe had to be the best of universes.

Which is obviously an ARBITRARY cause. Indeed, since potential infinity is the only consistent infinite, it would be impossible for humans to fully explain any object or natural phenomenon without recourse to a first cause that cannot be explained in terms of other causes (Corollary 17 of the First Cause, page 49). In the case of God (if there is one), if he is a consistent being, he could not do this either, just as he could not count the last natural number if, as we may suppose, even God cannot count a non-existent number. So the Corollary 17 of the First Cause applies to Him as well, which, as we shall see, has significant consequences for the origin of the universe itself.

One of these consequences is that Leibniz's theological objection to Newton's absolute space is inapplicable: one cannot always give a sufficient reason (a cause) for things being as they are and not otherwise, because in the end we will fall into an inevitable infinite regress of causes, from which it is only possible to get out by means of a first cause that cannot be explained in terms of other causes. Not even God could do that. But Leibniz, perhaps aware of this difficulty, proposed a first cause (the universe had to be the best of the universes) which, as we have just pointed out, is as arbitrary a cause as any other that cannot be explained by other causes. Actually, Leibniz would be quite satisfied with the Corollary 17 of the First Cause: he would only have to think of the universe as the physical object that it is. Since no object can be the cause of itself, every physical object, including the universe, must have a first cause external to the object itself.

What if the universe were eternal? Well, in that case its duration would be infinite, it would have, for example, an infinite number of seconds or any other arbitrary unit of time. That is, it would have an inconsistent duration (Corollary 4). What if the universe had arisen from a fluctuation of nothing? Well, then nothing would not be nothing, but something with the ability to fluctuate, and we would have to apply the Corollary 17 of the First Cause to that something with the ability to fluctuate. What if the present universe were a stage in a cyclic succession of universes being continuously created and destroyed? Well, in this case the number of cycles could only be finite (Corollary 4) and therefore there would be a first universe (Theorem 23 of the First Element) in the cyclic succession of universes to which the Corollary 17 of the First Cause could be applied.

The majority of contemporary physicists, all of them strictly relativistic, deny the existence of physical absolute space. According to them it is only a fiction useful to describe the evolution of the (always) relative positions of natural objects (see the final appendix to Chapter 23). At the same time, and according to these same physicists, space expands, bends, vibrates and transmits its own vibrations. And one wonders how something that does not exist can expand, deform, vibrate and transmit its own vibrations?

Since 2015, we have empirical evidence of gravitational waves, and this changes everything. The vibrations of space are no longer a theoretical matter, they are real, they interact with material objects (by changing the distances between the mirrors of the interferometers that detect them), and the interactions can be detected and measured. Therefore, space is real; it is a real and unique physical object; it is the same for all material objects; it is absolute; it is Newtonian. Chapters 23 and 24 discuss the physical consequences of absolute space and the nature of its substance, respectively.

12. Newton's bucket and absolute rotations

Abstract.-The content of this chapter links the famous Clarke-Leibniz discussion on Newton's bucket experiment with a new and independent argument confirming Newton's idea about the absolute nature of the rotation of the water in his famous bucket. Literally, the existence of billions of objects in the universe animated by absolute motion is demonstrated here: their rotations around their respective internal axes of rotation. Newton's thought experiment of the two balloons connected by a string is also recalled in this chapter. Finally, the problem of the relation between inertial mass, gravitational mass and preinertia is raised.

12.1 A real Newton's experiment

Newton's famous bucket experiment is, in my opinion, one of the most important in the history of physics, both for its results and for the discussions it sparked, discussions that still continue to this day [137, p. 43-108]. The experiment was designed to demonstrate the reality of absolute motion, and therefore of absolute space, as opposed to those who, like Leibniz, defended the relative nature of space. As is well known, Newton's position in favor of absolute motion prevailed for several centuries, until the beginning of the twentieth century, when the theories of relativity finally prevailed in a hegemonic manner that was and still is very hostile to dissent. But as the reader will see throughout this chapter, the last word on Newton's bucket has not yet been said.

First of all, let us recall Newton's own description of his famous experiment in the Definitions prefatory to Book I of his PHILOSOPHIAE NATURALIS PRINCIPIA MATHEMATICA, published in 1687 [260, p. 131-132] [259, p. 80-81]:

> The effects which distinguish absolute from relative motion are, the forces of receding from the axis of circular motion. For there are no such forces in a circular motion purely relative, but in a true and absolute circular motion, they are greater or less, according to the quantity of the motion. If a vessel, hung by a long cord, is so often turned about that

the cord is strongly twisted, then filled with water, and held at rest together with the water; after, by the sudden action of another force, it is whirled about the contrary way, and while the cord is untwisting itself, the vessel continues for some time in this motion; the surface of the water will at first be plain, as before the vessel began to move: but the vessel, by gradually communicating its motion to the water, will make it begin sensibly to revolve, and recede by little and little from the middle, and ascend to the sides of the vessel, forming itself into a concave figure (as I have experienced), and the swifter the motion becomes, the higher will the water rise, till at last, performing its revolutions in the same times with the vessel, it becomes relatively at rest in it. This ascent of the water shows its endeavour to recede from the axis of its motion; and the true and absolute circular motion of the water, which is here directly contrary to the relative, discovers itself, and may be measured by this endeavour. At first, when the relative motion of the water in the vessel was greatest, it produced no endeavour to recede from the axis; the water showed no tendency to the circumference, nor any ascent towards the sides of the vessel, but remained of a plain surface, and therefore its true circular motion had not yet begun. But afterwards, when the relative motion of the water had decreased, the ascent thereof towards the sides of the vessel proved its endeavour to recede from the axis; and this endeavour showed the real circular motion of the water perpetually increasing, till it had acquired its greatest quantity, when the water rested relatively in the vessel. And therefore this endeavour does not depend upon any translation of the water in respect of the ambient bodies, nor can true circular motion be defined by such translation. There is only one real circular motion of any one revolving body, corresponding to only one power of endeavouring to recede from its axis of motion, as its proper and adequate effect; but relative motions, in one and the same body, are innumerable, according to the various relations it bears to external bodies, and like other relations, are altogether destitute of any real effect, any otherwise than they may perhaps partake of that one only true motion. And therefore in their system who suppose that our heavens, revolving below the sphere of the fixed stars, carry the planets along with them; the several parts of those heavens, and the planets, which are indeed relatively at rest in their heavens, do yet really move. For they change their position one to another (which never

> happens to bodies truly at rest), and being carried together with their heavens, partake of their motions, and as parts of revolving wholes, endeavour to recede from the axis of their motions.

It is clear, then, that for Newton the force responsible for the separation of the water from the axis of rotation is present only in the absolute motion of rotation. The ascent of the water up the walls of the bucket is a proof of absolute motion, which in turn implies changes of position in an absolute space.

12.2 Criticism of Newton's bucket experiment

Chapter 11 recalled the epistolary (and theologically motivated) discussion between G.W. Leibniz and S. Clarke, in which Leibniz rejected Newton's absolute space by making use of his Principle of Sufficient Reason and his Principle of Identity of Indiscernibles (see pag. 119):

> Since (according to Leibniz) two different and indiscernible things cannot exist, and since in Newton's absolute space things could be located in several different and indiscernible ways, Leibniz argued that God would have had to choose one of these indiscernible ways, without any reason to choose one of them in preference to the others, which for Leibniz is not proper to God. Therefore, absolute space cannot exist.

However, the thought experiment of Newton's rotating globes (which is recalled in the next section), made Leibniz change his opinion slightly (quoted in [234, p. 44]:

> I grant there is a difference between an absolute true motion of a body, and a mere relative change of its situation with respect to another body. For when the immediate cause of the change is in the body, that body is truly in motion; and then the situation of other bodies, with respect to it, will be changed consequently, though the cause of that change be not in them.

Although Leibniz never came to admit absolute space nor did he renounce his relational position.

Some 200 years later, Ernst Mach (1838-1916) resumed his critique of Newton's bucket experiment from the same relational perspective as Leibniz. The most prominent and well-known aspect of Mach's critique was his conclusion that the motion of the water in Newton's bucket is a relative motion: the water actually rotates WITH RESPECT TO the background of the fixed[1] stars (BFS from now on). I have highlighted

---

[1] Obviously, the stars are not fixed, as one might have believed in E. Mach's time.

the words "with respect to" because it is from them that my criticism of Mach's criticism and that of all those authors who make use of the same semantic trick will be deduced.

Although the expression *spinning with respect to something* is not entirely wrong, it is not the best way to describe a spin or rotation. Indeed, bodies that rotate, actually rotate AROUND something, usually a straight line (axis of rotation) that can even be materialized for example with a thin wire. Each point of a body rotating AROUND an axis describes a circle AROUND a point of that axis, the axis being internal or external to the object rotating AROUND it. Naturally, in the case of the molecules of water of Newton's bucket, they all describe concentric circles around a vertical axis passing through the geometrical center of the bucket. All of which can be materialized with floating beads and a vertical wire passing through the center of the bucket. The most honest way to describe the motion of these floating balls (or water molecules) is to say that they move AROUND the wire following CIRCULAR TRAJECTORIES AROUND the wire (axis of rotation), and that they are pushed by a force that tends to separate them from the axis of rotation, which is why they will move away from the axis of rotation as far as possible taking into account the walls of the bucket and the complete swarm of molecules subjected to this force. The beads do not describe circles around BFS, in fact it is impossible to describe a circular trajectory around a surface, only around a central point is possible to describe a circular trajectory, simply because a circle is exclusively defined by a point (its center) and a fixed distance to the center (its radius). So to rotate around (with respect to, as Mach would say) a bi-dimensional or three-dimensional surface is meaningless. I think that if he had used the word "AROUND" instead of the expression "WITH RESPECT TO," Mach would have realized his mistake.

As is well known, the circular motion of a material object around an axis imparts a centrifugal force on the rotating body that tends to move it away from the axis of rotation. That force (which we now know is proportional to the distance to the axis of rotation) is responsible for the fact that a fluid such as the water in Newton's bucket, which is rotating around a vertical axis passing through the center of the bucket, tends to move away from that axis and up the walls of the bucket, creating the famous concave absolute surface of the water in Newton's bucket. Therefore, and according to Newton, the concave absolute surface of the water shows the presence of a force caused by the real, absolute motion of the water molecules AROUND the axis of rotation, not WITH RESPECT TO the BFS, a force that is the reaction to the force that must be continuously applied to continuously change the direction of motion of each rotating molecule of water. It is a proof of absolute motion, as the following argument will also prove. But before continuing, let us recall Galileo's words on a subject similar to the one under discussion here [125, p. 183-184]:

> Now, if in order to achieve the same effect in a precise way, it is just as important that the Earth alone should move, stopping all the rest of the Universe, as it is that the whole Universe should move with a single movement, who would want to believe that Nature (which, according to common agreement, does not do by the intervention of many things what it can do by means of a few) has chosen to make an immense number of very large bodies move, with inestimable speed, in order to achieve what can be obtained by the moderate movement of a single body around its own center?

Back to our discussion, consider the daily rotation of the Earth around its north-south geographic axis, without considering the precession and nutation motions of this axis. The trajectory T of any point P of the Earth during this rotation is the complete circle that P describes in about 24 hours, where the center of this circle is a unique point Q of the axis of rotation of our planet, and its unique radius is the distance between the two points P and Q. Because of preinertia [206, Link], observers on the Earth do not notice this rotation; what we do observe is that the Sun and the rest of the celestial bodies rotate around the center of the Earth. Although we have long since discovered that such daily rotational motions of celestial bodies observed from the Earth are only apparent, not real, it is necessary to give a formal, physical demonstration that this is the case, simply to be able to infer from such a case some conclusion about the nature of this motion. As will be seen below, it is possible to demonstrate that such rotational motions are indeed only apparent, not even relative, because, as we shall see, they are physically and logically impossible.

Indeed, in addition to the Earth, there are other planets in the solar system that also rotate around an internal axis under similar conditions to the Earth. From these planets, and for the same reasons as for the Earth, the Sun and the other celestial bodies appear to rotate around each of these planets. Consequently, each point of the Sun and the other celestial bodies would simultaneously describe different circular orbits with different centers of rotation, which would mean that each of the points of each of the celestial bodies would have to be in different places at the same time, describing different orbits. Furthermore, a star located, say, a billion light-years from Earth and on Earth's equatorial plane would have to move at a speed 3.3×10^{13} times greater than the speed of light, which we assume to be physically impossible.

Consequently, and since it is impossible for the same point to be in different places at the same time, describing different orbits, and since we also assume that it is impossible to exceed the speed of light, we must conclude that the rotations of the Sun and of all the celestial bodies observed from the Earth and from the rest of the planets of the solar system are logically and physically impossible. Therefore, they are not real, they do not exist, they are only apparent motions. The only things that exist are the planetary rotations around their respective internal axes of rotation. The same argument applies to all the planets of any other planetary system (star-planets) in the universe. It must be concluded that all the rotations around their respective internal axes of rotation of all the planets of all the planetary systems in the universe are absolute rotations, as Newton argued for the case of the rotation of the water in the bucket in his experiment. There are literally billions of objects in absolute motion (rotation) in the universe. A conclusion that should have been universally accepted for centuries, but still is not: for relativistic officialism, absolute motion does not exist.

12.3 A thought experiment: Newton's rotating globes

In the same text in which Newton presented his famous bucket experiment, he also presented another experiment, this time a thought experiment, which would also confirm the absolute nature of physical space. [260, p. 133] [259, p. 82]:

> It is indeed a matter of great difficulty to discover, and effectually to distinguish, the true motions of particular bodies from the apparent; be cause the parts of that immovable space, in which those motions are performed, do by no means come under the observation of our senses. Yet the thing is not altogether desperate: for we have some arguments to guide us, partly from the apparent motions, which are the differences of the true motions; partly from the forces, which are the causes and effects of the true motions. For instance, if two globes, kept at a given distance one from the other by means of a cord that connects them, were revolved about their common centre of gravity, we might, from the tension of the cord, discover the endeavour of the globes to recede from the axis of their motion, and from thence we might compute the quantity of their circular motions. And then if any equal forces should be impressed at once on the alternate faces of the globes to augment or diminish their circular motions, from the increase or decrease of the tension of the cord, we might infer the increment or decrement of their motions: and thence would be found on what faces those forces ought to be impressed, that the motions of the

> globes might be most augmented; that is, we might discover their hinder most faces, or those which, in the circular motion, do follow. But the faces which follow being known, and consequently the opposite ones that precede, we should likewise know the determination of their motions. And thus we might find both the quantity and the determination of this circular motion, even in an immense vacuum, where there was nothing external or sensible with which the globes could be compared. But now, if in that space some remote bodies were placed that kept always a given position one to another, as the fixed stars do in our regions, we could not indeed determine from the relative translation of the globes among those bodies, whether the motion did belong to the globes or to the bodies. But if we observed the cord, and found that its tension was that very tension which the motions of the globes required, we might conclude the motion to be in the globes, and the bodies to be at rest; and then, lastly, from the translation of the globes among the bodies, we should find the determination of their motions. But how we are to collect the true motions from their causes, effects, and apparent differences; and, vice versa, how from the motions, either true or apparent, we may come to the knowledge of their causes and effects, shall be explained more at large in the following tract. For to this end it was that I composed it.

In this case, Leibniz partially agreed with Newton. [233, p. 82-83]:

> I agree, however, that there is a difference between a true absolute motion of a body and a mere relative change of its situation by reference to another body. For when the immediate cause of the change is in the body, it is truly in motion and then the situation of the others in relation to it will consequently be changed even though the cause of this change is not in them.

Two hundred years later, E. Mach took up the problem of the spinning balloons to declare it outside our possibilities of analysis because it is based on an unreal situation of which we have no experience to serve as a guide, and then we cannot analyze the situation. But, we should answer him, this eliminates the possibility of thought experiments and the pure exercise of logic in the practice of science. Indeed, even if we have no experience with the two-balloon universe, we have logical tools to analyze the problem and deduce formal consequences that can be applied to the real universe if the real universe is formally consistent, which is possible to demonstrate on the basis of its directional evolution (Chapter 5).

12.4 Mass and Mach's Principle

Mach's Principle states that the inertial mass of material objects is produced by all the masses present in the universe. It would then be what we could call a collective property, so that if we were to leave a single body in the universe, it would have no inertial mass. Among contemporary physicists there is a division of opinion regarding Mach's Principle. In any case, even that collective property involves the basic and general concept of mass, which is probably a primitive concept, not definable in terms of other more basic concepts. Gravitational mass, on the other hand, does not seem to be a collective but an individual property of each of the material bodies, this mass is what modifies the properties of physical space making possible the gravitational interaction, one of the four fundamental interactions responsible for the evolution of the universe.

Preinertia is another particular property of every object in the universe, including photons (see Chapter 19), by virtue of which the object inherits the velocity vector of the material object(s) that sets it in motion. This property, despite making continuous unconsciously (implicit) use of it, contemporary physicists have not yet discovered it and, therefore, they cannot use it in its arguments and experiments. We have, then, three physical facts in search of explanation:

1. Inertial mass: The resistance of all material objects to change their state of motion.
2. Gravitational mass: The ability of all material objects to modify the physical properties of the real physical space[2] by creating gravitational fields around them.
3. Preinertia: The ability of any physical object to inherit the velocity vector of the proper reference frame where it is set in motion.

Mach's Principle proposes an explanation of the first fact: it is produced by the mass of the whole universe, it would be a sort of collective property. The other two facts can only be thought of as individual properties of each object. Thus, although much remains to be discussed on mass, the simplest explanation of the above three facts would be that all material objects have a property capable of modifying the properties of space, of offering a certain resistance to their own changes of motion, and of inheriting the velocity vector of the proper reference frame from which they start its own independent motion. This latter property manifests itself even in supposedly massless objects such as photons. Although there is a possibility that photons have a mass of the order of

[2]Space is in fact a real physical object as the empirical detection and measurement of its vibrations (gravitational waves) proves.

10^{-64} Kg, which could be called quantum mass m_q [209, Link p. 235]:

$$m_q = \sqrt{\frac{G\hbar^3 R_\infty^4}{c^5}} = \hbar t_p R_\infty^2 = 6.845023 \times 10^{-64} Kg \tag{1}$$

where t_p is the Planck time and R_∞ is the universal Rydberg constant, which is specific to each chemical element and varies slightly with its mass. In any case, I repeat here some of the mass-related questions that should be asked and the answers sought:

- Of an object in uniform motion, what determines and controls its linear trajectory, its successive positions along the successive instants?
- How does a body remember that it was pushed? Where lies the imprint of that action?
- What changed, if any, in its internal structure as a consequence of being set in motion?
- What distinguishes a ball that has been pushed from another that was not?
- Are space and time somehow affected by a ball set in motion?
- Knowing that a body A was pushed and other body B was not pushed, being initially A and B at relative rest, is it the same to say that A moves with respect to B as to say that B moves with respect to A?
- To put A in motion is the same as to put the rest of the universe in motion?
- Is there any absolute describable reality?
- If there is no reality describable in absolute terms, are there as many realities as there are relative forms of observing it? To observe what?
- Could the universe be described, as such an object, from outside the universe?
- Are we living beings with the capacity to reason but not to observe reality?
- Is the theory of special relativity the ultimate theory?
- What relation, if any, does exist between inertia and preinertia?
- Can be preinertial a massless particle?
- Are inertial and preinertial objects affected in the same way by gravitational fields?

- Is preinertia sufficient to explain light deviation by massive objects?
- Is preinertia a fundamental attribute, as mass, charge or spin, of elementary particles?
- If not, which fundamental attribute of elementary particle could account for preinertia?
- Does preinertia result from the interaction between matter and physical space?
- Are all waves preinertial?
- etc.