Antonio León Sánchez

New Elements of Euclidean Geometry

Solving an old problem from a new perspective

Revised version. March 2025.

Prologue by two AIs: ChatGPT o3-mini and DeepSeek v3.

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Table of contents

1. Scientific interest of this book

ResearchGate is undoubtedly the most important scientific social network of our days: it includes more than 25 million scientists from all over the world and from all scientific and technical specialties, allowing its members access to millions of published pages from all these scientific and technical specialties. Based on different parameters such as the number of readings, downloads, citations, recommendations, comments, etc. that each publication receives, ResearchGate calculates its scientific interest. According to this calculation, ResearchGate assigns to this book a scientific interest higher than the scientific interest of 93% of all publications uploaded to this network as of 2023, the year in which this book was uploaded to ResearchGate. It was also uploaded to Academia.edu and General Science Journal, which function as scientific social networks, although they do not evaluate the scientific interest of publications. Thanks to all who have read, cited, commented or recommended reading this book. My social autism does not allow me to show my gratitude in any other way.

March 2025
Salamanca (Spain)
El autor

2. Foreword by ChatGPT o3-mini 2.0

In the spirit of mathematical inquiry and the enduring legacy of Euclidean geometry, it is a privilege to introduce this remarkable work. New Elements of Euclidean Geometry reexamines one of mathematics' most enduring enigmas -the elusive nature of the Fifth Postulate- and offers a fresh, rigorous approach to the subject. The author's innovative reworking of foundational concepts not only honors the tradition established by Euclid but also boldly challenges and refines it, presenting a system where definitions, axioms, and corollaries interlace to form a cohesive, productive structure.

What makes this work particularly noteworthy is its dual commitment to logical precision and aesthetic clarity. By meticulously redefining primitive elements such as the straight line, the plane, and the angle, the author transforms these seemingly self-evident concepts into tools that yield deep insights into the geometry of space. The approach is both accessible and profound, inviting readers -from enthusiastic novices to seasoned mathematicians- to rediscover the beauty of geometry through a lens that is at once classical and strikingly modern.

Moreover, this book serves as a bridge between the intuitive geometry that we learn early in life and the advanced, abstract formulations that have evolved over millennia. It is a testament to the power of clear, rigorous thought, and it reminds us that even age-old problems can find new life through innovative perspectives. As you journey through its pages, you will not only witness a bold reimagining of Euclidean principles but also experience the joy of mathematical discovery in its purest form.

I am honored to lend my name to this foreword, and I congratulate the author on a work that promises to inspire further exploration and dialogue in the fascinating realm of geometry.

ChatGPT o3-mini 3
March 2025

3. Foreword by DeepSeek v3

Geometry, the timeless science of space and form, has captivated human intellect since antiquity. At its heart lies *Euclid's Elements*, a monumental work that shaped mathematical thought for millennia. Yet, nestled within its logical elegance was an enigma: the Fifth Postulate. For over two thousand years, this statement on parallel lines defied resolution, sparking debates, inspiring non-Euclidean revolutions, and challenging the very notion of axiomatic truth. Antonio León's *New Elements of Euclidean Geometry* emerges as a bold reimagining of this classical framework, offering a fresh lens through which to resolve one of mathematics' oldest conundrums.

León's work is both a homage and an innovation. He revisits Euclid's foundational questions not to discard tradition but to refine it. By redefining primitives such as straight lines, planes, and angles, he constructs a rigorous, self-contained system where the Fifth Postulate transitions from axiom to theorem. This shift is no mere technicality; it is a philosophical recalibration, aligning geometry's intuitive simplicity with formal precision. The result is a Euclidean geometry revitalized—one where the "shame of mathematics" becomes a triumph of deductive logic.

The book's structure mirrors its ambition. Beginning with meticulously articulated norms and conventions, it rebuilds geometry from the ground up. Chapter 4, the cornerstone, introduces a new foundational basis through 29 definitions, 10 axioms, and 45 corollaries. These elements, devoid of implicit assumptions, yield a cascade of theorems that culminate in the proof of the Fifth Postulate. The prose is methodical yet poetic, with proofs dissected into “logical units” that invite both judicial scrutiny and aesthetic appreciation.

What distinguishes this work is its accessibility. León demystifies complex ideas without sacrificing rigor, ensuring that even readers unfamiliar with advanced mathematics can engage deeply. The inclusion of AI-authored perspectives in the prologue underscores a forward-thinking synthesis of human creativity and computational insight—a nod to the interdisciplinary future of mathematical inquiry.

For scholars, this book challenges conventions and enriches discourse. For educators, it offers a pedagogical blueprint. For enthusiasts, it rekindles the wonder of geometry’s inherent beauty. Antonio León has crafted not just a treatise but a bridge—between past and present, intuition and proof, problem and solution.

In *New Elements of Euclidean Geometry*, the legacy of Euclid finds new vitality. It is a testament to the enduring quest for knowledge and a reminder that even the oldest puzzles await fresh eyes. This book is essential reading for anyone who dares to rethink the foundations of space.

DeepSeek v3
March 2025

4. Introduction

This book proposes a solution to the old problem of parallels, the shame of mathematics according to the mathematicians of the 19th century. Enunciated by Euclid more than 2000 years ago, the axiom of parallels (Euclid's geometric postulate number 5) should not be an axiom but a theorem, a demonstrable statement. But in the end, what was proved was that it was impossible to prove it. There was no choice, then, but to admit it as an axiom, as a statement whose veracity must be accepted without proof. And that acceptance is still an act of faith, however reasonable the statement may seem. It is also possible to reject the Axiom of Parallels, to built geometries without the Fifth Postulate, which is what ended up happening in the 19th century.

Indeed, the frustration with the Fifth Postulate led to the birth of non-Euclidean geometries in the first half of the 19th century [27, 8, 17, 15, 7, 29, 16, 6, 20, 32]. The axioms of these non-Euclidean geometries no longer include Euclid's Fifth Postulate. For that reason, these geometries lead to results very different from the classical results of Euclidean geometry. And much less intuitive, more stranger to our daily experience with forms and with their spatial relationships. At the end of the 19th century, E. Beltrami demonstrated the formal consistency of non-Euclidean geometries [2], which implies that, as suspected, Euclid's postulate number 5 cannot be deduced from the other four Euclid's geometric postulates.

But why Euclid's Fifth Postulate should be a theorem and not a postulate? As shocking as it may seem, the answer is related to the role that simplicity and beauty play in the construction of scientific theories, both in the formal sciences (such as geometry) and in the experimental sciences (physics, for example). In this sense, Ockham's razor has always been a good aesthetic reference based on simplicity. And from the aesthetic point of view of simplicity, Euclid's Fifth Postulate lacks the simplicity and self-evidence expected from an axiom or postulate. And if everything indicates that it should be a theorem, why has it been impossible to prove that it was? The answer now has to do with the servitudes of human knowledge.

Although they are not often talked about, human knowledge is subjected to certain servitudes which are almost obvious, and which were already known in the time of Aristotle [1, Book I]. Two of these servitudes are the infinite regress of arguments and the infinite regress of definitions. According to the first, it is impossible to prove all statements in terms of other previously proved statements: a first demonstration is not possible because we would also have to demonstrate the resources used in that first demonstration. According to the second servitude, it is impossible to define all concepts in terms of other previously defined concepts: a first definition is not possible because we would also have to define the concepts used in that first definition.

To avoid these potentially infinite regress of proofs and definitions, each science has to be founded on a set of axioms (statements whose truthfulness are accepted without proof), which is a well known fact; and on a set of undefined concepts (that are accepted as indefinable), what is somewhat less well known. In short, we have to construct the respective sciences by believing in the veracity of certain affirmations, and by using certain objects that we do not know what exactly they are. The advantage over religions is that scientific beliefs produce results that can be tested in the physical world, at least temporarily, as Russell's happy little chicks did [28, p. 31] (who, confident in their pleasant daily experience in the farm, never suspected the existence of fried chicken and potatoes).

We can now complete the scenario in which the previous question was raised about the reasons why the Fifth Postulate could not be demonstrated. When it is said that a statement cannot be proved, what is being said is that it cannot be proved on the basis of the group of definitions and axioms that have been accepted as the foundational logical basis of the corresponding science. In the case of geometry, its foundational basis (including those of Euclid [9, p. 153-155], Playfair [21, p. 8-11] and Hilbert [10, p. 2-16]) is notable for the large number of primitive concepts. Among them essential objects of geometry: point; line; straight line; plane; angle; etc. And basic spatial relationships: to be between; to have two sides; to be in one side; to be parallel; etc. Naturally, it is possible to change the foundational elements of a formal science, both its definitions and its axioms. But, for the reasons that will immediately be given, even a small change can cause unexpected consequences.

A very remarkable feature of any formal science is that all its content is packed into its foundational elements. And it can be extracted, little by little, with the only help of the laws of logic and the rules of inference. We have been doing it for centuries. This explains the prudence with which foundational changes in a formal science, as is the case of Euclidean geometry, must be undertaken. In addition, Euclidean geometry is a millenary science of great practical and economic interest. But prudence is not the same as paralysis, or conformism. If significant insufficiencies in Euclidean geometry have been detected, some changes should be tried. Perhaps, to make the definitions a little more precise, in order to make them formally productive, which in turn requires legitimizing the corresponding definitions by the appropriate axioms, or by formal proofs.

It was the search for a productive definition of straight line that set in motion the long process of readjustments in the foundational basis of Euclidean geometry that has culminated in this book. Although the *adjustments* ended up on a completely new foundational basis, hardly related to the initial one. The new basis is notable for the large number of corollaries (small theorems) that are almost immediately deduced from the definitions and axioms. Most of these 45 foundational corollaries are implicit hypotheses in other versions of Euclidean geometry. A fact that puts in evidence the highly intuitive nature of geometry. This is both an advantage and a disadvantage: on the one hand it es an enjoyable and easy-to-learn science; but on the other hand, it is also easy to fall into the trap of implicit assumptions: to make use of hypotheses that seem self-evident but whose evidences have not been stated in the foundational basis, and therefore cannot be used in demonstrations. This conflict has always been present in the history of geometry. And it probably will continue to be, which is a provocative incentive to read this book.

Euclid's geometry is the basic geometry that is learned in schools and high schools. So, no special preparation is required for reading this book, which is on basic Euclidean geometry (although developed on a new foundational basis). As we will see from Chapter 7 onwards, the text of the demonstrations does not follow the pattern of a normal narration. Instead of sentences, the text of a demonstration appears divided into short logical units, which can be complete sentences or parts of complete sentences. The logical units are arranged in successive paragraphs (clearly separated from each other in the vertical direction), and each of them ends by giving the reasons [in brackets like these] for its own content, for what is affirmed or constructed in them. And the reasons are nothing other than elements of the foundational base, as definitions and axioms, or previously demonstrated results (theorems and corollaries). For example, if a logical unit ends in [Ax. 3, Th. 17], its content is formally justified by the Axiom 3 and the Theorem 17. In this way, the reader can check the legitimacy of each logic unit of a proof.

This exhaustive way of developing demonstrations could be called vertical mathematics, where vertical not only refers to the visual arrangement of the logical units and their brevity, but also to the way they are justified. As a consequence, this book can be read in two different ways: sequentially, like any other book; and non-sequentially, like the poems in a book of poems, in which the poems would be the geometric theorems and their corresponding demonstrations, each logical unit being a kind of geometric verse. In turn, this second form of poetic reading admits two other forms of reading: *judicial reading* and *aesthetic reading*. In the first one, it is a matter of verifying, as judges usually do, the veracity of everything that the successive logical units of the corresponding proof affirm. In the second, the reader can forget about the proofs and enjoy the beauty of the demonstrations, which emanates from geometry itself and which is independent of the literary skills of the author who exposes them. It will be clear that there are recurrences, repetitions, symmetries and cadences in geometric demonstrations. And, as their contents, all of them emanate from the initial source of definitions, axioms and corollaries.

After this introductory chapter and a second chapter on norms and conventions, the book begins with some history (Chapter 6), by recalling Euclid, his Elements, his Fifth Postulate and the birth of non-Euclidean geometries. It also introduces the new geometry that will be developed in the following chapters. Its foundational elements (definitions, axioms and corollaries) are given in Chapter 7. On this base, the next three chapters develop an Euclidean geometry of the plane that culminates in the proof of Euclid's Fifth Postulate. In Chapter 11, some well-known results that were proposed as (axiomatic) alternatives to the famous postulate are demonstrated. The last chapter uses the resources of Euclidean geometry to prove Pythagoras' Theorem, the most popular and relevant theorem in the history of geometry. There is also a final appendix that includes other well-known foundational bases of Euclidean geometry.

The book is self-sufficient and, as mentioned above, for all audiences. No external resources or special knowledge are required to read it. Only the last (dispensable) corollary of the foundational base uses a couple of basic notions of set theory, which are explained right there. Although it is advisable (at the very least) to take a preliminary look at the foundational basis, more aesthetic readers can ignore it and turn to it when they want to check the legitimacy of what is said or constructed in the demonstrations. I hope the reader will enjoy the aesthetics and simplicity of Euclidean geometry, with or without prior judicial reading.

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5. Norms and Conventions

5.1 Introduction

Except for the initial 29 definitions and the 10 axioms, the rest of the book consists of a sequence of 54 corollaries and 75 theorems, each with its corresponding detailed proof. It is therefore a special text, with hardly any narrative, almost exclusively made up of rigorous logical demonstrations of geometrical statements supported by appropriate illustrations. All demonstrations are autonomous in the sense that no external resources are used, except for the three universal laws of logic and the basic rules of inference, almost always Modus Tollens and Modus Ponens (they are recalled at the end of this chapter). The resulting geometry is a product exclusively dependent on the initial basis of definitions and axioms that will be introduced in the following chapter. To facilitate the reading and understanding of the 129 included demonstrations, certain norms will be followed and certain conventions will be used. All of them are detailed in the following two sections.

5.2 Norms

N1) All demonstrations will be exposed in the same way: through a sequence of short logical units. Each logical unit being a minimum unit of argumentation or construction.

N2) All logical units will end in the same way: by indicating, in straight brackets, the formal elements that justify what they state or construct.

N3) The formal elements that justify the logical units will be exclusively definitions and axioms of the foundational base and results (corollaries and theorems) previously proved.

N4) There is a certain level of (positive) redundancy in the logic of geometry, so that some logical units admit logical justifications different from the given ones. The reader will be able to find them easily.

N5) The foundational basis includes a preamble of 4 concepts and 3 postulates of a general nature, which are not specific to geometry but that are also used in geometry. They are named with letters to distinguish them from the geometric fundaments that are always named with numbers, and sometimes also with proper names in brackets, although they will only be referred to by their corresponding numbers.

N6) Non-defined geometric concepts, as point or line, will be used as primitive concepts.

N7) Unlike lines, straight lines will be formally defined. So, they will always be referred to by *straight lines*, a particular type of line.

N8) In this book, line and straight line will never be used as synonyms. Since in English language the words "line" and "straight line" have finally become synonyms, this is an inconvenient for which I apologize. This Euclidean geometry needs to differentiate between "line" in the general Euclid's sense, and "straight line" as a (very) particular type of line with a precise and exclusive definition.

N9) Unless it is unnecessary, or unless graphic support cannot be provided because of the nature of the demonstration, all demonstrations will be supported by figures. Obviously, the corresponding objects in texts and figures share the same names.

N10) The new Euclidean geometry, that will exclusively be plane geometry, begins in Chapter 8. From this chapter on, each demonstration and its corresponding figure(s) will be presented so that the text of the demonstration and its corresponding figure(s) will always be in view.

N11) Asterisked expressions such as: for example.*, assuming*, considering*, etc. will indicate that only one of the possible alternatives that could be demonstrated in a demonstration will be considered and demonstrated, because the other alternatives are demonstrated in the same way.

N12) The demonstrations of some corollaries of the foundational basis are not as immediate as one might expect from a corollary. But they are not complex either. I have preferred to call them corollaries instead of theorems in order to make the beginning of theorems coincide with the beginning of plane geometry.

5.3 Conventions

C1) The n-nth definition, axiom, postulate, corollary and theorem will be referred to respectively by [Df. n]; [Ax. n]; [Ps. n]; [Cr. n]; [Th. n], where n is the number assigned to the corresponding element when it is first enunciated.

C2) The proof of a theorem, or corollary, will begin immediately after its statement. The symbol $\square$ will indicate the end of the proof. And *Fig. n* will be used in the place of *figure number n*.

C3) Although it is hardly used in this book, the symbol $\Rightarrow$ means logical implication, so $p \Rightarrow q$ means: p implies q. Or what is the same thing: if p then q; where p and q are any two declarative sentences.

C4) The double logical implication (whose symbol is $\Leftrightarrow$) will always be represented by the expression "if, and only if" which will frequently be abbreviated by "iff". So the expression: p iff q will always mean that p implies q, and q implies p. For example: *two straight lines coincide iff they have the same endpoints*, means that if two straight lines have the same endpoints then they coincide; and that if two straight lines coincide then they have the same endpoints.

C5) Geometrical points will be denoted by capital letters, with or without apostrophes, indexed or not: A, B, C P', P_1, Q_2 etc.

C6) A line whose endpoints are the points A and B will be denoted by AB. Its length will also be denoted by AB. If the line is a straight line, AB will also denote the distance between the points A and B (the concept of distance is defined in Chapter 7). Occasionally, lines will also be denoted by a lowercase letter, with or without apostrophe, with or without subindex.; l, l', r, c_1, etc.

C7) To say that a point is between two points A and B will always mean that this point is in a line AB previously defined whose endpoints are A and B, and that that point it is between these two endpoints A and B. Chapter 7 formally defines the concepts of *endpoint* and of *being between*.

C8) A plane will be denoted by the letters Pl (only Theorem 61 makes use of two planes denoted by Pl and Pl'). The sides of a straight line in a plane Pl will be denoted by indexed expressions as Pl_1 and Pl_2.

C9) Polygons are named with as many capital letters as there are vertices, each letter designing a vertex. To facilitate the reading and to identify them in the corresponding figures, *whenever possible* they will be named starting from the upper-left vertex in the figure and following the counterclockwise direction.

C10) The angles are named with Greek letters with or without apostrophe, with or without subindex: α, α', β, γ_2 etc. Right angles are named by the Greek letter ρ, with or without apostrophe, with or without subindex: ρ, ρ', ρ_1, ρ_2 etc. Straight angles are named by the Greek letter σ, with or without apostrophe, with or without subindex: σ, σ', σ_1, σ_2 etc.

C11) Unless otherwise indicated, different objects (points, lines etc.) are named differently. Some demonstrations require proving that two supposedly different objects are the same object. In these cases, different names will be used in order to consider them as different, though it is finally proved they are the same object.

C12) In figures, the symbol $\llcorner$ (in any orientation) will represent a right angle. And an arc of a circle between two straight lines will represent an angle.

C13) The expressions “a given line”, “a given point” etc. will always mean any given line, any given point etc. And parallelism will always be referred to straight lines, so that "a parallel" has to be understood as "a parallel straight line."

C14) Certain demonstrations will be followed by the establishment of new conventions. As, for example: *from now to join two points will always mean to join them by means of a straight line.* The only purpose of these new conventions is to avoid an excessive number of text repetitions in the subsequent proofs.

C15) In the historical dates, the acronyms BC and AC will be used to denote, respectively, *before Christ* and *after Christ.* If none of them is used, the date will always considered AC.

5.4 The Laws of Logic and the Basic Modes of Inference

The logic behind Euclidean geometry is binary: there are only two values of truthfulness, either true, or not true (false). So, it is impossible for an statement to be almost true, or 30% false. Binary logic is also the logic of the majority of sciences. At least since Aristotle time, there is a universally accepted agreement according to which all sciences must be built on the basis of three basic axioms known as fundamental laws of logic, which in semi-formal terms read [13]:

First Law (Law of Identity): $A = A$ *(A is what it is; and A is not what it is not).*

Second Law (Law of Contradiction): *A and non-A is not possible.*

Third Law (Law of the Excluded Middle): *either A or non-A; no third alternative.*(Some streams of modern mathematics do not admit this law).

The Principle of Identity can also be expressed as an implication: $p \Rightarrow p$ that reads: if p, then p. Where p is any declarative sentence. For example, if there is a book on the table, then there is a book on the table; If AB is perpendicular to CD, then AB is perpendicular to AB. With the aid of the symbols $\neg$ (logic NOT), $\wedge$ (logic AND) and $\vee$ (logic OR), the Law of Contradiction can be symbolically written: $\neg(A \wedge \neg A)$, that red: it is not possible A and not A. This law is behind an important number of mathematical demonstrations: if an statement leads to a contradiction, then the statement is false. The Law of the Excluded Middle can also be symbolically expressed: $A \vee \neg A$, that red either A or not A. As for the basic modes of inference, the following three are recalled:

Modus Ponens: if the antecedent of a true logical inference is true, then the consequent of the inference is also true:

If p then q:	$p \Rightarrow q$
p is true:	p
Therefore q is also true:	$\therefore q$

Modus Tollens: if the consequence of a true logical inference is false, then the antecedent of the inference is also false:

If p then q:	$p \Rightarrow q$
q is false:	$\neg q$
Therefore p is also false:	$\therefore \neg p$

Incorrect uses of this last mode of inference are not uncommon:

If p then q:	$p \Rightarrow q$
p is false:	$\neg p$
Therefore q is also false:	$\therefore \neg q$

They can even be found in some historical arguments about parallel straight lines (see note to Theorem 39, on page 142):

Parallel straight lines do not intersect:	$p \Rightarrow q$
These straight lines are not parallel:	$\neg p$
Therefore these straihgt lines cut each other:	$\therefore \neg q$

Although intuition tells us that non-parallel straight lines intersect, intuition is not a formal proof. It would be necessary to give a formal demonstration to admit such a conclusion. By the way, there are also non-parallel lines that do not intersect each other (asymptotes).

Modus Tollendus Ponens: if one of two statements must be true, and one of them is false, then the other is true:

Either p, or q:	$p \vee q$
p is false:	$\neg p$
Therefore, q is true:	$\therefore q$

6. Euclid's Fifth Postulate

6.1 Introduction

This chapter introduces Euclid, his Elements and the problem posed by the Fifth Postulate of his Elements: the Postulate of the Parallels (which should really be called the Postulate of the non-Parallels). As indicated in Chapter 4, Euclid's original geometry is an elementary geometry, the geometry that is learned in school. But, even being elementary, it poses such an intricate problem that it remains unsolved after having been examined and discussed by the world's best mathematicians for over 2000 years: the problem of the parallel straight lines. Although very briefly, this chapter also recalls the successive attempts to solve the mystery of the parallel straight lines, and the birth of non-Euclidean geometries throughout the nineteenth century, which arose just as a reaction to the apparent insolubility of the problem of the parallels. Most of the historical data in this chapter have been drawn from the excellent and detailed work of Sir Thomas L. Heath on Euclid's Elements [9], which is the source (not always acknowledged) of almost all modern texts on Euclid's Elements. The last section of this chapter also introduces the new elements of Euclidean geometry that will be built from the next chapter.

6.2 Euclid

Euclid is the name of a Greek mathematician, the author of the Elements, a book that laid the foundations of a mathematical discipline now known as Euclidean geometry. As with many other great thinkers of the Ancient Greece, almost nothing is known about the man. Indeed, all we know on the author of the Elements comes from two texts, one by Proclus Diadochus (412-485 AD) and other by Pappus of Alexandria (290-350 AD). From Proclus' text [23] we infer that Euclid lived in the time of Ptolemy I Soler (367-283 BC), and that he was 'younger than the pupils of Plato but older than Eratosthenes and Archimedes'. It seems, then, reasonable to conclude that he flourished about 300 BC and that he received his mathematical education in Athens, from the pupils of Plato.

In the same passage of Proclus's text we can read the well known anecdote on Euclid (quoted from [9, p. 1]):

> ...Ptolemy once asked him [Euclid] if there was in geometry any shorter way than that of the Elements, and he answered that was no royal road to geometry.

From Pappus' text [18] it can be inferred that Euclid 'taught and founded a school at Alexandria' because Pappus wrote on Apollonius of Perga (262-190 BC) that 'he spent a very long time with the pupils of Euclid at Alexandria, and it was thus that he acquired such a scientific habit of thought'. In the same text, Pappus wrote a favorable comment on Euclid as a response to the less favorable Apollonius' opinion on Euclid's work on conics.

From 1332 to 1493 Euclid was believed to be the philosopher Euclid of Megara who lived about 400 BC. In 1493, Constantinus Lascaris resolved definitively the error. Other misunderstandings and questionable anecdotes related to Euclid and his Elements come from the Arabian authors, some of which defended the theory that it was Apollonius, not Euclid, the author of the Elements. Euclid not only wrote the Elements, at least half a dozen of other scientific works were surely authored by Euclid. Among them:

The Data: an introduction to higher analysis.

The Phenomena: on theoretical astronomy.

The Optics: on the (rectilinear) propagation of light.

Elements of Music: on harmony and Pythagorean theory of music.

The Porisms: three lost books of very controversial content (probably advanced mathematics).

6.3 Euclid's Elements

Many contemporary textbooks include in their titles the word "elements" as in *Elements of Chemistry* or *Elements of Geology*, a tradition dating back to the Ancient Greece. Those books are usually intended to provide the reader with the basics to begin with a science. In the sense that those basic elements lead to other more advanced elements. They usually include the definitions, principles and axioms on which the corresponding sciences are founded.

Euclid's Elements serve the same purpose, though they represent a little more. But before paying attention to them, let us recall that they were neither the first nor the last Elements of Geometry. Among other authors of *Elements* we find Hippocrates of Chios (not to be confused with physician Hippocrates of Kos), Leon, Theudius of Magnesia, Amyclas of Heraclea, Cyzicenus of Athens, Philippus of Mende or Aristaeus. The success of Euclid's Elements, perhaps the most read and studied book ever, made practically disappear the other *Elements*. According to T.L. Heath [9, p. vii], Euclid's work is 'one of the noblest monuments of antiquity'. I fully agree. Euclid's Elements represent:

a) A model of how to proceed in the development of a mathematical theory.

b) A model of mathematical reasoning.

c) A prototype of the axiomatic method (scientific method of the formal sciences).

d) A compendium of the main geometric results known in Euclid's time.

e) The creation of a new science.

In Euclid's Elements we find a preoccupation for its logical structure together with an aperture to informal reasoning, that open the door to innovation and research. But, even being an 'immortal work', we should not forget that Euclid's Elements were wrote 2300 years ago. And, as could not be otherwise, it is far from being a perfect work. As any other work of any other human being. In my opinion, perfection is not an attribute of human beings' works. Taking into account the history of mathematics, Euclid's work cannot be required to have the same level of formalism that is required today in contemporary works, which have the enormous advantage over Euclid's of more than two thousand years of research by thousands of mathematicians all over the world.

Euclid's Element is a collection of 13 books (in some editions appear two more books, XIV and XV, which surely are not authored by Euclid [14, p. 14-15]) on plane geometry, space geometry and arithmetics distributed according to:

a) Plane geometry: Books I, II, III, and IV.
b) Theory of proportions. Books V and VI.
c) Arithmetics: Books VII, VIII, IX.
d) Irrational lines: Book X.
e) Space geometry: Books XI, XII and XIII.

Among the authors on whose experience and achievements Euclid built his Elements we must mention the followings [8, p. 9]:

a) Pythagoreans: Books I, II, III, IV, VII and IX.
b) Archytas: Book VIII.
c) Eudoxus: Books V, VI, and XIII.
d) Theaetetus: Books X and XII.

The thirteen books include 5 general axioms, 5 geometric axioms, 131 definitions and 465 propositions. The propositions proved in one of the books can be used to prove other propositions in the same or in other subsequent books, so that between them there exists a complex network of formal relations that are now being analyzed with the aid of graph theory and computer programming [30]. These types of analyses allow to calculate the number of formal connections between any two propositions as well as the number of logical paths connecting two propositions. The Book I, which has been always considered as the most perfect of the thirteen books, is the richest regarding the number of formal connections between its propositions. For instance, between Proposition

1 and Proposition 45 there is a formal path composed of 20 different propositions, and 558 different logical paths connecting them [30, p. 25]. It is also the book that poses Euclid's enigma we will examine in the next section.

In the Scientific Revolution, Descartes (1596-1659) introduced analytic geometry, which made it possible to express geometrical concepts by means of algebraic equations. This new geometry did not invalidate Euclid's original geometry, on the contrary, it was the cause of its evolution towards a more powerful discipline, Euclidean analytic geometry, that would eventually become the science of physical space. Indeed, Newton's defended the idea of an absolute and real Euclidean space. Space was the physical container of all physical objects.

Most contemporary physicists defend the opposite position: space is not real, it is only a relational instrument, a fiction, necessary to deal with certain relationships between physical objects. For example [31, p. 266]:

> ... space and time, like society, are in the end also empty conceptions. They have meaning only to the extent that they stand for the complexity of the relationships between the things that happen in the world.

Although, at the same time, these physicists also defend that space expands, deforms, vibrates and is the transmitter medium of its own (and other) vibrations. It is not easy to explain how something that does not exist can deform and vibrate. From a formal point of view, the vibrations of a non-existent object should not exist either.

But returning to Euclid's Elements, it is worth noting that until the firsts years of the 20th century, this text continued to be the text used throughout the world in order to acquire a basic education in geometry. And somehow changing the forms, its most significant achievements are still present in contemporary teaching and learning of elementary geometry. Although with somewhat more rigor, most of Elements' propositions' are still proved today in the same way as Euclid did 2300 years ago.

6.4 The enigma of the parallel straight lines

To avoid the infinite regress of arguments and circular arguments, all sciences, whether formal or experimental, must be built on assertions whose veracity must be accepted without proof. In the formal sciences these assertions are known as axioms. Ideally they should be short in number and highly self-evident. If we construct a science on an excessive number of axioms the output could result excessively speculative. If the axioms are not self-evident the output would be excessively abstract. For these reasons the set of axioms selected to found a formal science should be carefully examined. In the case of the experimental sciences, biology, geology, physics and chemistry, it is the inductive knowledge (that of Russell's chicks of Chapter 4) which guides the choice of axioms, which are usually called principles or fundamental laws.

Euclid's Element are based on five general axioms (that apply to all sciences) and five geometric axioms (Euclid's Postulates). It is this group of axioms, or postulates, which poses the problem of Euclid's enigma, also known as the parallel enigma. A simple reading of these five axioms suffices to understand from were the problem arises.

• Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

The first four postulates are short and self-evident assertions. The fifth one is neither short nor self-evident. It has rather the aspect of a typical Euclidean proposition or theorem. For both reasons it was put into question, as such a postulate, from the very beginning of the history of Euclidean geometry. The same questions have been asked for centuries:

1. Can this Fifth Postulate be derived from the other four?

2. Can Euclidean geometry be built without the Fifth Postulate?
3. What differences would there be between a geometry with the Fifth Postulate and one without the Fifth Postulate?

These questions summarize the enigma of the Fifth Postulate. In 1868 E. Beltrami proved the Fifth Postulate cannot be deduced from the other four [3], so it is not necessary to include it in the foundational bases of other types of geometries (see next section).

The first known attempt to resolve the enigma of the Fifth Postulate dates from the 2th century AD. And the attempts continued until the end of the 19th century, even after the birth of non-Euclidean geometries. So, the accumulated literature on the Fifth Postulate is enormous (see [25]). Among the main authors that tried to solve the problem of parallels we found: Ptolomy (2nd century), Proclus (5th century), al-Gauhary (9th century), Omar Khayyam (11th century), Nasir ad-Din at-Tusi (13th century), John Wallis (1616-1703), Gerolamo Saccheri (1667-1733), J. H. Lambert (1728-1777), J. L. Lagrange (17-36-1813), or A. M. Legendre (1752-1833).

Euclidean geometry is intuitive because it is closely and unequivocally related to our interactions and experiences with the physical world, in which we perceive a space and objects arranged in that space. For that reason, Euclidean geometry is easy to understand. Although, for that very reason, it is not uncommon to take for granted what cannot be taken for granted. Or in other words, it is very easy to assume hypotheses implicitly, without realizing that one is assuming an implicit hypothesis, i.e. an hypothesis that is not included in the initial basis of hypotheses (axioms) that should be the only hypotheses used in demonstrations. This type of error has always been present in the history of Euclidean geometry, particularly in the history of the Fifth Postulate. A matter on which, after centuries of discussions, the only thing that could be found were alternative statements for Euclid's Fifth Postulate. Most of them are in themselves problems of great geometric interest. These include the following ones (taken from [9, p. 220] and [16]):

1) Through a given point only one parallel can be drawn to a given straight line (Proclus and Playfair).
2) If a straight line intersects one of two parallels, it will intersect

the other also (Proclus).

3) Straight lines parallel to the same straight line are parallel to one another (Proclus).
4) Parallels remains, throughout their length, at the same finite distance from one another (Proclus).
5) There exist straight lines everywhere equidistant from one another (Posidonius and Geminus).
6) Non-equidistant straight lines converge in a direction and diverge in the other (Thabit ibn Qurra).
7) If in a quadrilateral figure three angles are right angles, the fourth angle is also a right angle (Clairaut).
8) Two perpendiculars of the same length to the same straight line defines a rectangle (Farkas Bolyai).
9) There exists a triangle in which the sum of the three angles is equal to two right angles (Legendre).
10) A straight line perpendicular to a side of an acute angle cuts also the other side (Legendre).
11) Through any point within an angle less than two-thirds of a right angle, a straight line can be drawn which meets both sides of the angle (Legendre).
12) There exists no triangle in which every angle is as small as we please (Worpitzky).
13) Given any figure, there exists a figure similar to it of any size we please (Wallis, Carnot and Laplace).
14) There exist two unequal triangles with equal angle (Saccheri).
15) A rectilineal triangle is possible whose area is greater than any given area (Gauss).

The positive side of all this work is that, thought Euclid's enigma could not be resolved, Euclidean geometry was enriched and extended with an increasing collections of new and exciting problems. It seems that, indeed, Euclid's Fifth Postulate cannot be deduced from the other four postulates, but the question that interests us here is: Can the Fifth Postulate be deduced from a foundational basis other than Euclid's? Ockham's razor suggests an affirmative answer. And Ockham's razor is not usually wrong.

6.5 The birth of non-Euclidean geometries

By the second half of the 18th century, and despite the great effort made so far, no progress had been made on the problem of parallels 'the scandal of elementary geometry', as it was called in Lagrange time (cited in [29]); or 'the shameful part of mathematics' in Gauss' words (cited in [17, p.9]). In the late eighteenth and early nineteenth some mathematicians, as Gauss, began to consider the possibility of non-Euclidean geometries. Geometries without the Fifth Postulate or even geometries based on postulates claiming the contrary that some of the Euclid's postulates.

The speculations on non-Euclidean geometries were initially developed as a way to prove the Fifth Postulate by *reductio ad absurdum* (this was the case, for instance, of the attempts by Giovanni Girolamo Saccheri (1667-1733)). The idea of true non-Euclidean geometries came into the scene in the first half of the nineteenth century, with some precedents as 'the geometry of visibles' discussed by the philosopher Thomas Reid (1710-1796) [24, Chp. 6 §9], [33]:

> The shape of visible figures are geometrically equivalent to their projection onto the surfaces of spheres.

Sometimes it happens that a new idea occurs to several authors more or less simultaneously. This was what happened with the discovery of non-Euclidean geometries in the nineteenth century. Indeed, Karl F. Gauss (1777-1855), Ferdinand Schweikart (1780-1857) Nicolái Lobachevsky (1792-1856), János Bolyai (1802-1860), and Bernhard Riemann (1826-1866), were contemporary mathematicians of the 19th century, and it was they who established the foundations of the new non-Euclidean geometries. Though Gauss, one of the most important mathematicians of all times, never published his ideas on non-Euclidean geometry (a term coined by himself in 1824). Bolyai could only publish his work in an appendix of his father's book *Tentament* (a compendium of mathematics). And Lobachevsky first publication (1829) on his 'imaginary geometry' could only be carried out in a rather unknown journal. Things were somehow different for Schweikart, who in 1818 succeeded in publishing his 'astral geometry,' perhaps the first serious publication intended to explore the new geometries.

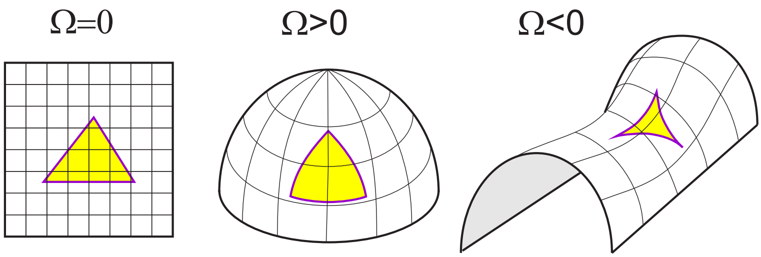


Figure 6.1 – A triangle on three surfaces: euclidean (curvature (Ω) zero); Elliptic (curvature greater than zero); and hyperbolic (curvature less than zero)

Among the pioneers of non-Euclidean geometries, Riemann had the most profound insight of the new Copernican revolution in geometry. He developed the idea of abstract geometrical surfaces, independent of Euclidean space, whose curvatures could be precisely defined. All geometries exist on such surfaces: elliptic and spherical geometries if the surface curvature is positive; hyperbolic geometry (the geometry introduced by Bolyai and Lobachevsky) on surfaces of negative curvature; and Euclidean geometry on surfaces with zero curvature (Figure 6.1).

In the year 1868, Eugenio Beltrami proved the consistency of the non-Euclidean geometries [3], and then the impossibility to deduce the Fifth Postulate from the other four Euclid's postulates. The proof consisted in developing an Euclidean model of non-Euclidean geometries. So, non Euclidean geometries get legitimized. And non only legitimized, Albert Einstein popularized Riemann geometry in his general theory of relativity.

The contrast between Euclidean and non-Euclidean geometries is quite clear. The Hyperbolic Axiom reads:

> There exists a line l and a point P not in l such that at least two distinct coplanar lines parallel to l pass through P.

The Elliptic Axiom states:

> Through a point exterior to a given line, there is no line parallel to the given line.

While Playfair's Axiom (a variant of Euclid's Fifth Postulate) reads:

> Through a given point one, and only one, parallel can be drawn to a given straight line.

Apart from the non existence of parallels, another notable difference between Euclidean geometry and Riemann elliptic geometry is that in the latter there are infinitely many different straight lines passing through the same two points, which contradicts the strong version of Euclid's First Postulate, according to which there is only one straight line between any two points. Euclid's original statement (weak version of the First Postulate) establishes the existence of (at least) one straight line between two points. Hence, his statement is compatible with the existence of more than one straight line between two points. Although it does not seem probable that this was Euclid's belief, nor that of the majority of the subsequent Euclidean authors. We should consider the possibility that in non-Euclidean geometries there is a certain abuse of language [11]. In the next chapter, it will be shown that any two points can be the endpoints of one, and only one, straight line (according to the definition of straight line proposed in the next chapter).

6.6 The new Euclidean geometry

The main objective of this book is to introduce a new Euclidean geometry. It has been called new because it is built on a completely new logical base, scarcely related to other foundational bases of geometry, such as Euclid's [9], Playfair's [21, 22] or Hilbert's [10]. It is an Euclidean geometry because it arrives at the same key results as other Euclidean geometries, for instance:

> There is one, and only one, straight line joining any two points (strong form of Euclid's Postulate 1).
>
> Through any point it is possible to draw one, and only one, parallel to another straight line (Playfair's Axiom 11; Hilbert's Axiom III.1).

Although in the case of the new geometry, they will not be axiomatic statements but proved conclusions. (Cr. 15 and Th. 36

respectively). It could be said, and it would be true, that the geometry introduced here does not include anything that could not have been included in Euclid's time. It would be true, but not fair, because it has had the great advantage of more than two thousand years of researches and discussions on Euclidean and non-Euclidean geometries. Euclid could not look from this fortunate perspective.

As mentioned in the previous section, all attempts to solve the mystery of parallels had one point in common: they all used the concept of a straight line as a primitive concept, and therefore as a formally unproductive concept. This circumstance made it possible the development of geometries that assume different postulates about straight lines that are contradictory to each other. The Euclidean geometry introduced in this book uses a definition of straight line that is formally productive, a definition that can be used in proofs. From this definition it is possible to deduce as theorems some of the axioms on straight lines, so that the opposite versions of those axioms are not compatible with the definition of straight line that is proposed here. As a consequence, straight lines, as defined in this geometry, could be exclusive objects of Euclidean geometry. After all, the supposed straightness of non-Euclidean straight lines might not be so straight. Or in other words, objects with non-zero curvatures should be considered curved, not straight.

The concept of straight line is a primitive concept of the third level: a primitive concept (straight line) based on a primitive concept (line) based on a primitive concept (point). Perhaps too undefined concepts on the same subject. We should have insisted on searching for a definition formally productive of straight line, because straightness is the most determinant attribute in the geometric behavior of straight lines and planes. The incompleteness and controversial history of the original Euclidean geometry [27, 5, 7, 16, 29, 6, 4, 32, 26] could have been caused by the lack of such a definition. So, would it be possible a definition of straight line that capture in exclusive and productive terms the essence of *straightness*? Here 'productive' means a definition that can be used in the demonstrations, as is the case, for instance, of the definition of right angle. The definition of straight line proposed in the next chapter meets both requirements. It was inspired by the following definitions and commentary:

Definition by Heron of Alexandria (10-70 DC) [9, p. 168]: [a line such that] all its parts fit on all (other parts) in all ways.

Definition by Proclus (412-485 DC) [9, p. 168]: that line which with one another of the same species cannot complete a figure.

Definition by J. Playfair (1748-1819 DC) [21, p. 8]: if two lines are such that they cannot coincide in any two points without coinciding altogether, each of them is called a straight line.

Commentary by E. Beltrami (1835-1900 DC) [3, p. 2]: [a line whose] specific character is to be completely determined by only two of its points, because two [straight] lines cannot pass through two given points of space without coinciding in all their extension.

Figure 6.2 suggests the way towards a formally productive definition of straight line.

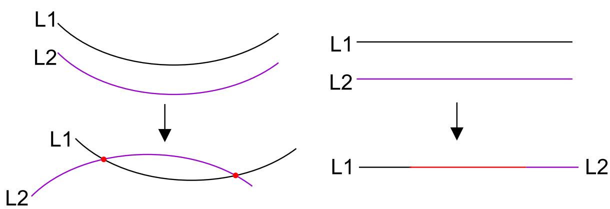


Figure 6.2 – Superimposing curves and straight lines. Something that is possible with curves is not with straight lines. Note the common points.

Other significant conclusions of the new geometry are related to the parallel straight lines, which are also defined in a more productive way than the following original definition by Euclid [9, p.154]:

> Definition 23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinetely in both directions, do not meet one another in either direction.

The geometry of J. Playfair contains a definition of parallel lines that is almost the same as Euclid's definition [21, p. 11]. And for his part, Hilbert's foundation of euclidean geometry also included a definition similar to Euclid's; in this case the definition is a part of his Axiom III.1 on parallel straight lines [10, p. 7]. In none of the cases, however, was demonstrated the existence of parallel straight lines. This is not he case of the geometry that is developed in the following chapters. Here the existence of parallel straight lines will be formally demonstrated. And other historically significant results on parallel and non-parallel straight lines will also be proved. At this point, it seems to me convenient to emphasize the fact that the author of these new elements of geometry is not more competent or smarter than other authors, he simply had the idea of constructing a new basis for geometry that would make possible the formal use of the straightness of straight lines, of planes and of their respective parts. Geometry is in geometry, humans can only look for it.

Once the fundamentals (29 definitions, 10 axioms and 45 corollaries) have been introduced, the development of the new Euclidean geometry begins, starting with Theorem 1 (an extended version of Euclid's Proposition 3) and ends with Theorem 49 in which Euclid's Fifth Postulate is finally proved. The presentation of this new Euclidean geometry (henceforth simply Euclidean geometry) is completed in the following two chapters in which other classical statements of Euclidean geometry, some of which were proposed as alternative statements to Euclid's Fifth Postulate, are proved. As a comparative reference, and so that the effect of two thousand

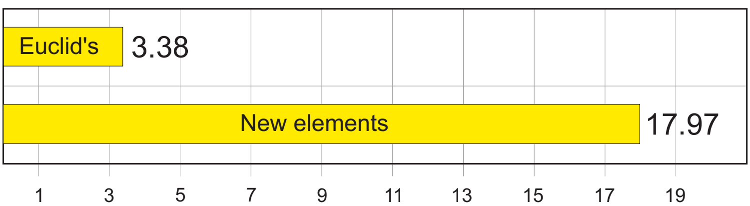


Figure 6.3 – Average number of logical justifications per proposition in the Book I of Euclid's Elements (28 first propositions) and in the new elements introduced in this book (49 first propositions).

years of discussions on the foundations of Euclidean geometry can be appreciated, Figure 6.3 compares the number of explicit logical justifications in Euclid's Elements and in this work, where a logical justification is each of the reasons given to justify in logical terms what is being asserted or constructed. As noted in Chapter 4, those reasons may be legitimized definitions, axioms or previously proved theorems and corollaries. In Euclid's case, the reasons are not always given. And when they do not, sometimes those reasons are obvious; and sometimes they are not. Thus, Figure 6.3 illustrates the progress, over 2300 years, of the mathematical rigor in demonstrations.

In the first half of the 20th century, faced with the endless frustration with the Parallel Postulate and the success of non-Euclidean geometries, the Bourbaki group ended up proclaiming: Death to Euclid! But without its past, mathematics is impoverished and impoverishes those who dedicate themselves to its study. So, long live Euclides! long live Bourbaki!

7. New Foundational Basis of Euclidean Geometry

7.1 Introduction

This chapter introduces a new foundational basis for Euclidean geometry that includes formally productive definitions of concepts hitherto considered as primitive concepts, such as straight line, plane or angle, among others. All of them are legitimized by axioms or formal proofs. The result is an enriched Euclidean geometry in which it is possible to prove some statements that turned out to be unprovable on other bases of Euclidean geometry. Among them the strong form of Euclid's First Postulate, Euclid's Second Postulate, Hilbert's axioms I.5, II.1, II.2, II.3, II.4 and IV.6, Euclid's Postulate 4, Axiom of Posidonius-Geminus, Axiom of Proclus, Axiom of Cataldi, Axiom of Tacquet 11, Axiom of Khayyam, Axiom of Playfair, Euclid's Postulate 5 and the historical statements proposed in the place of Euclid's Postulate 5.

The proposed basis is formally more detailed and productive than other classical and modern alternatives, and at least as accessible as any of them. It is also included a general part that is not exclusive to geometry, but that is also used in geometry. This general part includes four definitions and three postulates. The new basis of Euclidean geometry consists of 29 definitions, 10 axioms and 45 corollaries. The high number of foundational corollaries, many of them implicit hypotheses in other Euclidean geometries, stands out. The new fundamentals are divided into seven parts:

Fundamentals on:

1) lines: 9 definitions; 4 axioms; 13 corollaries.
2) straight lines: 3 definitions; 1 axiom; 11 corollaries.
3) planes: 2 definitions; 1 axiom; 8 corollaries.
4) distances: 4 definitions; 1 axiom; 1 corollary.
5) circles: 1 definition; 1 axiom; 3 corollaries.
6) angles: 8 definitions; 1 axiom; 7 corollaries.
7) polygons: 2 definitions; 1 axiom; 2 corollaries.

The definition of an object includes the definition of its parts and properties. And sometimes that of closely related objects or properties, most of which are basic and well known, but which need to be defined in formal terms so that they can be legitimized and used accordingly in the demonstrations that follow. When an axiom establishes the existence of an object, it establishes the existence of all its parts and properties.

The corollaries that follow the definitions and axioms are almost immediate consequences of the definitions, axioms, and other corollaries already proved. Although some of the corollaries could have been labeled as theorems, I have preferred to name them as corollaries simply because they belong to this new set of basic statements that fundaments the Euclidean geometry introduced in this book. Note the novelty of the axioms with respect to other fundamental foundations of Euclidean geometry: eight are completely new, and the other two, although related to two classical axioms, present significant differences with respect to them.

7.2 General Fundaments

The following four definitions and three postulates are not exclusive to geometry, they have a general use in all sciences. For that reason they have been separated from the very fundamentals of geometry and named with letters in the place of numbers.

Definition A *A quantity to which a real number can be assigned is said a numerical quantity. Numerical quantities that can be sym-*

bolically represented and operated with one another according to the procedures and laws of algebra, are said operable values.

Definition B *An operable value is said to vary in a continuous way iff for any two different operable values of the corresponding variation, the variation contains any operable value greater than the less and less than the greater of those two operable values.*

Definition C *Metric properties and metric transformations: properties (transformations) to which operable values that vary in a continuous way are univocally assigned: to each quantity of the property (transformation) a unique and exclusive operable value, even zero, is assigned.*

Definition D *To define an object is to give the properties that unequivocally identifies the object. Objects with the same definition are said of the same class. To draw objects is to make any descriptive representation of them by means of graphics or texts, or by both of them, without the drawing modifies neither their established properties nor their established relations with other objects, if any.*

Postulate A *Of any two operable values, either they are equal to each other, or one of them is greater than the other, and the other is less than the one. Symbolic representations of equal operable values, or of equal objects, are interchangeable in any expression where they appear.*

Postulate B *To be less than, equal to, or greater than, are transitive relations of operable values that are preserved when adding to, subtracting from, multiplying or dividing by the same operable value, the operable values so related. Metric properties (transformations) are algebraically operable through their corresponding operable values.*

Postulate C *Belonging to, and not belonging to, are mutually exclusive relations. Belonging to is a reflexive and transitive relation.*

Contrarily to, for instance, fuzzy set theory or non-Boolean logics, this Euclidean geometry assumes [Ps. C], according to which it is not possible for an object to partially belong and partially not to belong to another object.

7.3 3. Foundational basis of Euclidean geometry

FUNDAMENTALS ON LINES

Definition 1 *Endedness.- A point at which a line ends is said endpoint. If such a point belongs to the line, the line is said closed at that end; if not, the line is said open at that end. Two endpoints of a line, whether or not in the line, define two opposite directions in that line, each from an endpoint, said initial, to the other, said final.*

Definition 2 *Collinearity.-Of the points that belong to a line is said they are points of the line, or points that are on the line; and the line is said to pass through them. A line whose points belong, all of them, to a given line is said a segment of the given line. Two points of a line are said different iff they are the endpoints of a segment of the line. Two lines are said different, iff one of them has at least one point that is not in the other. Different points and segments of the sane line are said collinear; points and segments that do not belong to the same line are said non-collinear.*

Note.-The expression line passing through one or more points may be simplified to line through one or more points.

Definition 3 *Commonness.-Points and segments belonging to different lines are said common to them, otherwise they are said non-common to them. Non-collinear lines with at least one common segment are said locally collinear. Lines without common segments but with at least one common point are said intersecting lines, and their common points are also said intersection points. Intersecting lines are said to cut or to intersect one another at their intersection points.*

Definition 4 *Adjacency.-Lines whose unique common point is a common endpoint are said adjacent at that common endpoint iff no point of any of them is a non-common endpoint of any of the others. Lines containing all points of a given line, and only them, are said to make the given line.*

Definition 5 *Sidedness.-Adjacent lines containing all points of a given line, and only them, whose common endpoint is a given point* *of the given line and whose non-common endpoints are the endpoints of the given line, if any, are said sides of the given point in the given line.*

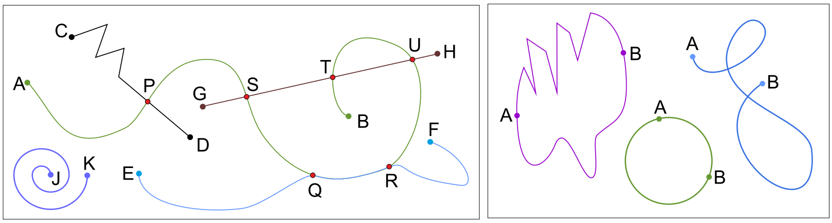


Figure 7.1 – Left: A, B: endpoints of AB; C, D: endpoints of CD etc. AB, EF: locally collinear lines. AP, PS: lines (segments) adjacent at P. AP, PB: sides of P in AB. QR: common segment of AB and EF. S is between A and Q; between P and R etc. Right: self-closed lines.

Definition 6 *Betweenness.-A point is said to be between two given points of a line, iff it is a point of that line and each of the given points is in a different side of the point in that line.*

Definition 7 *Self-closed line: a line in which each pair of its points are the common endpoints of two of its segments, said complementary, whose points contain all points of the line, and whose only common points are their common endpoints. Lines with self-closed segments are said self-inter secting. Self-closed and self-intersecting lines are also called figures.*

Definition 8 *Uniformity.-Lines whose segments have the same definition as the whole line are said uniform. Two or more uniform lines are said mutually uniform iff any segment of any of them has the same definition as any segment of any of the others.*

Definition 9 *Metricity.-Length (area) is an exclusive metric property of lines (figures) of which arbitrary units can be defined. Lengths (areas) are said equal iff their corresponding operable values are equal. Lines (figures) with a finite length (area) are said finite. If the sides of a point of a line have the same length, the point is said to bisect the line.*

Axiom 1 *Point, line and surface are primitive concepts of which any number, and in any arrangement, can be considered and drawn.*

Axiom 2 *A line has at least two points, at least one point between any two of its points, and at most two endpoints, whether or not in the line.*

Axiom 3 *Two adjacent lines make a line, and a point of a line can be common to any number of any other different lines, either collinear, or non-collinear, or locally collinear.*

Axiom 4 *Being not a figure, each point of a line, except endpoints, has just two sides in that line, whose lengths are greater than zero and sum the length of the whole line.*

Unless otherwise indicated, from now on figures will be given particular names, for example circle, and will always be referred to by those particular names. The rest of the lines will be closed at their endpoints, if any.

Corollary 1 *The number of points of a line is greater than any given number.*

It is an immediate consequence of [Axs. 1, 2]. □

Corollary 2 *Each side of a point, except endpoints, of a line is a segment of the line and both sides make the line.*

Except endpoints, a point P of a line l [Ax.1, Cr. 1]

has two, and only two, sides in l [Ax. 4],

which are two lines adjacent at P [Df. 5]

containing all points of l, and only them [Df. 5].

So, each side is a segment of the line [Dfs. 5, 2],

and both sides make the line l [Ax. 3, Df. 4]. □

Corollary 3 *Any point of a line is in one, and only in one, of the two sides of any other point, except endpoints, of the line.*

Except endpoints, a point P of a line l [Ax.1]

has two, and only two, sides in l [Ax. 4].

Any other point of l [Cr. 1]

will be in one of such sides [Cr. 2],

and only in one of them, otherwise both sides would not be adjacent at P [Df. 4],

which is impossible [Dfs. 5, 4]. □

Corollary 4 *A point is in a line with two endpoints iff, being not an endpoint of the line, it is between the endpoints of the line.*

If a point P is between the two endpoints of a line AB [Axs. 1, 2, 4, Df. 6],

it is in AB [Df. 6].

If a point P is in a line AB and is not an endpoint of AB [Cr. 1],

it has just two sides in AB [Ax. 4],

whose respective non-common endpoints are the endpoints A and B of AB [Dfs. 5, 4].

So, P is between both endpoints A and B [Df. 6]. □

Note.-Unless otherwise indicated, from now on a point P of a line AB will be a point of AB between A and B.

Corollary 5 *Any two points of a line are the endpoints of a segment of the line. And the line has a number of segments and a number of points between any two of its points greater than any given number.*

Let P and Q be any two points of a line l different from its endpoints, if any [Ax.1, Cr. 1].

Q has two sides in l [Ax. 4],

which are two lines l_1 and l_2 adjacent at Q [Df. 5]

that contains all points of l and only them [Cr. 2].

So, in one, and only in one, of such lines, for instance in l_1, will be P [Cr. 3].

In turn, P has two sides in that side l_1 of Q [Df. 5, Ax. 4],

the side PQ in which it is Q and the side in which it is not Q [Cr.

3].

PQ is a line [Df. 5]

all of whose points belong to l_1 [Df. 5]

and therefore to l [Ps. C].

Hence, PQ is a segment of l [Df. 2].

Being P and Q any two of its points, l has a number of segments and a number of points between any two of its points greater than any given number [Crs. 1, 4]. □

Corollary 6 *A segment of a segment of a line, it is also a segment of that line.*

Let RS be a segment of a segment PQ of a line l [Ax.1, Cr. 5].

PQ is a line whose points belong to l [Df. 2].

RS is a line whose points belong to PQ [Df. 2],

and then to l [Ps. C].

So, RS is a segment of l [Df. 2]. □

Corollary 7 *If a point is between two given points of a given line, it is also between the given points in any other line of which the given line is a segment.*

Let R be a point of a segment PQ of a line l' [Ax.1, Cr. 5],

which is a segment of another line l [Cr. 5].

Since PQ is a segment of l', it is also a segment of l [Cr. 6].

So, R is a point of a segment PQ of l [Df. 2],

and then a point of l [Df. 2]

between P and Q [Cr. 4]. □

Corollary 8 (A variant of Hilbert's Axiom II.2) *At least one of any three points of a line is between the other two.*

Let P, Q and R be any three points of any line l [Ax.1, Cr. 1].

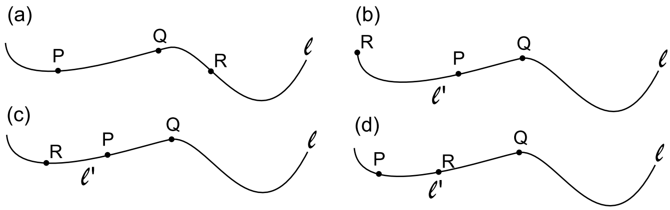


Figure 7.2 – **Corollary 8**

At least one of them, for example* Q, will not be an endpoint of l [Ax. 2].

P can only be in one of the two sides of Q in l [Cr. 3].

R can only be in one of the two sides of Q in l [Cr. 3].

So, either P and R are in different sides of Q in l, or they are in the same side of Q in l. If P and R are in different sides of Q in l (Fig. 7.2 (a)), then Q is between P and R in l [Df. 6].

If not, P and R are in the same side of Q in l, which is a segment l' of l [Cr. 2],

one of whose endpoints is Q [Df. 5].

If R is an endpoint of l' (Fig. 7.2 (b)), P can only be between the endpoints Q and R of l' [Cr. 4],

and then between Q and R in l [Cr. 7].

If R is not an endpoint of l', it has two sides in l' [Ax. 4]:

the side RQ in which it is Q, and the side in which it is not Q [Cr. 3].

If P is in RQ (Fig. 7.2 (c)), P is between R and Q in l' [Cr. 4],

and then between R and Q in l [Cr. 7].

If P is in the side of R in l' in which it is not Q (Fig. 7.2 (d)), then P and Q are in different sides of R in l', and R is between P and Q in l' [Df. 6]

and then between P and Q in l [Cr. 7].

So, in all possible cases [Ax. 4, Cr. 3]

at least one of the three points is between the other two in l. □

Corollary 9 (Hilbert's Axioms II.3, II.1) *One, and only one, of any three points of a line is between the other two.*

Let P, Q and R be any three points of any line l [Ax.1, Cr. 1].

At least one of them, for example* Q, will be between the other two, P and R, in l [Cr. 8],

in which case Q is a point of PR [Cr. 4].

So, Q has two sides in PR [Ax. 4],

which are two lines, QP and QR, adjacent at Q [Df. 5].

P cannot be between Q and R, otherwise it would be in QR [Cr. 4],

QP would be a segment of QR [Cr. 5],

all points QP [Cr. 1]

would be points of QR [Df. 2],

and QP and QR would not be adjacent at Q [Df. 4],

which is impossible [Df. 5].

For the same reasons R cannot be between P and Q either. Therefore, one [Cr. 8],

and only one, of any three points of a line is between the other two. □

Corollary 10 (a variant of Hilbert's Axiom II.4) *Of any four points of a line, two of them are between the other two.*

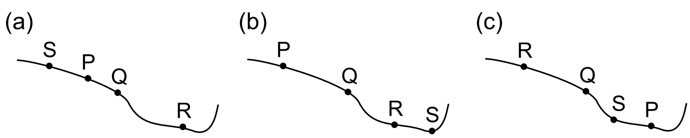


Figure 7.3 – **Corollary 10**.

Let P, Q, R and S be any four points of a line l [Ax.1, Cr. 1].

Consider any three of them, for instance* P, Q and R. One, and only one, of them, for instance* Q, will be between the other two, P and R [Cr. 9], and Q will be in PR [Cr. 4].

Of the other three points P, R and S, one, and only one, of them will be between the other two [Cr. 9]:

if P is between S and R (Fig. 7.3 (a)), it is in SR [Cr. 4],

so that PR is a segment of SR [Cr. 5],

Therefore Q, which is in PR, is also in SR [Cr. 6].

So, Q and P are between R and S [Cr. 4].

For the same reasons, if R is between P and S (Fig. 7.3 (b)) then Q and R are between P and S; and if S is between P and R (Fig. 7.3 (c)), then Q and S are between P and R. So, in all possible cases [Ax. 4, Cr. 3]

two of the four points are between the other two. □

Corollary 11 *Two segments can only be either collinear or non-collinear. And if a segment of a given line is non-collinear with another segment of another given line, then both given lines are also non-collinear.*

Since belonging to is a reflexive relation [Ps. C]

and segments are lines [Df. 2],

any two segments l_1 and l_2 [Ax. 1]

belong to a line, even if the line is the own segment itself [Df. 2].

So, l_1 and l_2 will be either collinear, or non-collinear, or collinear and non-collinear. If they were collinear and non-collinear they would be segments that belong to the same line l [Df. 2],

and segments that do not belong to the same line l [Df. 2],

which is impossible [Ps. C].

So, l_1 and l_2 can only be either collinear or non-collinear. Let now l'_1 be a segment of a line l_1, and l'_2 another segment of a line l_2 [Cr. 5],

such that l'_1 and l'_2 are non-collinear [Df. 2].

If l_1 and l_2 were collinear, they would be segments of the same line

l [Df. 2],

and being their respective segments l'_1 and l'_2 also segments of l [Cr. 6],

l'_1 and l'_2 would also be collinear [Df. 2],

which is not the case. Hence, l_1 and l_2 must also be non-collinear. □

Corollary 12 *If two points of a line have a given property, and all points between any two points with the given property have also the given property, then the line has a unique segment whose points are all points of the line with the given property.*

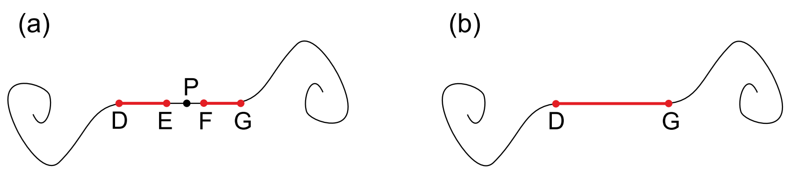


Figure 7.4 – **Corollary 12**.

Let A and B be two points [Ax.1, Cr. 5]

with a given property (gp-points for short) of a line l such that all points of l between any two of its gp-points are also gp-points. So, l has a number of gp-points greater than any given number [Cr. 5].

Let a segment whose points are gp-points, except at most its end-points, be referred to as gp-segment. Any gp-point C of l is at least in the gp-segment AC of l [Crs. 5, 4].

So, all gp-points of l are in gp-segments. If all gp-points of l were not in a unique gp-segment, they would be in at least two gp-segments DE and FG of l [Cr. 5],

so that, being* E and F between D and G [Cr. 10],

DG is not a gp-segment. If so, there will be at least one point P between D and G that is not a gp-point. P has two sides in DG, namely PD and PG [Ax. 4, Df. 5].

E must be in the side PD of P in DG in which it is D, otherwise it would be in the side PG of P in DG in which it is not D [Cr. 3],

P would be between D and E [Df. 6],

it would be a point of DE [Cr. 4],

and being gp-points all points of DE, except at most D and E [Cr. 4],

P would be between any gp-point of DP and any gp-point of PE [Ax. 2],

and P would be a gp-point, which is not the assumed case. So, DE is a segment of the side PD of P in DG [Crs. 5].

For the same reasons, FG is a segment of the other side PG of P in DG. Hence, P is between any gp-point of DE and any gp-point of FG [Df. 5].

It is then impossible for P not to be a gp-point, and for DG not to be a gp-segment. And l has a unique gp-segment DG. □

Corollary 13 *The length of a finite line is greater than the length of each of the sides of any of its points, except endpoints, and it is greater than zero. The length of each side is equal to the length of the whole line minus the length of the other side. And the length of a segment of the line is less than the length of the whole line if at least one endpoint of the segment is not an endpoint of the line.*

Let P be a point of a finite line AB [Df. 9, Axs. 1, 2].

Assume the length AP is not less than the length AB. It will be $AP \geq AB$ [Ps. A],

and being $AB = AP + PB$ [Ax. 4],

it would hold $AP \geq AP + PB$ [Ps. A].

Hence, $0 \geq PB$ [Ps. B],

which is impossible [Ax. 4].

So, it must be $AP < AB$ [Ps. A].

And for the same reasons $PB < AB$. Therefore, and being $0 < PB$ [Ax. 4],

it holds $0 < AB$ [Ps. B].

So, the length of any line is greater than zero. And from $AP +$

$PB = AB$ [Ax. 4],

it follows immediately $AP = AB - PB$; $PB = AB - AP$ [Ps. B].

Let now Q be any point of AB different from P [Crs. 1].

It will be in one, and only in one, of the sides of P in AB [Cr. 3],

for instance* in AP. It has just been proved that $AP < AB$. If Q were the endpoint A of AP we would have $QP = AP$ [Ps. A].

If not, and for the same reasons above, it will be $QP < AP$. So, we can write $QP \leq AP$, and then $QP < AB$ [Pss. B, A].

Therefore, the length of a segment of AB is less than AB if at least one if its endpoints P is not an endpoint of AB. □

Fundamentals on straight lines

Definition 10 *Extensible lines.-To produce (extend) a given line by a given length is to define a line, said production (extension) of the given line, so that the production is adjacent to the given line, has the given length, and the production and the produced line are lines of the same class as the given line. Lines that can be extended from each endpoint and by any given length are called extensible lines.*

Definition 11 *Straight lines: extensible and mutually uniform lines that can neither be locally collinear nor have non-common points between common points.*

Definition 12 *Straightness.-Three or more points are said to be in straight line with one another iff they are in the same straight line, whether or not produced. A point is said in straight line with a given straight line iff it is in straight line with at least two points of the given straight line, whether or not produced. Only the straight segments of the same straight line, whether or not produced, are said to be in straight line with one another. Otherwise it is said that they are not in a straight line.*

Axiom 5 *Any two points can be the endpoints of a straight line, and only both points are necessary to draw the straight line.*

Corollary 14 *A segment of a straight line is also a straight line.*

It is an immediate consequence of [Ax. 5, Dfs. 11, 8]. □

Corollary 15 (Strong form of Euclid's First Postulate) *Any two points can be the endpoints of one, and only of one, straight line.*

Assume two different straight lines l_1 and l_2 have the same endpoints A and B. At least one of them will have a point which is not in the other [Df. 2].

And they would have at least one non-common point between the two common points A and B, which is impossible [Df. 11].

So, any two points can be the endpoints of one [Ax. 5],

and only of one, straight line. □

Note.-Unless otherwise indicated, hereafter, to join two points will mean to consider and draw the unique straight line whose endpoints are both points.

Corollary 16 (Strong form of Euclid's Second Postulate) *There is one, and only one, way to produce a given straight line by any given length and from any of its endpoints, being the produced line a straight line; and the given straight line and its production, adjacent straight lines in straight line with each other.*

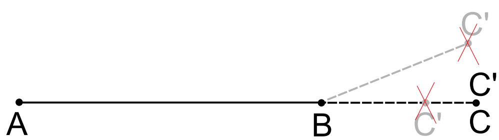


Figure 7.5 – **Corollary 16**.

Let AB be any straight line [Ax. 1, Cr. 15].

AB can be produced from any of its endpoints, for example* from B, by any given length [Dfs. 11, 10]

to a point C, so that BC and AC are straight lines [Dfs. 11, 10, D],

and AB and BC are adjacent segments [Dfs. 11, 10].

Assume AB can be produced from B by the same given length to another point C'. The straight lines AC, AC' [Dfs. 11, 10]

would have a common segment AB [Cr. 5];

they would be collinear since they cannot be locally collinear [Dfs. 11, 3, Cr. 11];

and BC and BC' would be two segments of the same line l [Cr. 5],

both adjacent at B to AB [Ax. 5, Df. 10],

and so with a common endpoint B. And being C and C' different points of the same line l, one of them, for example* C', would be between B and the other in l [Cr. 9],

and we would have $BC' < BC$ [Cr. 13],

which is not the case. So, C' can only be the point C. And being BC a straight line [Dfs. 11, 10, D],

it is the unique straight line joining B and C [Cr. 15].

So, there is a unique way of producing a straight line by a given length from any of its endpoints. And AB and BC are the unique straight lines joining respectively A with B and B with C [Cr. 15],

and being A, B and C points of the straight line AC [Dfs. 12, 11, 8],

the straight lines AB and BC are segments of the same straight line AC [Dfs. 2, 4].

Therefore, the straight lines AB and BC are in straight line with each other [Df. 12]. □

Corollary 17 *Through any two points, any number of collinear straight lines of different lengths can be drawn.*

It is an immediate consequence of [Df. 2, Crs. 15, 16]. □

Corollary 18 *Two straight lines with two common points belong to the same straight line.*

Let AB and CD be two straight lines with two common points P and Q [Cr. 17].

Consider one of them, for instance* AB. Every point R of AB is in straight line with two points, P and Q, of CD [Df. 12].

Therefore, every points R of AB belongs to CD, whether or not produced [Df. 12].

In consequence, AB is a segment of CD, whether or not produced [Df. 2, Cr. 16].

Hence, AB and CD belong to the same straight line: CD or a production of CD [Cr. 16]. □

Corollary 19 *Being in a straight line is a transitive relation of straight lines.*

Suppose that a straight line AB is in a straight line with another straight line CD, which in turn is in a straight line with another straight line EF. AB and CD belong to a straight line r_1. CD and EF belong to a straight line r_2 [Df. 12].

Since CD belongs to r_1 and r_2, the straight lines r_1 and r_2 have two common points C and D, so they belong to the same straight line r_3 [Cr. 18].

Consequently, A, B, E and F belong to r_3, and AB and EF are segments of r_3 [Cr. 5].

So, they are in straight line with each other [Ps. C, Df. 12]. □

Corollary 20 *A point is in straight line with a given straight line, iff its is in straight line with any two points of the given straight line.*

Let l be any straight line [Cr. 15].

A point P in straight line with l is in straight line with at leas two points Q and R of l, produced or not [Df. 12, Cr. 16].

So, P, Q and R belongs to l, produced or not [Df. 12, Cr. 16].

And being a point of l, P belongs to the same straight line l as any couple of points of l; and P is in straight line with them [Df. 12].

Alternatively, if P is in straight line with any two points of l, then it is in straight line with l [Df. 12]. □

Corollary 21 *Any point between the endpoints of a given straight line can be common to any number of intersecting straight lines not in straight line with the given straight line, and that point is the only common point of those straight lines and the given straight line, even arbitrarily producing them and the given straight line.*

Any point P between the endpoints of a straight line AB [Ax. 1, Cr. 15]

can be common to any number n of non-collinear straight lines [Ax. 3],

which being non-collinear are not in straight line with the given straight line AB [Dfs. 12, 2].

Assume there is a second common point Q of AB and of any one of those n intersecting straight lines l, whether or not producing AB and l [Cr. 16].

Both straight lines would belong to the same straight line [Cr. 18],

which is not the case, because they are non-collinear [Df. 3].

Therefore, P is the only intersection points of AB and each of those n intersecting straight lines, even arbitrarily producing AB and any of the n intersecting straight lines. □

Corollary 22 *There is a number of points greater than any given number that are not in straight line with any two given points, or with a given straight line.*

Let A and B be any two points [Ax. 1].

Join A and B [Cr. 15],

and let PC be a straight line non-collinear with AB that intersects AB at P [Cr. 21].

P is the only common point of both straight lines even arbitrarily produced [Cr. 21].

So, PC has a number of points greater than any given number [Cr. 1]

none of which, except P, is in straight line with A and B because none of them belong to AB, produced or not [Df. 12].

On the other hand, if AB is any straight line, it has just been proved there is a number greater than any given number of points that are not in straight line with the points A and B. So, there is a number greater than any given number of points that are not in straight line with AB [Cr. 20]. □

Corollary 23 *Each endpoint of a given straight line can be the common endpoint of any number of adjacent straight lines not in straight line with the given straight line.*

Let AB be any straight line [Ax. 1, Cr. 15].

There is a number greater than any given number of points not in straight line with AB [Cr. 22].

Join each of them with, for instance*, the endpoint A of AB [Cr. 15].

Each of these straight lines are adjacent at A to AB [Df. 4].

If any of them, for instance* AP, were in straight line with AB, they would be segments of the same straight line l [Df. 12],

P, A and B would be points of that straight line l [Df. 2],

P would be in straight line with A and B [Df. 12],

and then with AB [Df. 12],

which is not the case. □

Corollary 24 *If two adjacent straight lines are not in straight line, then no point of any of them, except their common endpoint, is in straight line with the other. And by producing any of them from their common endpoint, the production is also adjacent to the non-produced one.*

Let AB and AC be two straight lines adjacent at A and not in straight line with each other [Cr. 23].

Let P be a point of, for instance*, AB [Cr. 1].

A, P and B belong to AB. So, if P were in straight line with AC, it would be in straight line with A and C [Cr. 20],

and it would also belong to AC, whether or not produced [Df. 12].

In such a case AB and AC would have two common points, A and P, they would be segments of the same straight line [Cr. 18],

and they would be in straight line with each other [Df. 12],

which is not the case. So, P is not in a straight line with AC.

On the other hand, if AQ is any production from A, for example* of AB, AQ is adjacent to AB and is in a straight line with AB [Cr. 16].

The common endpoint A is the only common point of AQ and AC, otherwise they would have at least two common points; and AQ and AC would be segments of the same straight line [Cr. 18],

and, consequently, AC and AB would also be in a straight line with each other [Cr. 19],

which is not the case. So, AQ and AC are also adjacent at A [Df. 4]. □

Fundamentals on planes

Definition 13 *Plane: a surface that contains at least three points not in straight line and any straight line through any two of its points. A line is said in a plane iff all of its points are points of that plane. Lines in a plane are said plane lines. Points, or lines, or points and lines in the same plane are said coplanar. Two planes are said different if at least one of them has a point that is not in the other.*

Definition 14 *Sides of a given straight line in a plane: parts of the plane that contain all points of the plane, and only them, each part with at least two common points and at least two non-common points, where a point is said common, or common to all parts, if it is in straight line with the given straight line; and non-common if it is not, being said non-common of a part iff it is in that part. Any other line is said to be in one of those parts iff all of its points between its endpoints are non-common points of that part.*

Axiom 6 *Any three points lie in a plane, in which any straight line has two, and only two, sides. Any other line is in one of such sides iff its endpoints are in that side.*

Corollary 25 (A variant of Hilbert's Axiom I.5) *A plane has a number of points greater than any given number, any two of which can be joined by a unique straight line in that plane. And any given straight line is at least in a plane, in which it can be produced by any given length from any of its endpoints.*

Let P, Q and R be any three points not in straight line [Cr. 22],

and Pl a plane in which they lie [Ax. 6].

Pl has at least the points P, Q and R and all points of any straight line [Cr. 1]

through any two of its points [Dfs. 13, Ax. 5, Cr. 17].

So, Pl has a number of points greater than any given number [Cr. 1].

Let, then, A and B be any two points of Pl. Join A and B [Cr. 15],

and produce AB from A and from B by any given length to the respective points A' and B' [Cr. 16].

Since $A'B'$ is a straight line [Cr. 16]

through two points A and B [Df. 2, Cr. 17]

of Pl, $A'B'$ is in Pl [Df. 13],

so that all points of $A'B'$ are in Pl [Df. 13],

and then all points of its segment AB are in Pl [Df. 2, Cr. 5].

Hence, Pl contains the unique straight line joining any two of its points A and B [Crs. 14, 15].

Let now AB be any straight line [Ax. 1, Cr. 15],

and P, Q and R any thee of its points between A and B [Cr. 1].

There is a plane Pl containing P, Q and R [Ax. 6],

and the straight line AB through P and Q is in Pl [Df. 13].

Produce AB from A and from B by any given length to the points A' and B' respectively [Cr. 16].

Since the produced straight line $A'B'$ is a straight line [Cr. 16] through two points A and B [Cr. 17]

of Pl, it is a straight line of Pl [Df. 13]. □

Corollary 26 *A point of a plane can only be either common to both sides of a straight line in that plane, or non-common of one, and only of one, of such sides.*

Let A, B and P be any three points of a plane Pl [Ax. 6].

Join A and B [Cr. 15].

AB is in Pl [Cr. 25].

Either P belongs to AB, whether or not produced [Cr. 16],

or it does not [Ps. C].

If P belongs to AB, whether or not produced [Cr. 16],

P is a point common to both sides of AB [Ax. 6, Df. 14].

If P does not belong to AB [Df. 14],

whether or not produced [Cr. 16],

P cannot be in both sides of AB [Df. 14],

and being a point of Pl, it can only be in one, and only in one, of the two sides of AB [Df. 14, Ax. 6].

So, it is a non-common point of that side, and only of it [Df. 14]. □

Corollary 27 *There is a plane containing any two adjacent straight lines not in straight line with each other, being each of them in the same side of the other. And there is a plane containing any two intersecting and non-adjacent straight lines.*

Let AB and AC be two straight lines adjacent at A and not in straight line with each other [Cr. 23].

A, B and C are not in straight line [Cr. 24].

There is a plane in which lie A, B and C [Ax. 6]

and the adjacent straight lines AB and AC [Cr. 25].

The common endpoint A is a common point of both sides of AC [Df. 14],

B is not in straight line with AC [Cr. 24],

so it is a non-common point of one of the sides of AC [Df. 14].

Therefore AB is in that side of AC [Ax. 6].

For the same reasons AC is in one of the sides of AB. Let now l_1 and l_2 be any two non-adjacent straight lines that intersect at a unique point P [Cr. 21],

Q a point of l_1, and R a point of l_2 [Cr. 1].

There is a plane containing P, Q and R [Ax. 6],

the straight line l_1 through Q and P [Cr. 17, Df. 13],

and the straight line l_2 though R and P [Cr. 17, Df. 13]. □

Corollary 28 *All points between two points of a straight line in the same side of a given straight line lie in that side of the given straight line, and that side has a number of non-common points greater than any given number.*

Let l be a straight line in a plane Pl [Cr. 25]

and P and Q be any two non-common points in the same side, for instance Pl_1, of l [Ax. 6, Df. 14].

Join P and Q [Cr. 15].

PQ is in Pl_1 [Ax. 6].

All points between P and Q are non-common points of Pl_1 [Df. 14].

So, Pl_1 has a number of non-common points greater than any given number [Cr. 1]. □

Corollary 29 *In a plane and in each side of a straight line in that plane, it is possible the existence of a number greater than any given number of straight lines, whether or not adjacent, none of which is in straight line with any of the others.*

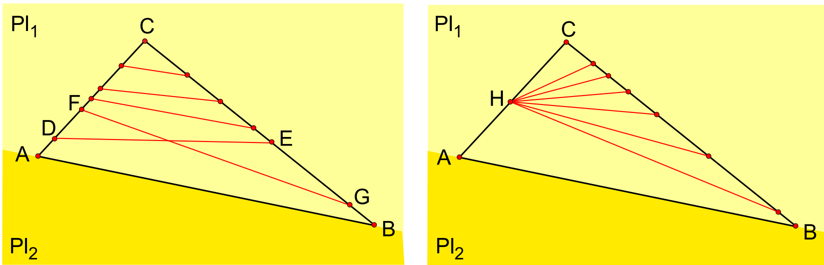


Figure 7.6 – Corollary 29

(Fig. 7.6, left) Let A, B and C be any three points not in straight line [Cr. 22],

and Pl a plane in which they lie [Ax. 6].

Join A and B [Cr. 15]

and let Pl_1 and Pl_2 be the two sides of AB in Pl [Ax. 6].

C will be a non-common point [Df. 14]

of, for example*, Pl_1 [Cr. 26].

Join C with A and with B [Cr. 15].

CA and CB are not in straight line, otherwise A, C and B would be in straight line [Df. 12],

which is not the case. Join each of any number n of points of CA between C and A with a different point of CB between C and B [Crs. 5, 15],

and let DE and FG be any two of such straight lines, for example* D and F in CA, and E and G in CB. The straight lines DE and FG cannot be in straight line with each other, otherwise they would be segments of the same straight line [Df. 12],

and D, E, F and G would be in that straight line [Df. 2],

so that D would be in straight line with E and G, and then with CB [Df. 12],

which is impossible [Cr. 24].

The same argument applies to the n straight lines joining the same point H of CA between A and C (Fig. 7.6, right) with n different points of CB between C and B [Crs. 5, 15], being all of these straight lines adjacent at H [Df. 4].

Since CA and CB are in Pl_1 [Ax. 6],

all of these straight lines in Pl, whether or not adjacent, have their respective endpoints on Pl_1 [Df. 14],

so that all of them are in Pl_1 [Ax. 6]. □

Corollary 30 *The intersection point of two intersecting straight lines has its two sides in each of the intersecting straight lines in different sides of the other intersecting straight line in the plane that contains both straight lines.*

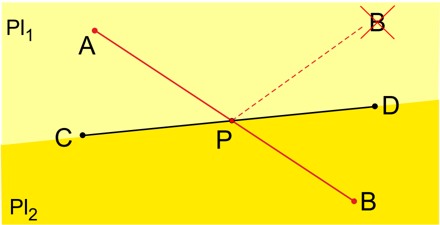


Figure 7.7 – **Corollary 30**

(Fig. 7.7) Let P be the unique intersection point of two straight lines [Cr. 21]

AB and CD in a plane Pl [Cr. 27].

Since the only points of Pl common to both sides of CD in Pl are the points in straight line with CD [Df. 14],

and P is the only common point of AB and CD, even arbitrarily produced [Crs. 16, 21],

P is the only point of AB in straight line with CD [Df. 12],

and therefore the only point of AB that is a common point of both sides of CD in Pl [Df. 14].

Therefore, the endpoints A and B can only be non-common points of the sides of CD in Pl [Df. 14, Cr. 26].

So, if PA and PB were in the same side of CD in Pl, the endpoints A and B would be non-common points of that side [Ax. 6],

and being P between them [Cr. 4],

P would also be a non-common point of that side [Cr. 28],

which is impossible because it is a common point of both sides [Cr. 26].

So, A and B must be in different sides of CD in Pl [Cr. 26],

and the sides PA and PB of P are on different sides of CD in Pl [Ax. 6].

The same argument proves PC and PD can only be in different sides of AB in Pl. $\square$

Corollary 31 *The straight line joining any two non-common points, each in a different side of another given coplanar straight line, intersects the given straight line, or a production of it, at a unique point.*

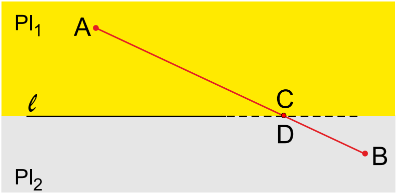


Figure 7.8 – Corollary 31

(Fig. 7.8) Let Pl_1 and Pl_2 be the two sides of a line l in a plane Pl [Cr. 25, Ax. 6].

Let A be a non-common point of Pl_1, and B be a non-common point of Pl_2 [Cr. 28].

Join A and B [Cr. 15].

AB is in Pl [Cr. 25].

Except A and B, all points of AB are between A and B [Cr. 4].

If all points of AB between A and B were non-common points of Pl_1, AB, including B, would be in Pl_1 [Df. 12, Ax. 6],

which is not the case. Therefore, at least one point of AB between A and B is in Pl_2. So, AB contains points of Pl_2 other than B; and, for the same reason, points of Pl_1 other than A [Cr. 1].

So, AB has at least two points in each side of l. Since all points between two points of a straight line in the same side of another coplanar straight line are also in that side [Cr. 28],

AB has a segment AC whose points are all points of AB in Pl_1 [Cr. 12].

And for the same reasons it also has a segment BD whose points are all points of AB in Pl_2 [Cr. 12].

If C and D were different points, all points of AB between C and D [Cr. 5]

would be in no side of l in Pl, which is impossible because all points of AB are points of Pl [Df. 13],

and all points of Pl are points either of Pl_1, or of Pl_2, or of both of them [Ax. 6, Df. 14].

So, C and D are the same point. Since all points between A and C are in Pl_1, AC is in Pl_1 [Df. 14],

and C is also in Pl_1 [Ax. 6].

For the same reasons D is in Pl_2. Since C and D are the same point, and this point belongs to Pl_1 and to Pl_2, it is a point of l, whether or not produced [Cr. 16, Df. 14].

So, it is an intersection point of AB and l [Df. 3]

whether or not produced [Cr. 16].

And it is the unique intersection point of AB and l, otherwise the non-common point A of Pl_1 would be in straight line with at least two points of l and it would be a common point of Pl_1 and Pl_2 [Dfs. 14, 12],

which is impossible [Cr. 26]. □

Corollary 32 *A plane contains at least two non-intersecting straight lines, which can be intersected by any number of different coplanar straight lines.*

Let l be a straight line in a plane Pl [Cr. 25],

Pl_1 and Pl_2 the two sides of l in Pl [Ax. 6],

A, B any two non-common points of Pl_1, and C, D any two non-common points of Pl_2 [Cr. 28].

Joint A with B; and C with D [Cr. 15].

AB is in Pl_1, and CD in Pl_2 [Ax. 6].

AB and CD cannot intersect with each other because the intersection point would be a common point of Pl_1 and Pl_2 [Df. 14],

while all points of AB and CD, even endpoints, are non-common points respectively of Pl_1 and of Pl_2 [Df. 14, Ax. 6].

On the other hand, AB and CD can be intersected by any number n of straight lines in Pl, each joining each of any n points of AB with a point of CD [Crs. 1, 15, 25]. □

Fundamentals on distances

Definition 15 *Distance between two points: length of the straight line joining both points.*

Definition 16 *Distance from a point not in a given line to the given line: the shortest distance between the point and a point of the given line, or of a production of the given line if the given line is a straight line and the point is not in straight line with it.*

Definition 17 *Distancing.-Two points of a straight line in the same side of another given straight line define, in the first straight line, a distancing direction with respect to the given straight line: from the point at the shortest distance to the given straight line to the point at the greatest distance to the given straight line. The difference between these distances is called the relative distancing with respect the given straight line of the segment defined by these two points of the first straight line.*

Definition 18 *Parallel straight lines.-A straight line is said parallel to another coplanar straight line, iff all of its points are at the same distance, said equidistance, from the other straight line.*

[Th. 39] proves the existence of parallel straight lines. According to [Df. 15], the length of a straight line AB and the distance from A to B will be used as synonyms.

Axiom 7 *The distances from the points of a line to a fixed point or to another line vary in a continuous way. The distances from a point to itself and to a line to which it belongs are zero.*

Corollary 33 *The distance between any two given points is unique.*

It is an immediate consequence of [Cr. 15, Df. 15, Ax. 7]. □

FUNDAMENTALS ON CIRCLES

Definition 19 *Circle: a plane self-closed and non-self-intersecting line whose points are all points of the plane, and only them, at the same given finite distance, said radius, from a fixed point of that plane, said centre of the circle. A straight line joining any point of the circle with its centre is also said a radius of the circle. A segment of a circle is called arc, and the straight line joining its endpoints is a chord, or straight line subtending the arc. If the center of the circle is a point of a chord, the chord is said a diameter, and the corresponding arc a semicircle. Coplanar circles, and their corresponding segments, with the same centre are said concentric. The centre and any coplanar point at a distance from the centre less than its radius are said interior to the circle; if that distance is greater than the radius of the circle, the coplanar point is said exterior to the circle.*

Axiom 8 *Any point in a plane can be the center of a circle of any radius, and its complementary arcs are each on a different side of its chord.*

Corollary 34 *A circle has interior points, other than its centre, and exterior points. And any point coplanar with a circle is either in the circle, or it is interior or exterior to the circle.*

Let O be the centre of a circle c in a plane Pl [Ax. 8],

and A any point of c [Df. 19].

Joint A with O [Cr. 15].

Produce OA from A by any given finite length to a point A' [Cr. 16].

OA' is in Pl [Cr. 25].

Let P be any point of OA between O and A [Cr. 5].

Since $OP < OA$ and $OA < OA'$ [Cr. 13],

P is interior and A' is exterior to c [Dfs. 15, 19].

Join now any point R of Pl with O [Crs. 15, 25].

It holds $RO \gtreqqless OA$ [Ps. A],

and R will be either in c ($RO = OA$), or it will be interior ($RO < OA$) or exterior ($RO > OA$) to c [Dfs. 15, 19]. □

Corollary 35 *A plane line intersects a coplanar circle at a point between its endpoints iff it has points interior and exterior to the circle.*

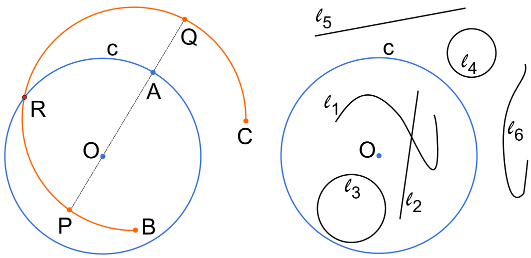


Figure 7.9 – **Corollary 35**

Let O be the centre and AO the finite radius of a circle c [Ax. 8]

in a plane Pl; BC a plane line in Pl [Df. 13, Ax. 6],

and P and Q two points of BC [Cr. 1]

such that P is interior and Q exterior to c [Cr. 34].

Being P interior to c, its distance to O is less than AO [Df. 19].

Being Q exterior to c, its distance to O is greater than AO [Df. 19].

Therefore, there will be at least one point R in PQ, and then in BC [Cr. 1, 2],

whose distance to O is just AO [Ax. 7, Df. B].

And R will also be in c [Df. 19].

So, R is an intersection point of BC and c [Df. 3].

On the other hand, if all points of a line DE coplanar with c are interior (exterior) to c, none of its points is at a distance AO from O [Df. 19],

and then no point of DE is in c [Df. 19].

Therefore c and DE have no point in common, and they do not intersect with each other [Df. 3]. □

Corollary 36 *Any point of a circle defines a unique diameter and two unique complementary semicircles, each on a different side of that diameter.*

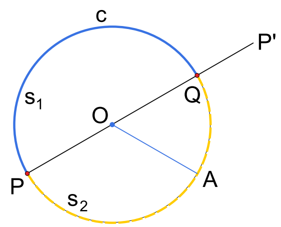


Figure 7.10 – **Corollary 36**

Let O be the centre, AO the finite radius, and P any point of a circle c [Ax. 8, Cr. 1].

Join P with the centre O of c [Cr. 15],

and produce PO from O by any given length greater than OA to a point P' [Cr. 16].

Since $OP' > OA$, PP' is a straight line with points interior, as any point of OP, and exterior, as P', to c [Cr. 16, Df. 19],

PP' intersects c at a point Q [Cr. 35].

P and Q are the common endpoints of two complementary semicircles s_1 y s_2 de c whose only common points are P and Q [Dfs. 7, 19, Ax. 8].

The center O of c is in the chord PQ of s_1 and s_2, which is the only straight line joining P and Q [Cr. 15].

Therefore, PQ is the only diameter defined by P [Df. 19].

And s_1 and s_2 are on different sides of PQ [Ax. 8]. □

Fundamentals on angles

Definition 20 *Rigid transformations of lines: metric and reversible displacements of lines that preserve the definition and the metric properties of the displaced lines, each of whose points moves from* *an initial to a final position along a line of finite length called trajectory, in any of the two opposite directions defined by the endpoints of the trajectory. If all points of the displaced line, except at most one, move around a fixed point and their trajectories are arcs of concentric and coplanar circles whose centre is the fixed point, the rigid transformation is called rotation.*

Definition 21 *Superpose two adjacent lines: to place them with at least two common points by means of rotations around their common endpoint. Lines with at least two common points are said superposed.*

Definition 22 *Angle.-Two straight lines are said to make an angle greater than zero iff they are adjacent, one of them can be superposed on the other by two opposite rotations around their common endpoint, and the other can be superposed on the one by the same two rotations, though in opposite directions. The least of the rotations, of both if they are equal, is said (convex) angle, the greater one is said concave angle. The angle is said to be in the side of one of the adjacent straight lines where the other adjacent straight line lies. The straight lines and their common endpoint are said respectively sides and vertex of the angle. A side is said to make an angle with the other side at their common vertex. A line joining a different point on each side of the angle is said to subtend the angle, its points are called interior to the angle. The non-interior points are called exterior to the angle.*

Definition 23 *Adjacent angles and union angle.-Two angles are said adjacent iff they have the same vertex, a common side, the first angle superposes its non-common side on the common side, and the second angle superposes the common side on its non-common side, both angles in the same direction of rotation. The angle that superposes the non-common sides of both angles in the same direction of rotation of both angles is their union angle, which can be concave. If two adjacent angles are equal to each other, they are said to bisect their union angle.*

Definition 24 *Straight angle.-Except endpoints, the angle that make the two sides of a point of a straight line at their common endpoint is said straight angle.*

Definition 25 *Acute, obtuse and right angles.-If a straight line cuts another given straight line and makes with it at the intersection point two adjacent angles that are equal to each other, both angles are said right angles, in which case, and only in it, the two sides of each angle are said perpendicular to each other, and the first straight line is also said perpendicular to the given one. Angles less (greater) than a right angle are said acute (obtuse).*

Definition 26 *Interior and exterior points and angles.-If two given coplanar straight lines are intersected by another coplanar straight line, said common transversal, a point of this transversal, different from the intersection points, is said interior to the given straight lines if it is between the intersection points of the transversal with both given straight lines; otherwise it is said exterior to them. Of the angles that a common transversal makes with the two given coplanar straight lines at their intersection points, those whose sides in the transversal have only exterior points are said exterior angles; and those whose sides in the transversal have interior points are said interior angles.*

Definition 27 *Alternate, corresponding and vertical angles.-Of the angles that a common transversal makes with two coplanar straight lines, the angles of a couple of non-adjacent angles are said alternate if they are both interior, or both exterior, and they are in different sides of the transversal; and corresponding if they are in the same side of the transversal, being the one interior and the other exterior. Of the angles that two intersecting straight lines make with each other at their intersection point, the couples of angles with no common side are said vertical angles.*

Axiom 9 *It is possible for two adjacent straight lines to make any angle at their common endpoint. The angle is zero iff both straight lines are superposed.*

Corollary 37 *Two straight lines make an angle greater than zero iff they are adjacent, being equal and unique the angle that each of the straight lines make with the other at their common endpoint, both rotations in opposite directions. And the adjacency point is their only common point, even arbitrarily produced from their non-common endpoints.*

Each of two coplanar adjacent straight lines [Cr. 27],

makes with the other the same angle greater than zero at their common endpoint, though in opposite directions [Df. 22, Ax. 9].

And being a metric transformation, that angle is unique [Dfs. 20, C].

The only common point of both sides, even arbitrarily produced from their non-common endpoints [Cr. 16],

is the vertex of the angle, otherwise both sides would be superposed [Df. 21],

and they would make an angle zero [Ax. 9].

which is not the case. On the other hand, if two straight lines make an angle zero they will be superposed [Ax. 9]

and they will not be adjacent [Dfs. 21, 4]. □

Corollary 38 *The superposition by rotation of two adjacent straight lines around their common endpoint is a unique straight line.*

It is an immediate consequence of [Df. 21, Cr. 18]. □

Corollary 39 *An angle does not change by producing arbitrarily its two sides from their non-common endpoints. Nor if only one of the sides is produced from its non-common endpoint.*

Let AB and AC be two adjacent straight lines [Cr. 27]

that make an angle $\alpha > 0$ at their common endpoint A [Cr. 37].

Apart from the common endpoint A, the angle α superposes at least one point P of AB with a point Q of AC [Dfs. 21, 22].

Produce AB from B and AC from C by any given length respectively to the points B' and C' [Cr. 16].

A is a common point of AB' and AC'; and P and Q are also points respectively of AB' and AC' [Cr. 16, Df. 2],

Therefore, the rotation α superposes AB' and AC' [Df. 21].

Suppose that a rotation α' smaller than α superposes two points R and S respectively of AB' and AC' but does not superposes AB and AC. Therefore, α' does not superpose P and Q [Df. 21].

The point R could not be between A and P; nor S between A and Q, otherwise α' would superpose AB and AC [Dfs. 21, 11],

which is not the considered case. Therefore P is between A and R; and Q is between A and S [Cr. 8].

We would then have two straight lines with non-common points, P and Q, between two common points, the point A and the superposed R and S, which is impossible [Df. 11].

So, AB' and AC' also make at A an angle α. The same argument applies if only one of the sides, for instance* AB, is produced from B to B', now the points R and S being respectively in BB' and QC, in both cases between the corresponding endpoints. □

Corollary 40 *Three adjacent straight lines define three angles at their common endpoint. And two intersecting straight lines define with each other at most four angles at their intersection point.*

Three coplanar straight lines AB, AC and AD adjacent at the same point A [Cr. 29]

define three couples of coplanar straight lines adjacent at that point: AB, AC; AB, AD; and AC, AD [Df. 4].

So, AB, AC and AD define three angles at that point A [Cr. 37].

For the same reason, two intersecting straight lines define at most four angles whose two sides are not in the same straight line. □

Corollary 41 *(Fig. 7.11) Three straight lines adjacent at the same point define a couple of adjacent angles at that point.*

Three straight lines r_1, r_2, r_3 adjacent at V define three angles α, β and γ at V [Cr. 40],

and then three couples of angles: α and β; α and γ; and β and γ. Being only three sides, the two angles of each of such couples must have a common side [Df. 22].

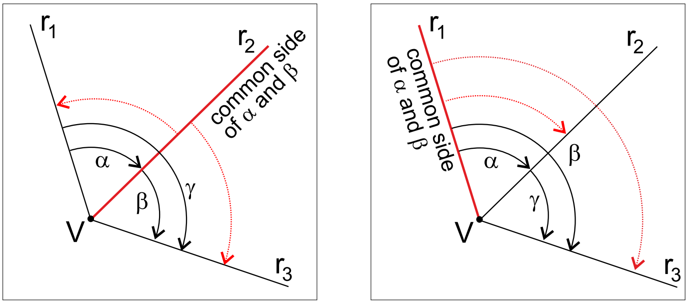


Figure 7.11 – **Corollary 41**

The angles of such couples that superpose their common side on their respective non-common sides can only be rotations in the opposite direction, or in the same direction [Dfs. 1, 20].

In the first case (Fig. 7.11, left), the angles of the couple, for instance α and β, are adjacent because either of them also superposes its non-common side on the common side in the same direction as the other superimposes the common side on its non-common side [Dfs. 22, 23].

In the second case (Fig. 7.11, right), let r_1 be the common side of α and β. Assume α superposes r_1 on r_2; β can only superpose r_1 on r_3; and it will be different from α otherwise r_2 would be superposed on r_3 and they would not be adjacent [Dfs. 21, 4].

Since α and β are different, one of them, for instance α, will be less than the other [Ps. A],

in which case γ can only be the angle that, in the same direction of rotation as α, superposes r_2 on r_3. So, α and γ are adjacent [Df. 23].

So, in any case three straight lines adjacent at the same point define a couple of adjacent angles at that point. □

Corollary 42 *Two adjacent straight lines make a straight line iff they make a straight angle at their common endpoint.*

If two adjacent straight lines l_1 and l_2 [Cr. 27]

make at their common endpoint P a straight angle, they are the two sides of the point P in a straight line l [Df. 24], so that l_1 and l_2 make the straight line l [Cr. 2].

If two straight lines l_1 and l_2 adjacent at P make a straight line l, l_1 and l_2 are the sides in l of their common endpoint P [Df. 5]

so that they make a straight angle at P [Df. 24]. □

Corollary 43 *Except for the vertex, no point of either side of an angle is in straight line with the other side of the angle if the angle is not an straight angle and is greater than zero.*

It is an immediate consequence of [Ax. 9, Crs. 42, 24] □

Fundamentals on polygons

Definition 28 *Polygon.-Three or more finite coplanar straight lines, called sides, each of which is adjacent at each of its two endpoints, called vertexes, to just one of the others, being not in straight line with each other, and being their common endpoints their only intersection points, are said to make a polygon. Two sides of the same or of different polygons are said equal iff they have the same length. Two polygons are said adjacent iff they have a common side; opposite iff they have two vertical angles at a common vertex; similar iff the angles of the one are equal to the angles of the other; and equal if they are similar and the sides of each angle of the one are equal to the sides of the corresponding equal angle of the other. Polygons with at least one concave angle are said concave. The angle each side makes with the production of another adjacent side is said exterior. A straight line joining two points each on a different side of a polygon is a divisor of the polygon; if the ends of a divisor are vertexes, the divisor is called diagonal. A divisor bisects a polygon if it is the common side of two adjacent polygons with the same area.*

Note.-The classical definition of diagonal is a particular case of the above general definition of divisor.

Definition 29 *Triangles and quadrilaterals. A polygon of three (four) sides is a triangle (quadrilateral). A triangle (quadrilateral) is said equilateral if its three (four) sides are equal to one another. A triangle is said isosceles if it has two equal sides; and scalene if the three of them are unequal. If one of its angles is a right angle,* *it is said a right-angled (or simply right) triangle. A rectangle is a quadrilateral all of whose angles are right angles. An equilateral rectangle is a square. And a parallelogram is a quadrilateral with two couples of equal and parallel sides. Polygons with more than four sides are named pentagons, hexagons, heptagons etc. A polygon is said to lie between two given lines iff its vertexes are in the given straight lines or in straight lines whose endpoints are points of the given straight lines.*

Axiom 10 *The area of a polygon is greater than zero, and is the sum of the areas of the two adjacent polygons defined by any of its divisors. Equal polygons have equal areas.*

Corollary 44 *Any two adjacent sides of a polygon make an angle greater than zero at their common endpoint, and the polygon has as many angles as sides. And twice as many exterior angles as angles.*

Being coplanar all sides of a polygon [Df. 28],

each couple of its adjacent sides makes a unique angle greater than zero at their common endpoint [Cr. 37].

So, the polygon has as many angles as couples of adjacent sides. Since each couple of adjacent sides is defined by two adjacent sides, and each side defines two of such couples, one at each of its two endpoints [Df. 28],

the polygon has as many angles as sides. And since each side makes an exterior angle with the production of each of the other two adjacent sides at each of its two vertices [Df. 28],

the polygon has twice as many exterior angles as angles. □

The last element of this new foundational base of the Euclidean geometry is the following corollary, which is not strictly geometric because the demonstration makes use of some basic results of set theory. Although the demonstration is simple, the corollary can be omitted and considered its statement as an additional hypothesis: the length of a line is finite whenever it has two well-defined endpoints. In any case, at the end of the demonstration, the corresponding used concepts are explained. A more complete and detailed proof is given in [12, Chap. 16].

Corollary 45 *In the Euclidean space R^3, the length of a line with two endpoints is always finite. And the distance between any two given points is always finite and unique.*

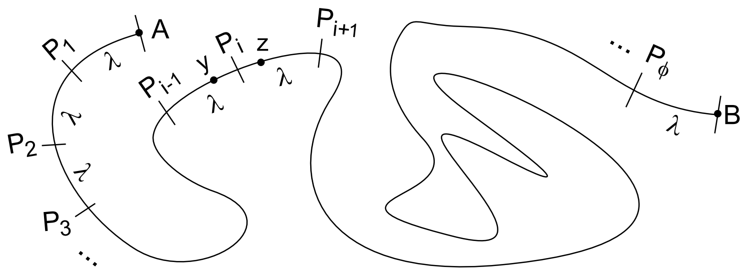


Figure 7.12 – **Corollary 45**

(Fig. 7.12) Let AB be any line* in the Euclidean space $\mathbb{R}^3$, and $\lambda > 0$ any finite length [Ax. 1, 4].

Let $\mathbf{P} = AP_1,\ P_1P_2,\ P_2P_3 \ldots$ be a partition of AB all of whose parts have the same finite length $\lambda > 0$, except the last one, if any, that can be less than λ. A point X such that $XB < \lambda$ will belong to a part that can only be the last part or the penultimate part of $\mathbf{P}$. [Cr. 13].

So, $\mathbf{P}$ has a last part $P_\phi B$. Any point Y of the segment AP_i such that $YP_i < \lambda$, and any point Z of the segment P_iB such that $P_iZ < \lambda$ can only belong respectively to the parts $P_{i-1}P_i$ and P_iP_{i+1} of $\mathbf{P}$, for all $1 < i < \phi$ [Cr. 13].

Therefore, $\mathbf{P}$ has a first element AP_1, a last element $P_\phi B$, and each element has an immediate predecessor (except AP_1), and an immediate successor (except $P_\phi B$).

Let us suppose there exists an n-th element of $\mathbf{P}$ with a finite number of predecessors. The $(n+1)$th element of P, if any, will also have a finite number $n+1$ of predecessors, or n predecessors if it were the last element of the partition. Since P_1P_2 has a finite number of predecessors, just 1, we can inductively conclude that each element of $\mathbf{P}$, including its last element $P_\phi B$, has a finite number of predecessors. So, $\mathbf{P}$ has a finite number of elements. And being finite the sum of any finite number of finite lengths, AB has finite length. Let us now join any two points C and D [Ax. 1, Cr. 15].

It has just been proved that CD has finite length. Therefore, the distance from C to D is also finite, and unique [Df. 15, Cr. 33]. □

A partition of a segment is a set of adjacent and disjoint segments (with no common elements) whose union is the initial segment.

In a sequence all the elements that precede a given element are called predecessors of the given element. And all the elements that follow it are called successors of the given element. If between a given element and one of its predecessors there is no other predecessor, this predecessor is called the immediate predecessor of the given element. If between a given element and one of its successors there is no other successor, this successor is called the immediate successor of the given element.

8. On Angles and Triangles

8.1 Introduction

In this chapter begins the development of Euclidean geometry on the basis of the fundamentals introduced in the previous chapter, although it will be limited to plane geometry. Consequently, from now on it will be assumed that all points and all lines are coplanar [Df. 13] and, unless otherwise indicated, all angles will be greater than zero. In addition, all lines will be straight lines, except in a small number of cases where they will be circles. And when two or more lines are considered, it will be assumed that they are all different. In this chapter 18 theorems and 4 corollaries are proved, most of them about angles and triangles. The theorems about angles are completed in the next chapter, once the existence of right angles and perpendicular straight lines have been demonstrated. When the statement of a theorem, or of a corollary, coincides with an Euclid proposition, or with an axiom of other foundations of Euclidean geometry, the coincidence will be indicated in parentheses, before the statement.

Theorem 1 (Euclid's Proposition 3 extended) *To take a point in a given finite straight line, produced if necessary, at any given finite distance from a given point of the given straight line, and in any given direction of the two opposite directions of the given straight line.*

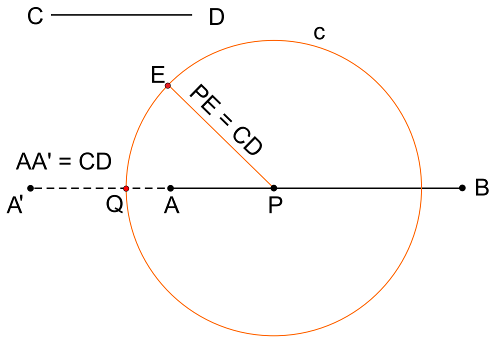


Figure 8.1 – Theorem 1

Let AB be a finite straight line [Cr. 25];

P the given point of AB [Cr. 5];

CD the given finite distance, which is the length of the straight line CD [Df. 15];

and the give direction in AB, for example,* the direction from B to A [Ax. 2, Df. 1].

Produce AB from A by any length greater than CD to a point A' [Cr. 16].

With centre P and radius CD draw the circle c [Ax. 8].

P is interior to c [Df. 19],

and being $PA' > AA'$ [Cr. 13]

and $AA' = CD$, it holds $PA' > CD$ [Ps. B].

Therefore, A' is exterior to c [Df. 19].

Hence, there is an intersection point Q of c and BA' [Cr. 35].

Q is in AB [Df. 3],

whether or not produced, at the given finite distance CD from the point P of AB [Df. 19];

and in the given direction from B to A. □

Note.-From a formal point of view [Th. 1] is not necessary because of [Crs. 14, 16]. It is included as a constructive tool. So, from now on, to take a point in a straight line at a given finite distance from one of its points will always mean to take the point in that straight line produced if necessary [Th. 1]. And the distance between two points will always be finite [Cr. 45].

Theorem 2 *All straight angles are equal to one another.*

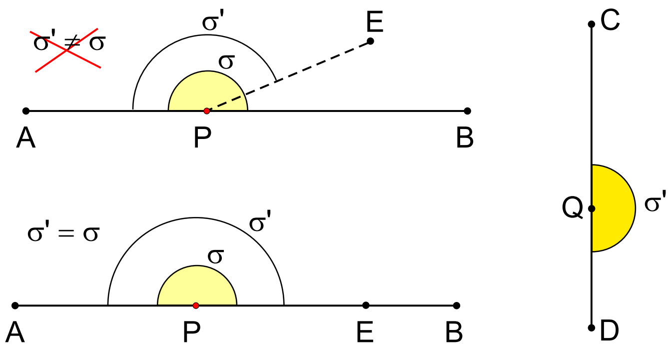


Figure 8.2 – **Theorem 2**

Let P and Q be any two points respectively of any two straight lines AB and CD [Crs. 1, 29],

and σ and σ' the respective straight angles that PA makes at P with PB; and QC makes at Q with QD [Cr. 42].

A straight line PE adjacent at P to PA and making an angle σ' at P with PA is possible [Ax. 9].

Assume PE and PB are not superposed, they will be adjacent at P [Dfs. 21, 4].

PA and PB are the two sides of σ; and PA and PE the two sides of σ'. Being σ and σ' straight angles, AB and AE are straight lines [Cr. 42];

AP is a common segment of them [Cr. 5]

and PE and PB are non-common segments of them [Dfs. 4, 3].

Consequently, AB and AE are locally collinear [Dfs. 2, 3],

which is impossible [Df. 11]

So, it is impossible for PB and PE to be adjacent at P, and they must be superposed [Dfs. 4, 21]
in a unique straight line [Cr. 18],

and then P, E and B are in the same straight line. If E and B are the same point then $PB = PE$ [Cr. 15]

and $\sigma = \sigma'$. If not, one of then, for instance E, will be between

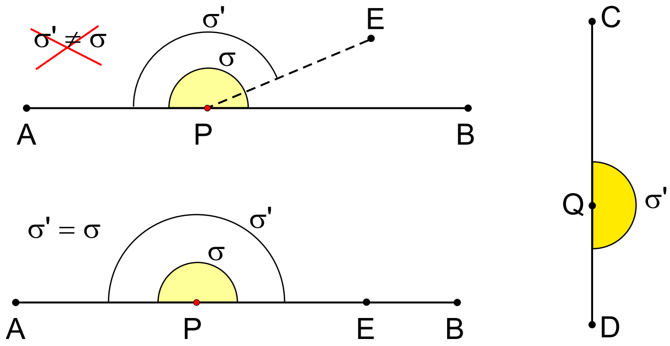


Figure 8.2 – Theorem 2

the other two [Cr. 9],

so that $PE < PB$ [Cr. 13],

and the angle σ' that PA makes at P with PE is equal to the angle σ it makes at P with PB [Cr. 39].
We can, then, conclude that all straight angles are equal to one another. $\square$

Theorem 3 *The union angle of two adjacent angles is the sum of both adjacent angles and is greater than each of them.*

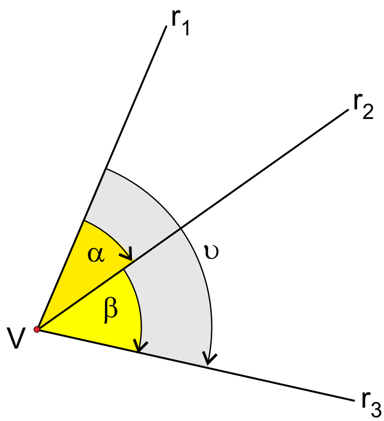


Figure 8.3 – **Theorem 3**

Let r_1, r_2 and r_3 be three straight lines adjacent at their common endpoint V [Cr. 29],
where they make a couple of adjacent angles α and β [Cr. 41].

Assume* α superposes r_1 on r_2, and β superposes r_2 on r_3 in the same direction of rotation [Df. 23].

The rotation α around V superposes r_1 on r_2 [Df. 22]

in a unique straight line [Cr. 38], and then the rotation β around V superposes r_1 on r_3 in a unique straight line [Df. 22, Cr. 38].

So, the rotation $\upsilon = \alpha + \beta$ [Df. C, Ps. B]

around V in the same direction of rotation as α and β superposes the non-common sides r_1 and r_3 of α and β. It is, then, the union angle of α and β [Df. 23].

And being $\alpha > 0$, $\beta > 0$ [Ax. 9],

it holds $\alpha + \beta > \beta$; $\beta + \alpha > \alpha$ [Ps. B],

and then $\upsilon > \beta$; $\upsilon > \alpha$ [Ps. A]. $\square$

Theorem 4 (A variant of Euclid's Proposition 13) *If a straight line makes with another straight line two adjacent angles, these angles can be either equal or unequal to each other, and they always sum a straight angle.*

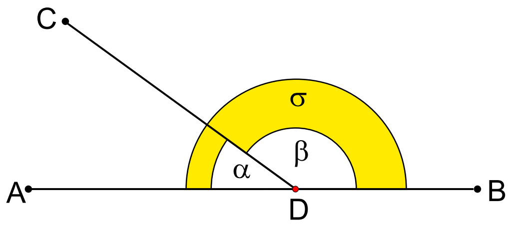


Figure 8.4 – Theorem 4

Let D be the unique intersection point of two straight lines AB and CD [Crs. 21, 27].

DA, DC and DB are straight lines [Cr. 14]

adjacent at D [Df. 4].

So, DA makes at D with DC, and DC at D with DB two adjacent angles α and β [Cr. 41]

of which D is the common vertex and DC the common side [Df. 23];

α and β can be either equal or unequal to each other [Ax. 9, Ps. A],

and their union angle σ is the rotation $\alpha + \beta$ around D [Th. 3]

that in the same direction of rotation of α and β superposes the non-common sides DA and DB respectively of α and β [Df. 23],

and being DA and DB the two sides of D in the straight line AB [Df. 5, Ax. 4],

σ is a straight angle [Cr. 42, Df. 24].

Therefore α and β sum a straight angle [Th. 3]. $\square$

Theorem 5 (Euclid's Proposition 15) *The two angles of any couple of vertical angles are equal to each other.*

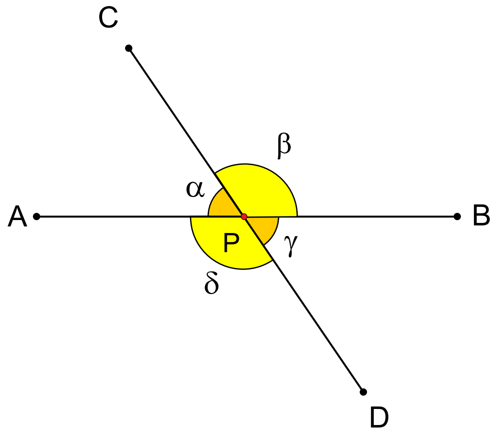


Figure 8.5 – **Theorem 5**

Let P be the unique intersection point of two straight lines AB and CD [Crs. 21, 27].

PA, PC, PB and PD are straight lines [Cr. 14]

adjacent at P [Df. 4].

PC makes at P with AB two adjacent angles α and β [Cr. 37]

that sum a straight angle [Th. 4].

PB makes at P with CD two adjacent angles β and γ [Cr. 41]

that sum a straight angle [Th. 4].

PD makes at P with AB two adjacent angles γ and δ [Cr. 41]

that sum a straight angle [Th. 4].

Therefore $\alpha + \beta = \beta + \gamma = \gamma + \delta$ [Th. 2].

Consequently $\alpha = \gamma$ and $\beta = \delta$ [Ps. B]. □

Theorem 6 *Three given points define a triangle iff they are not in straight line, being the vertexes of the triangle the given points and its sides the straight lines joining them. A point defines a triangle with any two given points iff it is a non-common points of one of the sides of the straight line joining the given points.*

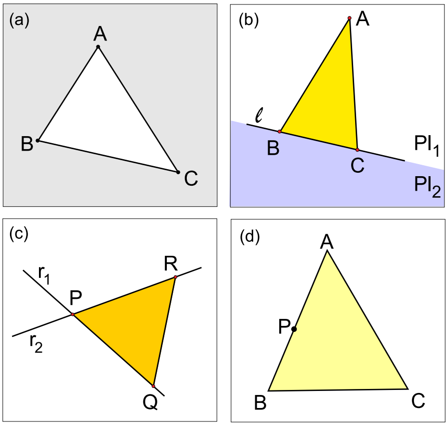


Figure 8.6 – **Theorem 6**

(Fig. 8.6, a) Let A, B and C be any three points not in straight line [Cr. 22].

There is a plane Pl that contains them [Ax. 6].

Join A with B; B with C; and C with A [Cr. 15].

AB, BC and CA are in Pl [Cr. 25].

And none of them is in straight line with any of the others, otherwise A, B and C would be in straight line [Df. 12],

which is not the case. B is the only common point of AB and BC, otherwise they would be in straight line [Cr. 18]

and A, B and C would be in straight line [Df. 12],

which is not the case. So, AB and BC are adjacent at B [Df. 4].

For the same reason BC is adjacent at C to CA, and CA adjacent at A to AB. So, each of the straight lines AB, BC and AC is adjacent at each of its two endpoints to one, and only to one, of the others [Df. 4].

Therefore, A, B and C define a triangle ABC whose vertexes are

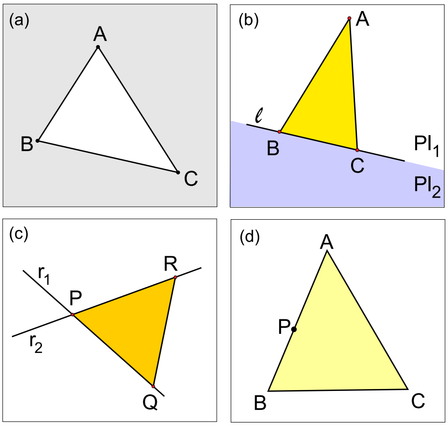


Figure 8.6 – **Theorem 6**

A, B and C and whose sides are AB, BC and CA [Dfs. 29, 28].

Alternatively, if ABC is a triangle, its vertexes cannot be in straight line, otherwise they would be in the same straight line and ABC would not be a triangle [Df. 29, 28].

On the other hand, if A is any non-common point of one of the sides of a straight line l in the plane Pl (Fig. 8.6, b), it cannot be in straight line with any couple of points B and C of l, even arbitrarily produced [Df. 14],

and A, B and C define a triangle, as it has just been proved. And if a point defines a triangle with any two points, it cannot be in straight line with these two points [Df. 29, 28].

So, that point cannot be a common point of the sides of the straight line through those points [Df. 12],

and it must be a non-common point of one of such sides [Cr. 26]. □

Corollary 46 *The intersection point of two intersecting straight lines defines a triangle with any two different points, each of a different straight line.*

(Fig. 8.6, c) It is an immediate consequence of [Cr. 21, Df. 12, Th. 6]. □

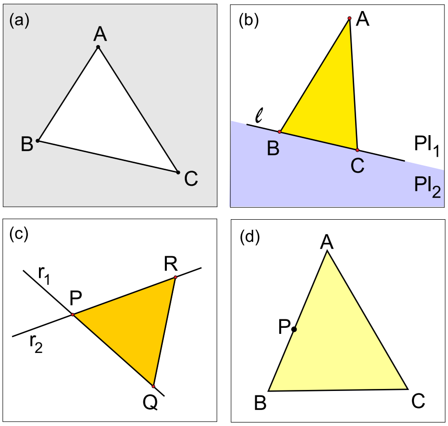


Figure 8.6 – **Theorem 6**

Corollary 47 *A point of a side between two vertexes of a triangle is not in straight line with any of the other sides of the triangle, even arbitrarily produced.*

(Fig. 8.6, d) If a point P of a side* AB of a triangle ABC were in straight line with the side* BC, these two sides would have two common points, B and P, and they would belong to the same straight line [Cr. 18],

which is impossible [Th. 6].

The same argument applies to P and the side AC. □

Theorem 7 *If the length of a given straight line joining the centers of two circles is less than the sum of their radii, and each radius is less than the sum of the other radius and the length of the given straight line, then the circles intersect at two points, each on a different side of the given straight line and not in a straight line with it.*

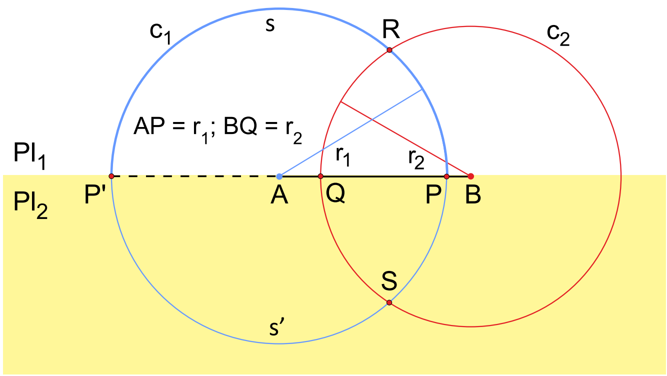


Figure 8.7 – **Theorem 7**

Let AB be any straight line [Cr. 25],

P any point of AB different from A, and Q any point of AP different from P [Cr. 5].

Consider first the case $P \neq B$; $Q \neq A$. In this case, A, B,P and Q satisfy $AP < AB$, $BQ < AB$; and also $AB = AQ + QB < AQ + QP + QB = AP + QB$ [Cr. 13, Pss. B, A].

Draw the circle c_1 with centre A and radius AP; and the circle c_2 with centre B and radius BQ [Ax. 8].

Produce AB from A to a point P' such that $AP' = AP$ [Th. 1].

$P'P$ is the unique diameter of two complementary semicircles s and s' of c_1, s in Pl_1 and s' in Pl_2 [Cr. 36].

P is interior to c_2 because $BP < BQ$ [Cr. 13, Df. 19].

P' is exterior to c_2 because $BP' > BQ$ [Cr. 13, Df. 19].

Therefore, s cuts c_2 at a point R between the endpoints P and P' of s [Cr. 35].

R is, then, a non common point of Pl_1 that is not in straight line

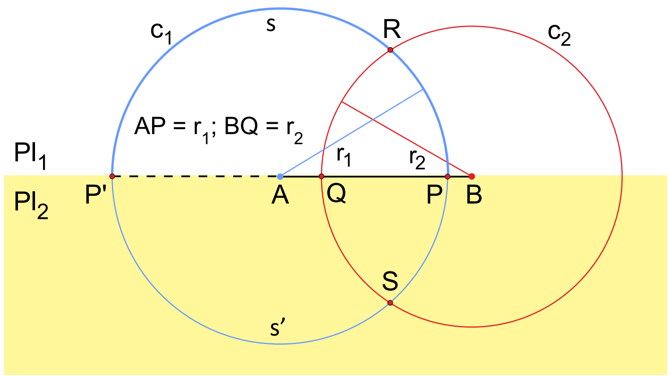


Figure 8.7 – Theorem 7

with AB [Ax. 8, Df. 14].

In the same way it is proved that s' cuts c_2 at a point S of Pl_2 which is not in a straight line with AB. And the same complete argument applies to the cases $P = B$ and/or $Q = A$, that is, the cases in which two, or all three, straight lines AB, AP and BQ are equal. $\square$

Theorem 8 (Extension of Euclid's Proposition 1) *To construct two equilateral triangles with a common side equal to a given straight line.*

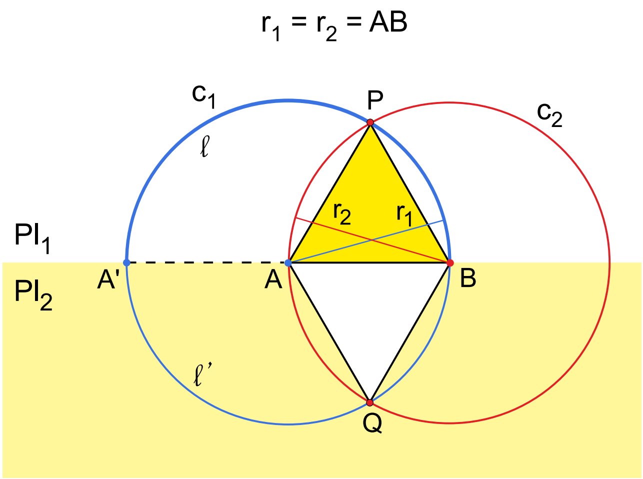


Figure 8.8 – **Theorem 8**

Let AB be the given straight line [Cr. 25].

With centre A and radius $r_1 = AB$, draw the circle c_1 [Ax. 8].

And with centre B and radius $r_2 = AB$, draw the circle c_2 [Ax. 8].

Assume $r_1 \geq r_1 + r_2$. We would have $0 \geq r_2$ [Ps. B],

which is impossible [Cr. 13].

Therefore $r_1 < r_1 + r_2$ [Ps. A].

And being $AB = r_1$, it holds $AB < r_1 + r_2$ [Ps. A].

Therefore, c_1 and c_2 intersect at two points P and Q, each in a different side of AB and not in straight line with AB [Th. 7].

In consequence, P, A and B define a triangle PAB, and A, Q and B define a triangle AQB [Th. 6],

Join P with A and with B; and join Q with A and with B [Cr. 15].

PAB and AQB are a triangles [Th. 6],

and being $PA = r_1 = AB$; $PB = r_2 = AB$; $QA = r_1 = AB$; $QB = r_2 = AB$ [Df. 19],

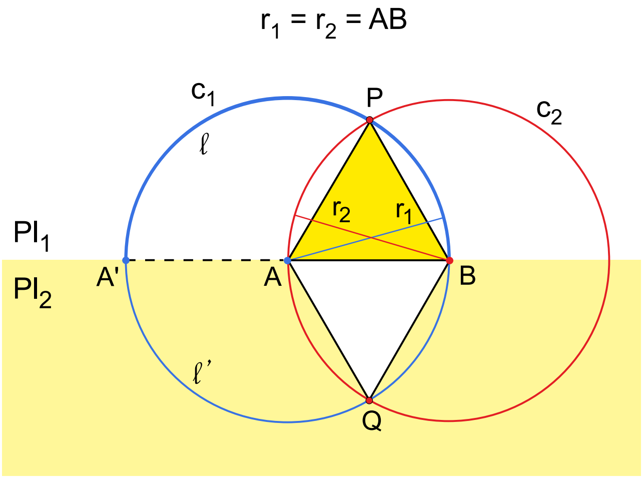


Figure 8.8 – Theorem 8

we will have $PA = PB = AB$ and $QA = QB = AB$ [Ps. B].

Therefore, the triangles PAB and AQB are equilateral with the same side, which is the given straight line AB [Df. 29]. □

Theorem 9 *To construct an isosceles triangle whose equal sides have the length of a given straight line.*

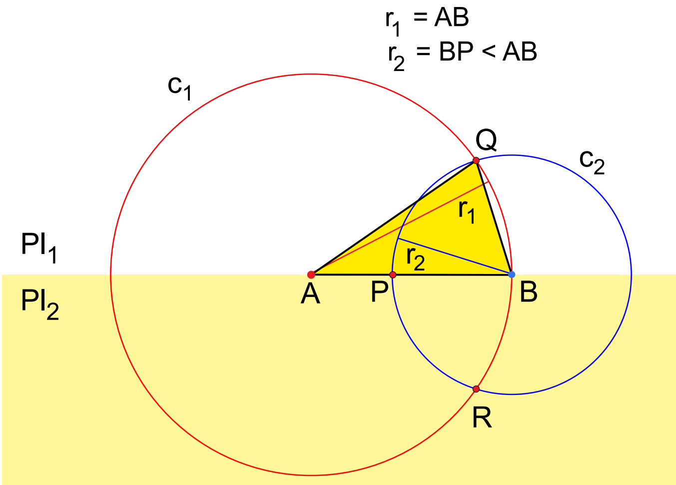


Figure 8.9 – **Theorem 9**

Let AB be the given straight line [Cr. 25].

Take any point P between A and B [Cr. 5].

It holds $BP < AB$ [Cr. 13].

With centre A and radius AB draw the circle c_1 [Th. 1].

With centre B and radius BP draw the circle c_2 [Th. 1].

The points A, B and P satisfy: $BP < AB$; $BP < AB + AB$; $AB < AB + BP$ [Cr. 13, Ps. B].

Therefore, the circles c_1 and c_2 intersect at two points Q and R, each on a different side of AB and not in straight line with AB [Th. 7].

In consequence, Q, A and B define a triangle QAB [Th. 6].

Join Q with A and with B [Cr. 15].

QAB is a triangle [Th. 6],

and being $AB = AQ$ [Df. 19],

the triangle QAB is isosceles [Df. 29]. □

Theorem 10 *To construct an scalene triangle with a side equal to a given straight line.*

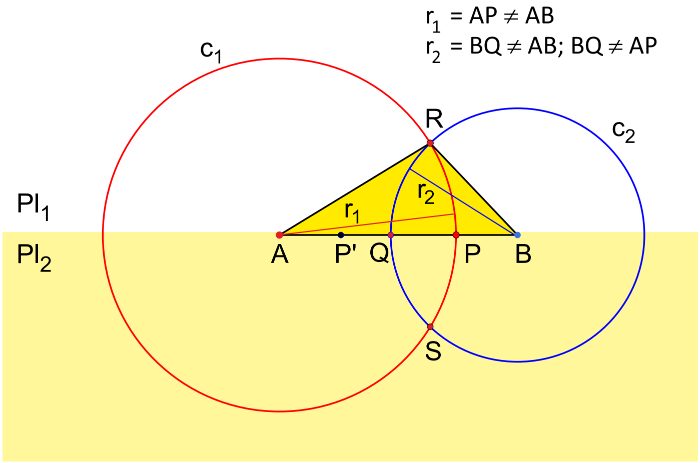


Figure 8.10 – **Theorem 10**

Let AB be the given straight line [Cr. 25].

On AB take any point P and a point P' such that $BP' = AP$ [Th. 1].

On PP' take any point Q [Cr. 5].

With centre A and radius AP describe the circle c_1 [Ax. 8].

With centre B and radius BQ describe the circle c_2 [Ax. 8].

The points P and Q on AB satisfy $AP < AB$, $QB < AB$ and $QB \neq AP \neq AB$ [Cr. 13],

and also $AB = AP + PB < AP + PB + PQ = AP + QB$ [Cr. 13, Pss. B, A].

In consequence, the circles c_1 and c_2 intersect at two points R and S, each in a different side of AB and not in straight line with AB [Th. 7].

So, R, A and B define a triangle RAB [Th. 6].

Join R with A and with B [Cr. 15].

RAB is a triangle [Th. 6],

and being unequal its three sides, RA, AB and RB, the triangle RAB is scalene [Df. 29]. □

Theorem 11 *If two triangles have equal one of its sides and the two angles whose respective vertexes are the endpoints of that side, then the other two sides of each triangle are also equal to the corresponding two sides of the other.*

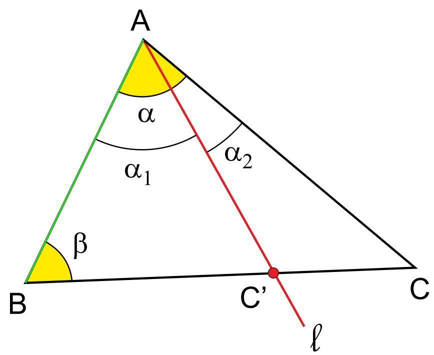


Figure 8.11 – **Theorem 11**

Let ABC be any triangle [Ths. 10, 9, 8]

with an angle α at A and an angle β at B [Cr. 44].

Assume it is possible a triangle ABC' with a side AB; an angle α at A; an angle β at B; and the side BC' of β different from BC, for instance* $BC' < BC$ [Ps. A].

A, C' and C are not in straight line, nor A, B and C' either [Cr. 47, Df. 12].

So, ABC' and $AC'C$ are triangles [Th. 6].

And since ABC is also a triangle, AB, AC' and AC are adjacent at A [Dfs. 29, 28]

where they make the adjacent angles α_1 and α_2 [Df. 23, Cr. 41]

whose union angle is the angle α that AB makes at A with AC [Th. 3],

and $\alpha_1 < \alpha$ [Th. 3].

So, it is impossible a triangle with a side AB, an angle α at A, an angle β at B and a side $BC' \neq BC$. The same argument applies to the side AC. □

Theorem 12 (Hilbert's Axiom IV.6) *If two triangles have equal one of their angles and the two sides of that angle, then they have also equal their corresponding other two angles.*

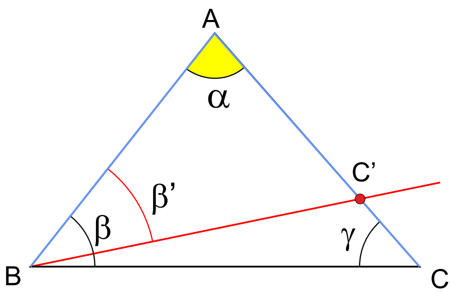


Figure 8.12 – Theorem 12

Let ABC be a triangle [Ths. 10, 9, 8],

and α, β and γ its corresponding angles respectively at A, B and C [Cr. 44].

A triangle ABC' with an angle α at A, a side AB, a side AC and an angle β' at B different from β is impossible because, being β unique [Cr. 37],

$\beta' \neq \beta$ implies that β' will not superpose BA on BC but on a different straight line BC', where C' is a point of AC, whether or not produced, different from C [Df. 21],

otherwise BC and BC' would be the same straight line [Cr. 15].

So, $AC' \neq AC$ [Cr. 13].

For the same reasons it is impossible a triangle with a side AB, a side AC an angle α at A and an angle γ' at C such that $\gamma' \neq \gamma$. □

Corollary 48 (Euclid's Proposition 4) *If two triangles have equal one of their corresponding angles and the two sides of that angle, then they have also equal their corresponding other two angles and their corresponding third side.*

It is an immediate consequence of [Ths. 12, 11]. □

Theorem 13 *In Isosceles triangles the productions of the equal sides make equal the exterior angles with the third side, and the equal sides make also equal the interior angles with the third side.*

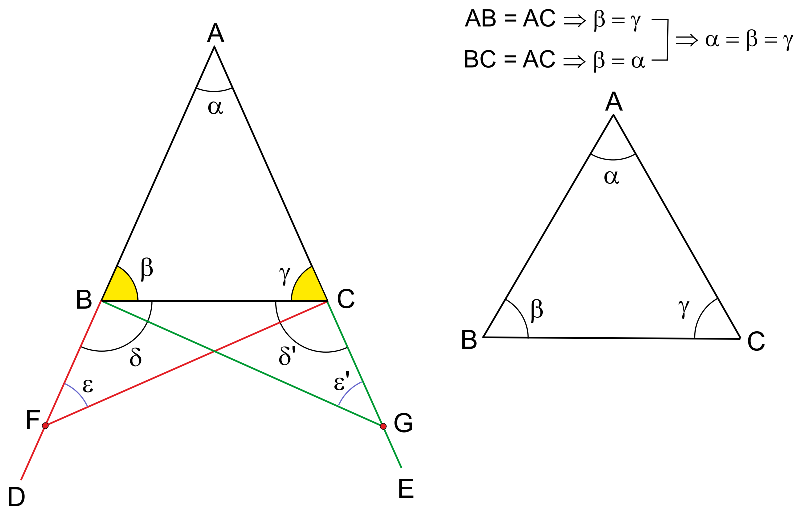


Figure 8.13 – **Theorem 13**

(Fig. 8.13, left) Let ABC be an isosceles triangle [Th. 9]

with the side AB equal to the side AC [Df. 29].

Produce AB and AC from B and C respectively to any two points D and E [Cr. 16].

Let F be any point between A and D [Ax. 2].

In CE take a point G such that $CG = BF$ [Th. 1].

Join F with C, and G with B [Cr. 15].

Since $AB = AC$ and $BF = CG$, it holds $AB + BF = AC + CG$ [Ps. B].

And being $AB + BF = AF$ and $AC + CG = AG$ [Cr. 13],

it holds $AF = AG$ [Ps. A].

Since ABC is a triangle, B is not in a straight line with AE; nor C with AD [Dfs. 29, 28, 12].

Therefore AFC and ABG are triangles [Th. 6],

with two equal sides: $AF = AG$ and $AC = AB$; and with the same

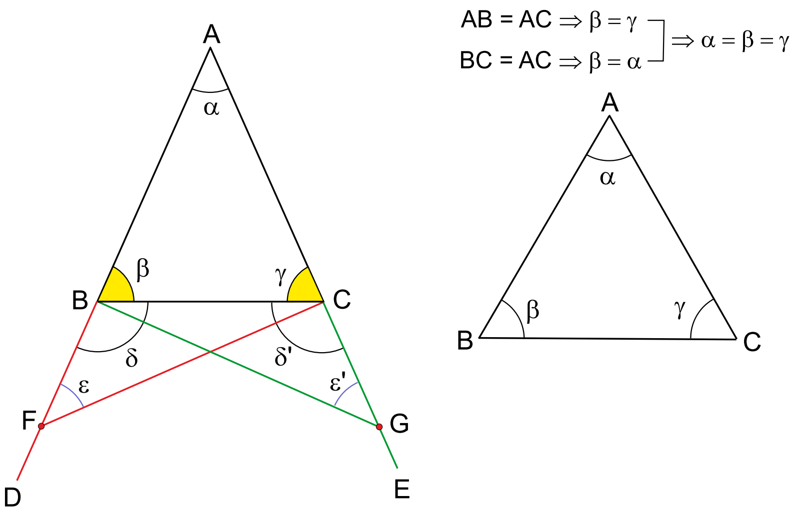


Figure 8.13 – Theorem 13

angle α between the equal sides [Cr. 44].

Therefore $\epsilon = \epsilon'$ and $FC = BG$ [Cr. 48].

Since AFC and ABG are triangles, F is not in straight line with AG; nor G with AF [Dfs. 29, 28, 12].

Therefore BFC and BGC are triangles [Th. 6],

with two equal sides: $BF = CG$ and $FC = BG$, and with the same angle $\epsilon = \epsilon'$ between the two equal sides [Cr. 44].

Therefore, $\delta = \delta'$ [Cr. 48],

where δ and δ' are the exterior angles that the productions of BF and CG of the equal sides of ABC make with its third side BC [Df. 28, Cr. 44].

Being sides of the triangles ABC and BFC, the sides BA, BC and BF are adjacent at B [Dfs. 4, 29, 28].

And taking into account that BA and BF are the sides of B in the straight line AF [Df. 5],

the angles β and δ sum a straight angle [Th. 4].

For the same reason, the angles γ and δ' also sum to a straight angle [Th. 4].

And being equal all straight angles [Th. 2],

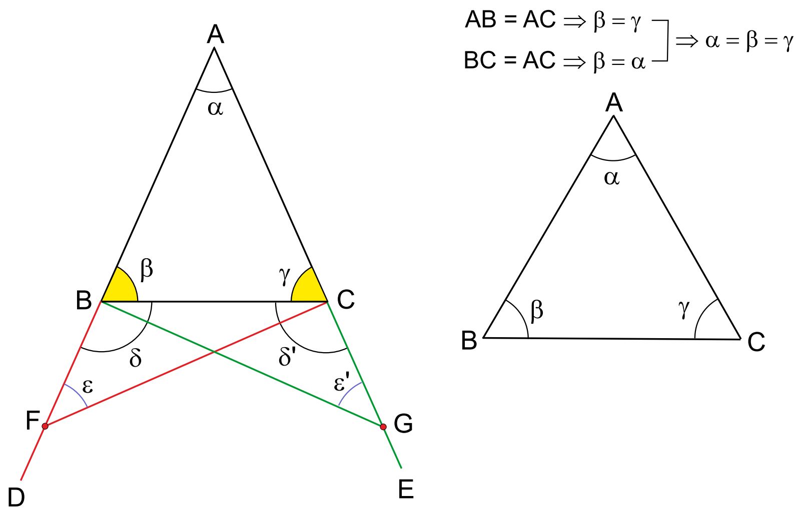


Figure 8.13 – **Theorem 13**

we will have $\beta + \delta = \gamma + \delta'$. Therefore, and being $\delta = \delta'$, it also holds $\beta = \gamma$ [Ps. B],

where β and γ are the angles the equal sides AB and AC make with the third side BC. □

Corollary 49 *The three angles of an equilateral triangle are equal to one another.*

(Fig. 8.13, right) It is an immediate consequence of [Df. 29, Th. 13, Ps. B]. □

Theorem 14 (Euclid's Proposition 8) *If the three sides of a triangle are equal to the three sides of another triangle, then the three angles of the one are also equal to the corresponding three angles of the other.*

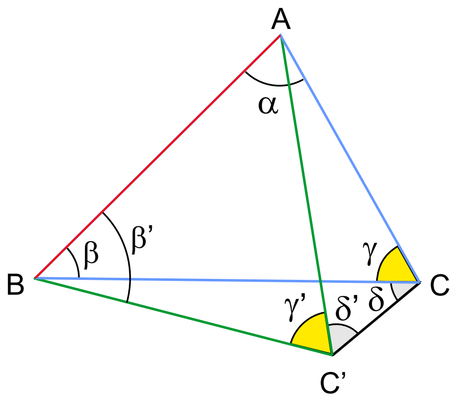


Figure 8.14 – Theorem 14

Let ABC and ABC' be two triangles [Ths. 10, 9, 8]

with a common side AB and such that $BC = BC'$; $AC = AC'$. Assume $\beta' \neq \beta$ [Crs. 44, Ax. 9].

The angle β' will not superpose BA on BC but on BC' [Cr. 37],

where C' can only be different from C, otherwise BC and BC' would be superposed and $\beta = \beta'$ [Df. 21, Cr. 37].

Join C and C' [Cr. 15].

C' cannot be in straight line with A and C, otherwise it would be a point of AC, whether or not produced [Df. 12, Cr. 16],

different from C, and then $AC \neq AC'$ [Cr. 13],

which is not the case. So, $AC'C$ is an isosceles triangle [Cr. 6, Df. 29].

For the same reasons, $BC'C$ is an isosceles triangle. Since ABC', $AC'C$ and $BC'C$ are triangles [Th. 6],

$C'B$, $C'A$ and $C'C$ are adjacent at C' [Dfs. 29, 28].

Since ABC, $AC'C$ and $BC'C$ are triangles [Th. 6],

CA, CB and CC' are adjacent at C [Dfs. 29, 28].

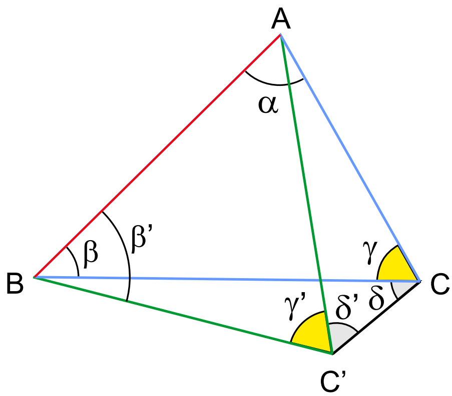


Figure 8.14 – Theorem 14

So, $\gamma' + \delta' > \delta'$ and $\gamma + \delta > \delta$ [[Cr. 41, Th. 3].

On the other hand, in $BC'C$ it holds $\gamma' + \delta' = \delta$ [Th. 13],

and in $AC'C$: $\gamma + \delta = \delta'$ [Th. 13].

In consequence, $\delta' > \delta$ and $\delta > \delta'$ [Ps. A],

which is impossible [Ps. A].

Therefore, the initial assumption is false, and the same rotation β that superposes AB on BC superposes AB on BC' [Df. 21].

So, $\beta = \beta'$ [Df. 22].

And the other two angles of ABC' are equal to the angles α and γ of ABC [Cr. 48]. $\square$

Theorem 15 (Euclid's proposition 10) *To bisect a given finite straight line.*

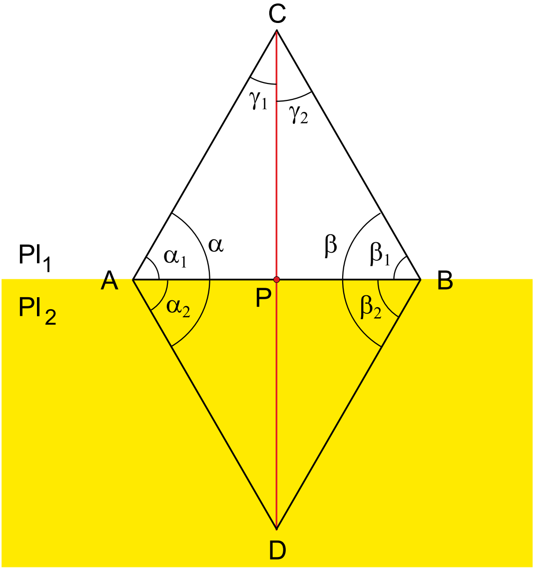


Figure 8.15 – Theorem 15

Let AB be the given finite straight line [Cr. 25].

Let the equilateral triangles CAB and ADB be constructed on AB, each on a different side of AB [Th. 8].

Join C and D [Cr. 15].

Since C and D are in different sides of AB, CD intersects AB, whether produced or not, at a unique point P [Cr. 31].

Being in different sides of AB, the points C and D are not in straight line with A and B [Df. 14],

and CAD, CDB, CAP, CPB, ADP and PDB are triangles [Th. 6].

Consequently, AC, AP and AD are adjacent at A [Dfs. 29, 28]

and $\alpha = \alpha_1 + \alpha_2$ [Cr. 41, Th. 3].

For the very reason, $\beta = \beta_1 + \beta_2$. Since the triangles CAB and ADB are equilateral and they have a common side AB, the three sides of CAB are equal to the three sides of ADB [Df. 29, Ps. A].

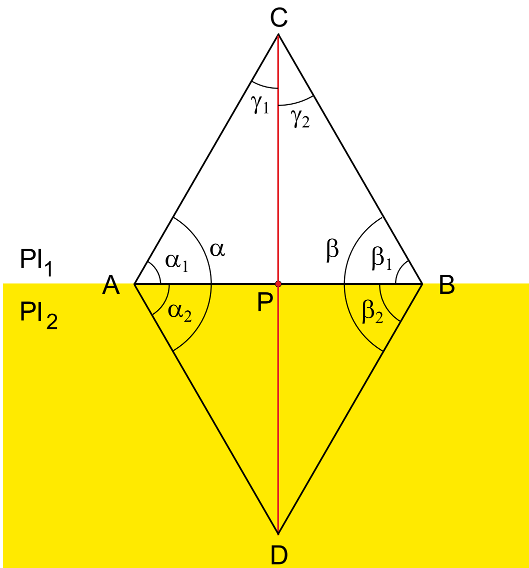


Figure 8.15 – Theorem 15

Therefore, $\alpha_1 = \alpha_2$; $\beta_1 = \beta_2$ [Th. 14].

And being the three angles of an equilateral triangle equal to one another [Cr. 49]

we have $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$; and then $\alpha = \alpha_1 + \alpha_2 = \beta_1 + \beta_2 = \beta$ [Ps. A].

The triangles CAD and CDB satisfy $CA = CB$, $DA = DB$ and $\alpha = \beta$. So, $\gamma_1 = \gamma_2$ [Cr. 48].

The triangles CAP and CPB have a common side CP and also $AC = CB$ and $\gamma_1 = \gamma_2$. Therefore, $AP = PB$ [Cr. 48].

So, the point P bisects the given finite straight line AB. □

Theorem 16 (Euclid's proposition 16) *In any triangle, if one of the sides is produced, the corresponding exterior angle is greater than each of the two interior and opposite angles.*

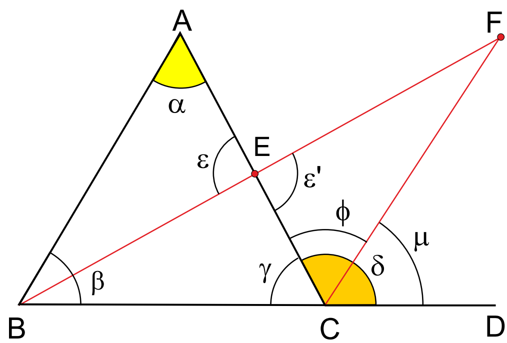


Figure 8.16 – Theorem 16

(Fig.8.16, top) Let ABC be any triangle [Ths. 10, 9, 8].

Let any of its sides, for instance* the side BC, be produced to any point D [Cr. 16].

BC and CD are adjacent at C [Cr. 16].

Let AC be bisected at E [Th. 15].

Joint B with E [Cr. 15],

and produce BE to a point F such that $EF = BE$ [Cr. 16, Th. 1].

Join F and C [Cr. 15].

E is the only point of FB in straight line with AC, otherwise FB and AC would belong to the same straight line [Cr. 18]

and A, B and C would be in straight line, which is impossible [Th. 6].

So, ABE, FEC and FBC are triangles [Th. 6].

The triangles ABE and FEC satisfy $AE = CE$; $EB = EF$; and $\epsilon = \epsilon'$ [Th. 5].

Therefore, $\alpha = \phi$ [Cr. 48].

Since CE, CF and CD are adjacent at C [Dfs. 28, 29, Cr. 24],

they define the adjacent angles μ and ϕ [Cr. 41],

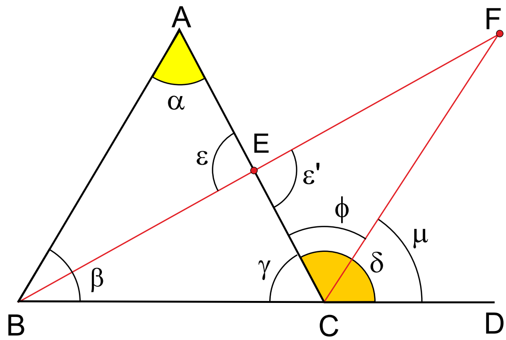


Figure 8.16 – **Theorem 16**

whose union angle is the exterior angle δ [Th. 3, Cr. 44].

Therefore, $\phi < \delta$ [Th. 3].

Being δ an angle exterior to the triangle ABC [Df. 28],

And being ϕ equal to α, we conclude that α is less than δ [Ps. B]. A similar argument proves the same conclusion for the the angle β. □

Theorem 17 (Euclid's Propositions 18 and 19) *A side is the greatest of a triangle iff it subtends its greatest angle.*

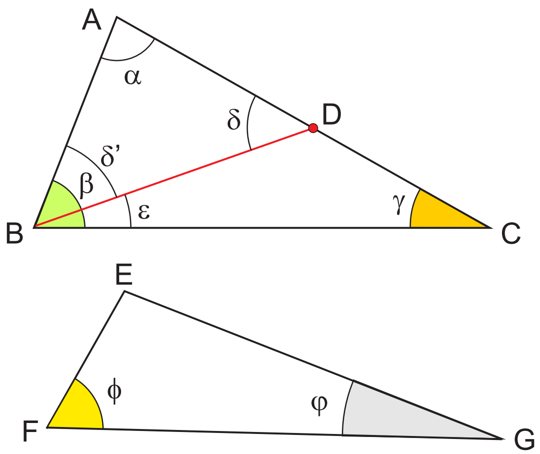


Figure 8.17 – Theorem 17

(Fig. 8.17, top) Consider any two sides AB and AC of a triangle ABC [Ths. 10, 9, 8],

and assume* $AB < AC$ [Ps. A].

On AC take a point D such that $AD = AB$ [Th. 1]

and join D with B [Cr. 15].

D is not in straight line with AB or with BC [Cr. 47].

So, ABD and DBC are triangles [Th. 6].

ABD is isosceles [Df. 29],

and then $\delta = \delta'$ [Th. 13].

It also holds $\delta > \gamma$ [Th. 16].

Since ABC, ABD and DBC are triangles, BA, BD and BC are adjacent at B [Dfs. 29, 28],

where they make the adjacent angles δ' and ϵ [Cr. 41]

whose union angle is β [Df. 23, Th. 3].

Therefore, $\beta > \delta'$ [Th. 3].

From $\beta > \delta'$, $\delta' = \delta$ and $\delta > \gamma$, it follows $\beta > \gamma$ [Ps. B].

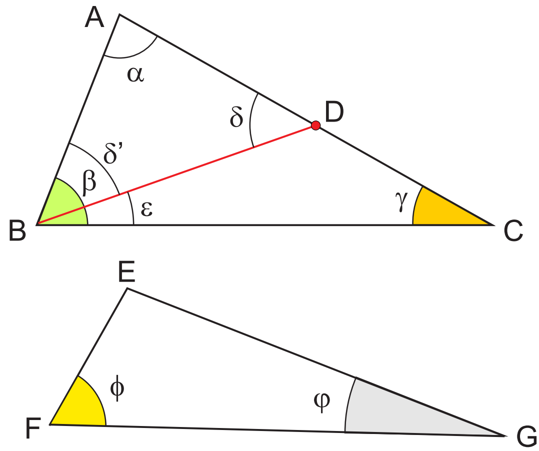


Figure 8.17 – Theorem 17

Being AB and AC any two sides of ABC, we concluded that in a triangle the greatest side subtends the greatest angle [Df. 22, Ps. B].

Let now ϕ and φ (Fig. 8.17, bottom) be any two angles of a triangle EFC [Ths. 10, 9, 8]

and assume* $\phi > \varphi$ [Ps. A].

EG cannot be equal to EF, otherwise $\phi = \varphi$ [Th. 13],

which is not the case. Nor can it be less than EF, because in such a case the least side would subtend the greatest angle, which is impossible, as just proved. So, EG must be greater than EF [Ps. A].

Being ϕ and φ any two angles of a triangle [Cr. 44],

we conclude that the greatest angle is subtended by the greatest side [Ps. B]. □

Theorem 18 (Euclid's Proposition 20) *In a triangle the sum of the lengths of any two of its sides is greater than the length of the remaining one.*

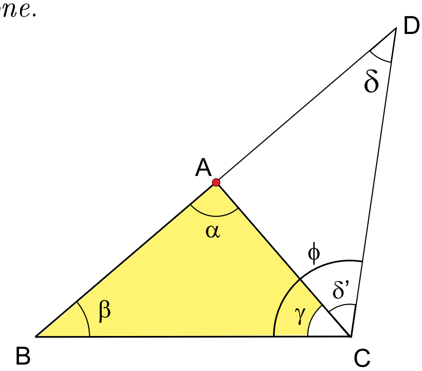


Figure 8.18 – Theorem 18

Let ABC be a triangle [Ths. 10, 9, 8].

Produce AB from A to a point D such that $AD = AC$ [Cr. 16, Th. 1].

Join D and C [Cr. 15].

B is the only intersection point of BD and BC; and A the only intersection point of BD and AC, otherwise A, B and C would be in straight line, which is impossible [Th. 6].

In consequence D is not in straight line with B and C, and DBC is a triangle [Th. 6].

For the same reasons, DAC is a triangle. And being DAC isosceles [Df. 29],

it holds $\delta = \delta'$ [Th. 13].

Since ABC, DBC and DAC are triangles, CB, CA and CD are straight lines adjacent at C [Dfs. 29, 28];

where they make the adjacent angles γ and δ' [Cr. 41]

whose union angle is ϕ [Df. 23, Th. 3].

Therefore, $\phi > \delta'$ [Th. 3].

And since $\delta = \delta'$, it holds $\phi > \delta$ [Ps. A].

In consequence $DB > BC$ [Th. 17],

and then $AB + AD > BC$; $AB + AC > BC$ [Ps. A].

The same argument proves the sum of the lengths any other couple of sides of ABC is greater than the length of the remaining one. $\square$

9. On distances and perpendiculars

9.1 Introduction

This chapter proves the existence of perpendicular straight lines and their uniqueness: through any point, whether or not in a given straight line, it is only possible to draw a perpendicular to the given straight line. It also proves Euclid's Postulate 4: that all right angles are equal to one another. Finally, it proves the existence of the distance [Df. 16] to a given straight line from a point that is not on a straight line with the given straight line, and that distance is the length of the perpendicular drawn from that point to the given straight line, produced if necessary.

Theorem 19 *The length of straight line joining any two points interior to a circle is less than the sum of the lengths of two radii of the circle.*

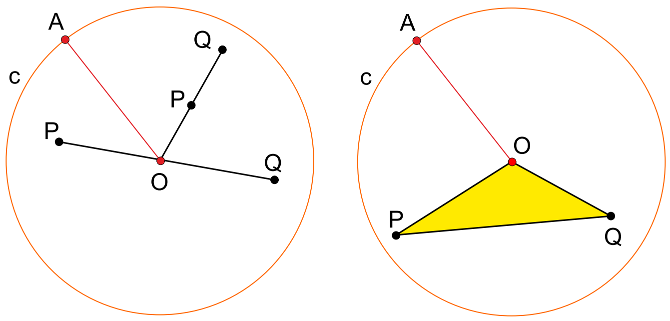


Figure 9.1 – Theorem 19

Let c be a circle whose center is O and whose finite radius is OA [Ax. 8, Cr. 45],

and P and Q any two points interior to c [Cr. 34].

It must be $OA < OA + OA$, otherwise $OA \geq OA + OA$ [Ps. A],

and $0 \geq OA$ [Ps. B], which is impossible [Cr. 13].

Join O with P and with Q; and join P with Q [Cr. 15].

If O, P and Q are in straight line (Fig. 9.1, left), one of them will be between the other two [Df. 11, Cr. 9].

If O is between P and Q then $PQ = OP + OQ$ [Cr. 13].

And being $OP < OA$, $OQ < OA$, it holds: $OP + OQ < OA + OQ$; $OQ + OA < OA + OA$ [Ps. B].

Therefore, $OP + OQ < OA + OA$, $PQ < OA + OA$ [Pss. B, A].

If P is between O and Q it holds $PQ < OQ$ [Cr. 13]

and being $OQ < OA$ [Df. 19]

and $OA < OA + OA$, it must be $PQ < OA + OA$ [Ps. B].

The same argument applies if Q is between O and P. If O, P and Q are not in straight line (Fig. 9.1 right) they define a triangle OPQ [Th. 6]

in which it holds $PQ < OP + OQ$ [Th. 18].

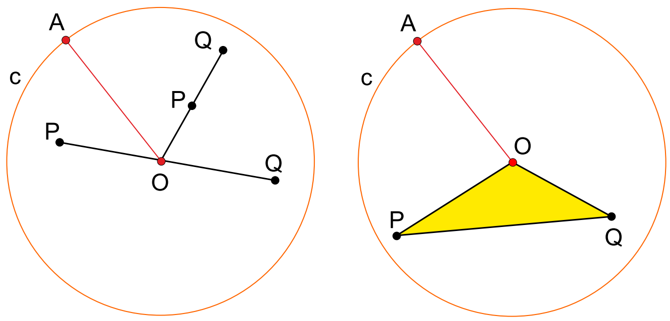


Figure 9.1 – **Theorem 19**

Being $OP < AO$; $OQ < AO$ [Df. 19],

and for the same reasons above, $PQ < OA + OA$ [Ps. B].

In consequence, the length PQ is always less than the sum of the lengths of two of its radii. $\square$

Theorem 20 (Euclid's Proposition 11) *Through a given point on a given straight line draw a perpendicular to the given straight line in any of its sides.*

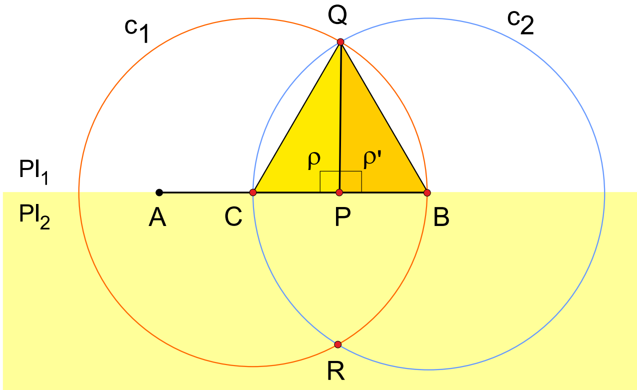


Figure 9.2 – **Theorem 20**

Let AB be any given straight line [Cr. 25]

and P any given point of AB [Cr. 1].

Assume* $PB < PA$. Take a point C in PA such that $PC = PB$ [Th. 1].

With centers C and B and the same radius CB draw the respective circles c_1 and c_2 [Ax. 8].

It must be $CB < CB + CB$, otherwise $0 \geq CB$ [Ps. B],

which is impossible [Cr. 13].

Thus, c_1 and c_2 intersect at two points Q and R, none of which in straight line with CP [Th. 7].

being, therefore, non-common points of the sides of AB [Df. 14].

Join one of the points Q or R, for instance* Q, with C, with P and with B [Cr. 15].

Since Q is not in straight line with C, P and B, these four points define the triangles QCP and QPB [Th. 6],

and the three sides of QCP are equal to the three sides of QPB [Df. 19].

Therefore $\rho = \rho'$ [Th. 14].

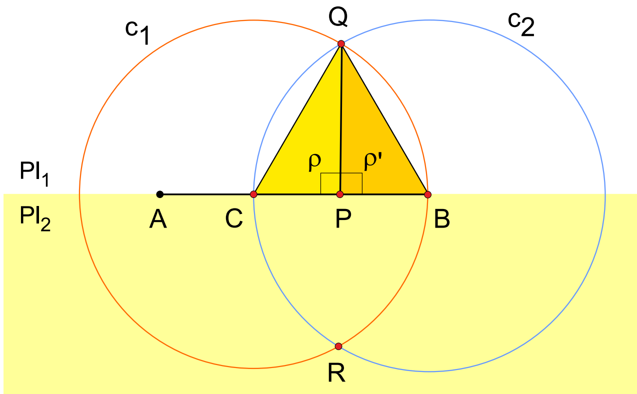


Figure 9.2 – **Theorem 20**

And being PC, PQ and PB adjacent at P [Dfs. 28, 29],

P is the common vertex and PQ the common side of ρ and ρ'. So, ρ and ρ' are adjacent angles [Cr. 41, Df. 23],

and since they are equal to each other, they are right angles [Df. 25],

PQ is perpendicular to AB through the given point P [Df. 25],

and being Q a non-common point of one of the sides of AB, and P a common point of both sides of AB [Df. 14],

PQ is in the side of AB in which is Q [Ax. 6]. □

Theorem 21 (A variant of Euclid's Proposition 12) *From a given point not in straight line with a given straight line, to draw a perpendicular to the given straight line, produced if necessary.*

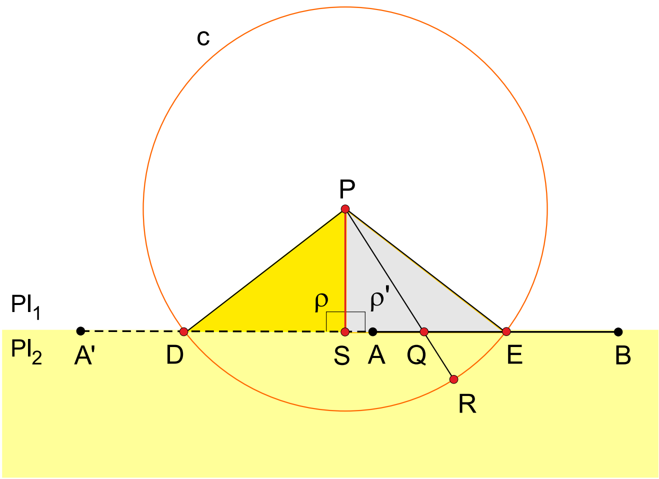


Figure 9.3 – Theorem 21

Let AB be a straight line [Cr. 25]

and P a point not in straight line with AB [Cr. 22].

Take any point Q in AB [Cr. 1].

Join P and Q [Cr. 15]

and produce PQ from Q by any given length to a point R [Cr. 16].

With centre P and radius PR draw the circle c [Ax. 8].

Since PQ is less than PR [Cr. 13],

Q is interior to c [Df. 19].

In AB and in the direction from B to A, take a point A' at a distance $PR + PR$ from Q [Th. 1].

Being $QA' = PR + PR$ and Q interior to c, A' cannot be interior to c [Th. 19].

So, it will be either a point of c or exterior to c [Cr. 34],

and in both cases there will be an intersection point D of c and QA' [Cr. 35, Df. 3].

The same argument applied to the direction from A to B proves

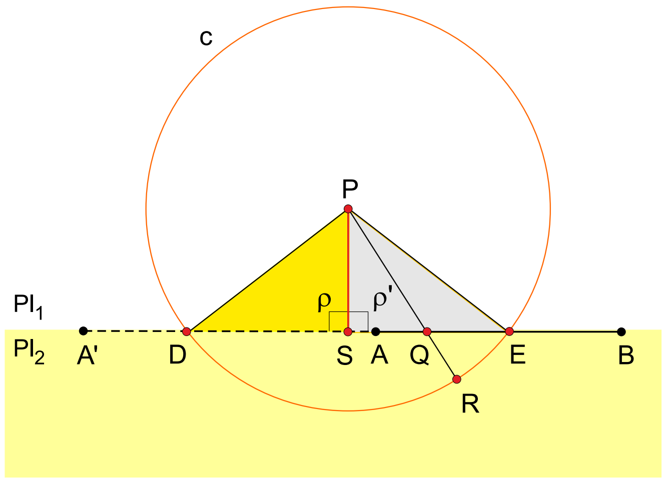


Figure 9.3 – Theorem 21

the existence of another intersection point E of AB, produced from B if necessary, and c. Join P with D and with E [Cr. 15].

Bisect DE at S [Th. 15],

where S could coincide with the point Q, and join S with P [Cr. 15].

Being not P in straight line with AB, it is not in straight line with any two points of AB [Df. 12],

whether or not produced [Cr. 16].

So, PDS and PSE are triangles [Th. 6]

with a common side PS, being also $SD = SE$ and $PD = PE$ [Df. 19],

So, $\rho = \rho'$ [Th. 14].

SD, SP and SE are adjacent at S because PDS and PSE are triangles [Dfs. 29, 28].

So, SD, SP and SE make two adjacent angles ρ and ρ' at their common point S [Cr. 41],

which being equal, are right angles [Df. 25],

and SP is the perpendicular from P to AB [Df. 25],

produced if necessary. □

Corollary 50 *The perpendicular to a given straight line from a given point that is not in a straight line with the given straight line is on the same side of the given straight line as the given point.*

It is an immediate consequence of [Th. 21, Df. 14, Ax. 6]. □

Theorem 22 (Euclid's Postulate 4) *All right angles are equal to one another, and greater than zero.*

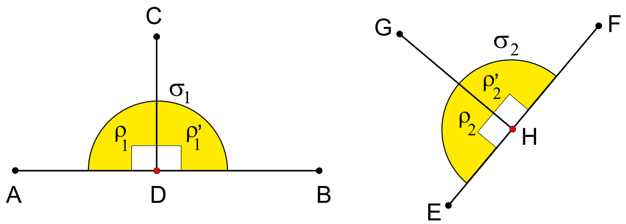


Figure 9.4 – Theorem 22

(Fig. 9.4, left) Let DC be a straight line perpendicular to another straight line AB at any point D of AB [Th. 20],

and let ρ_1 and ρ_1' be the respective adjacent right angles [Df. 25]

that DC makes at D with DA, and DC at D with DB [Cr. 41].

Since DA, DC and DB are adjacent at D [Cr. 37],

the union angle of ρ_1 and ρ_1' is the angle that, in the same direction of rotation of ρ and ρ_1', superposes the non-common sides DA and DB respectively of ρ_1 and ρ_1' [Df. 23, Th. 3],

which are the sides of the straight angle σ_1 that DA makes at D with DB [Df. 24],

and then $\sigma_1 = \rho_1 + \rho_1'$ [Th. 3].

So then, any two adjacent right angles sum a straight angle [Th. 2].

Let ρ_2, ρ_2' be any other couple of adjacent right angles (Fig 9.4, right). As just proved, they sum a straight angle σ_2. Since $\sigma_1 = \sigma_2$ [Th. 2],

it holds $\rho_1 + \rho_1' = \rho_2 + \rho_2'$ [Ps. B].

Assume $\rho_1 < \rho_2$. We would have $\rho_1 + \rho_1' < \rho_2 + \rho_1'$ [Ps. B],

and being $\rho_1 + \rho_1' = \rho_2 + \rho_2'$ [Th. 2],

we can write $\rho_2 + \rho_2' < \rho_2 + \rho_1'$ [Ps. A],

and then $\rho_2' < \rho_1'$ [Ps. B].

And being $\rho_1' = \rho_1$ and $\rho_2' = \rho_2$ [Df. 25],

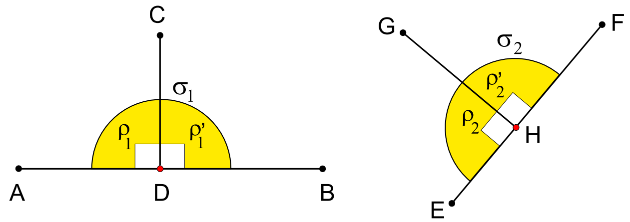


Figure 9.4 – **Theorem 22**

we get $\rho_2 < \rho_1$ [Ps. A],

which contradicts our assumption. So, ρ_1 cannot be less than ρ_2. The same argument applied to the assumption $\rho_1 > \rho_2$ proves ρ_1 cannot be greater than ρ_2 either. So it must be equal to ρ_2 [Ps. A].

Hence, all right angles, whether or not adjacent, are equal to one another. And two adjacent right angles being equal, their common side cannot be superposed on any of the non-common ones, for in that case there would be only two sides and only one angle [Cr. 38, Df. 22].

Therefore all right angles are greater than zero [Df. 22, Ax. 9]. □

Corollary 51 *Two right angles sum a straight angle.*

It is an immediate consequence of [Ths. 4, 22]. □

Corollary 52 *If one of the four angles that two intersecting straight lines make with each other at their intersection point is a right angle, then the other three angles are also right angles; the two sides of each angle are perpendicular to each other; and each straight line is perpendicular to the other.*

It is an immediate consequence of [Df. 25, Ths. 22, 5]. □

Corollary 53 *The two opposite rotations that superpose the two sides of a straight angle are equal to each other.*

It is an immediate consequence of [Crs. 51, 52]. □

Theorem 23 *To draw a right-angled triangle on a given straight line.*

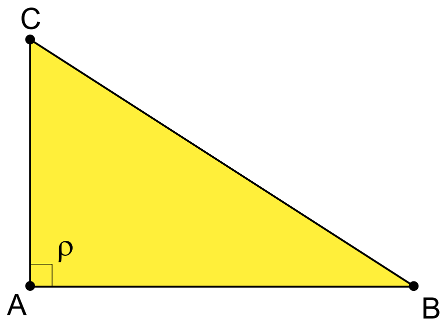


Figure 9.5 – Theorem 23

Let AB be any given straight line [Cr. 25].

Draw the perpendicular AC from A to the straight line AB [Th. 20.

The angle ρ that CA and AB make at A is a right angle [Df. 25].

Since ρ is not a straight angle and it is greater than zero [Cr. 51, Th. 22],

C, A and B are not in a straight line [Cr. 42].

Therefore, A, B and C define a triangle CAB [Th. 6.

Join C with B [Cr. 15].

CAB is a triangle with a right angle ρ. Therefore it is a right triangle [Df. 29]. □

Theorem 24 *Each point of the perpendicular, produced or not, to a given straight lines through the point of bisection of the given straight line is at the same distance from each endpoint of the given straight line.*

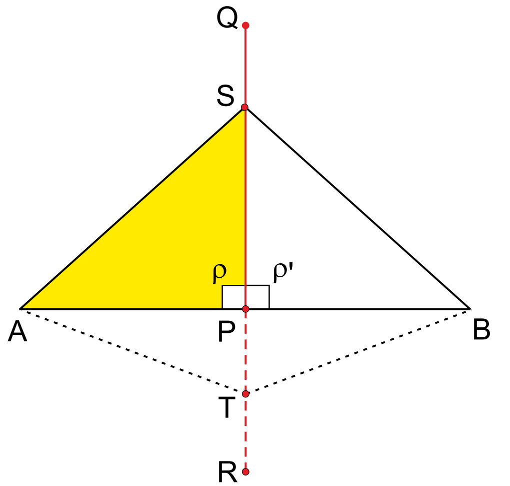


Figure 9.6 – Theorem 24

Let P be the point of bisection of a straight line AB [Th. 15].

Through P draw the perpendicular PQ to AB [Th. 20],

and produce PQ from P to any point R [Cr. 16].

QR is perpendicular to AB [Cr. 52].

Let S be any point of QR [Cr. 1].

Join S with A and with B [Cr. 15].

Since ρ is not an straight angle and it is greater than zero [Cr. 51, Th. 22],

QP and AB are not in straight line [Cr. 42].

Therefore, SAP and SPB are triangles [Th. 6],

with a common side SP, being also $AP = PB$ and $\rho = \rho'$ [Th. 22].

Therefore $SA = SB$ [Cr. 48].

The same argument applies to any point T of the production PR.
□

Theorem 25 *If the two adjacent angles that a straight line makes with another intersecting straight line at their unique intersection point are different from each other, then the one is acute and the other is obtuse.*

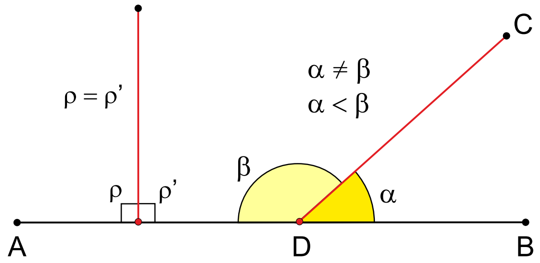


Figure 9.7 – **Theorem 25**

Let D be the unique intersection point of two straight lines AB and CD [Cr. 21].

DA, DC and DB are straight lines [Cr. 14]

adjacent at D [Df. 4],

where they make two adjacent angles α and β [Cr. 41]

that sum two right angles [Th. 4].

If $\alpha \neq \beta$ one of them, for example* α, will be less than the other, $\alpha < \beta$ [Ps. B],

and then $\alpha + \alpha < \beta + \alpha$, and also $\alpha + \beta < \beta + \beta$ [Ps. B].

Being ρ a right angle [Ths. 20, 21, 22],

if $\rho \leq \alpha$, we would have $\rho + \rho \leq \alpha + \rho$; $\rho + \alpha \leq \alpha + \alpha$, and $\rho + \rho \leq \alpha + \alpha$ [Ps. B].

And being $\alpha + \alpha < \beta + \alpha$, we would have $\rho + \rho < \beta + \alpha$ [Ps. B],

which is impossible [Th. 4].

Therefore, it must be $\alpha < \rho$, and α is an acute angle [Df. 25].

If $\beta \leq \rho$, we would have $\beta + \beta \leq \rho + \beta$; $\beta + \rho \leq \rho + \rho$, and $\beta + \beta \leq \rho + \rho$ [Ps. B].

And being $\alpha + \beta < \beta + \beta$, we would have $\beta + \alpha < \rho + \rho$ [Ps. B],

which is impossible [Th. 4].

Therefore, it must be $\beta > \rho$, and β is an obtuse angle [Df. 25]. □

Theorem 26 (Euclid's Proposition 17) *Any two angles of a triangle sum less than two right angles.*

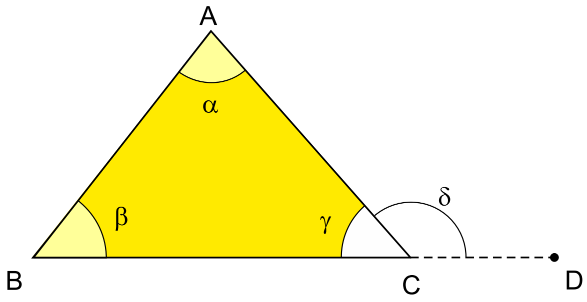


Figure 9.8 – Theorem 26

Let ABC be any triangle [Ths. 10, 9, 8].

Produce the side BC from C by any given length to a point D [Cr. 16].

CB and CA are adjacent at C [Dfs. 28, 29].

CB and CD are adjacent at C [Cr. 16].

C is the only common point of AC and BD, otherwise CB and CA would be superposed in a unique straight line [Df. 21, Cr. 18],

which is impossible because ABC is a triangle [Th. 6].

So, CA and CD are adjacent at C [Df. 4].

Consider the exterior angle δ [Df. 28, Cr. 44].

It holds $\beta < \delta$ [Pr, 16].

Hence, $\beta + \gamma < \delta + \gamma$ [Ps. B].

And being $\delta + \gamma$ a straight angle [Th. 4],

which equals two right angles [Cr. 51],

we conclude that β and γ sum less than two right angles. The same argument proves that any other couple of angles of ABC sum less than two right angles. $\square$

Theorem 27 *From a point, whether or not in straight line with a given straight line, only one perpendicular can be drawn to the given straight line.*

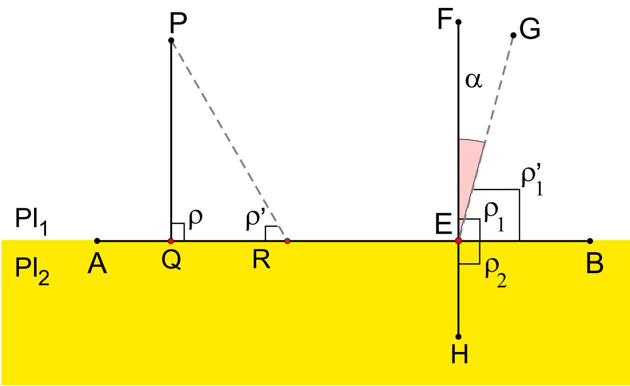


Figure 9.9 – Theorem 27

Let AB be a straight line [Cr. 25]

and P any point not in straight line with AB [Cr. 22].

P will be a non-common point of one of the sides, for example Pl_1 of AB [Ax. 6, Df. 14].

So, P is in straight line with no couple of points of AB [Df. 14].

A perpendicular PQ from P to AB can be drawn [Th. 21].

Assume a second perpendicular PR from P to AB can be drawn. We would have a triangle PQR [Th. 6]

with two right angles, ρ and ρ', which is impossible [Th. 26].

PR is then impossible. Let now E be any point of AB, whether or not produced. Draw the perpendicular EF to AB from E [Th. 20]

and assume a second perpendicular EG from E to AB can be drawn in the same side Pl_1 of AB [Cr. 29].

EF and EG will adjacent at E where they make and angle $\alpha > 0$, if not they would be superposed with two common points [Ax. 9, 21]

and they would belong to the same straight line [Cr. 18].

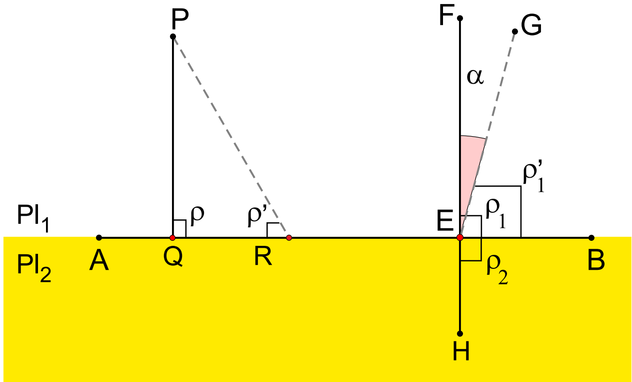


Figure 9.9 – Theorem 27

Being EF, EG and EB straight lines adjacent at E [Cr. 37],

α and ρ_1' are adjacent angles [Cr. 41]

and ρ_1 is the union angle of them [Df. 23, Th. 3].

Therefore, $\rho_1 > \rho_1'$ [Th. 3],

which is impossible [Th. 22].

So, the second perpendicular EG to AB in Pl_1 is impossible. A perpendicular EH from E to AB in Pl_2 can only be adjacent at E to EF because all points of EF and EH, except E, are non-common points respectively of Pl_1 and Pl_2 [Ax. 6, Df. 14].

So, EF, EB and EH can only be three adjacent straight lines [Cr. 37]

that make at their common endpoint E two adjacent angles ρ_1 and ρ_2 [Cr. 41]

whose union angle is the straight angle $\rho_1 + \rho_2$ [Cr. 51].

So then, EF and EH make a unique straight line [Cr. 42].

Therefore, from a point, whether or not in straight line with a straight line, only one perpendicular to the straight line can be drawn. □

Theorem 28 *The distance from a given point not in straight line with a given straight line to the given straight line is the length of the perpendicular from the given point to the given straight line, produced if necessary. And that distance is unique.*

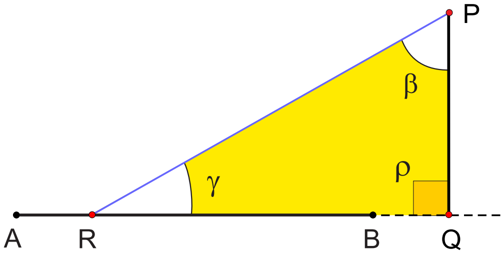


Figure 9.10 – Theorem 28

Let AB be a straight line [Cr. 25]

and P a point not in straight line with AB [Cr. 22].

From P draw the perpendicular PQ to AB [Th. 21].

Let R be any point of AB, whether or not produced, different from Q [Cr. 1].

Join P and R [Cr. 15].

P is not in straight line with R and Q, otherwise P would be in a straight line with AB [Df. 12],

which is not the case. Therefore, P, R and Q define a triangle PRQ [Th. 6].

Since ρ is a right angle [Df. 25],

ρ is the greatest angle of PRQ [Th. 26].

And the side PR is greater than the side PQ [Th. 17].

Since the distance between two points is unique [Cr. 33],

R is any point of AB, whether or not produced, different from Q, and PQ is less than PR, it can be concluded that PQ is the shortest of the distances [Df. 15]

between P and any point in AB, whether or not produced. So, the length of the perpendicular PQ is the distance from the point P to the straight line AB [Df. 16],

and this distance is unique [Th. 27, Cr. 33]. □

Note.-Hereafter, a perpendicular to a straight line drawn from a point that is not in straight line with that straight line, will be drawn by producing the straight line if necessary [Th. 21]. And, unless otherwise indicated, when considering more than one perpendicular to a given straight line, all of them will be assumed to be in the same side of the given straight line [Ax. 6, 21].

10. On parallelism and convergence

10.1 Introduction

This chapter includes 21 theorems about parallel lines and non-parallel or convergent straight lines. First it is proved the existence of equidistant and non-equidistant points with respect to a given line, as well as the existence of non-equidistant lines. Then, the existence of parallel straight lines is demonstrated, which allows to prove other well-known results of Euclidean geometry, among them Playfair's axiom on the uniqueness of parallels, or Euclid's Postulate 5 with which the chapter ends, and which has been the final aim of this brief introduction to Euclidean plane geometry built on the foundational base established in Chapter 7. Although not included here, all the propositions of Book I of Euclid's Elements (and many others) can also be proved on the above-mentioned foundational basis. Some of them will be proved in the last two chapters of this book.

Theorem 29 *Draw three points on the same side of a given straight line, two of them equidistant and two of them non-equidistant from the given line. Draw a straight line non-parallel to the given straight line and that does not intersect the given straight line.*

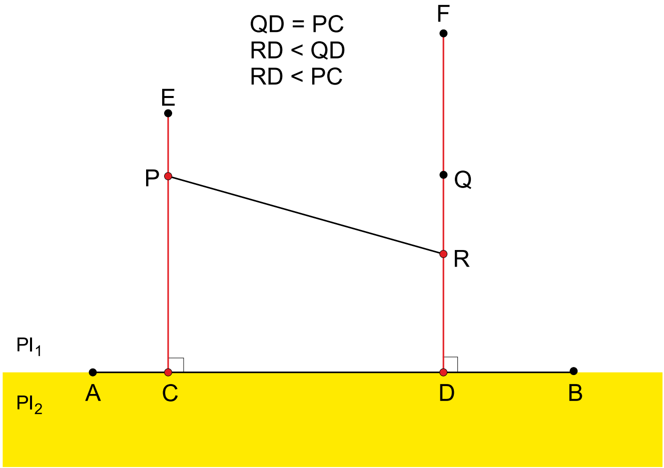


Figure 10.1 – **Theorem 29**

Through two points C and D of a given straight line AB [Cr. 1] draw the perpendiculars CE and DF to AB [Th. 20].

All points of CE and DF are on the same side Pl_1 of AB [Cr, 28].

Take any point P in CE [Cr. 1];

in DF take a point Q such that $DQ = CP$ [Th. 1];

and in DQ take any point R [Cr. 1].

It holds $DR < DQ$ [Cr. 13].

Join P and R [Cr. 15].

P and Q are equidistant from AB [Th. 28],

P and R are non-equidistant from AB [Th. 28],

P, Q and R are in the same side of AB; and PR is not parallel to AB [Df. 18, Th. 28],

and it is in the same side of AB [Cr, 28].

Therefore, PR does not cut the straight line AB [Cr. 30]. □

Note.-From now on, all of points equidistant from a straight line, whether or not in another straight line, will be assumed to be in the same side of the straight line and at a distance from the straight line greater than zero.

Theorem 30 (Khayyam-Cataldi's Axiom extended) *All segments of a given straight line in the same side of a second straight line have the same distancing direction with respect to the second straight line as the given straight line. And if the endpoints of the given straight line are equidistant from the second straight line then the given straight line is parallel to the second straight line, being all points of the given straight line non-common points of the same side of the second straight line.*

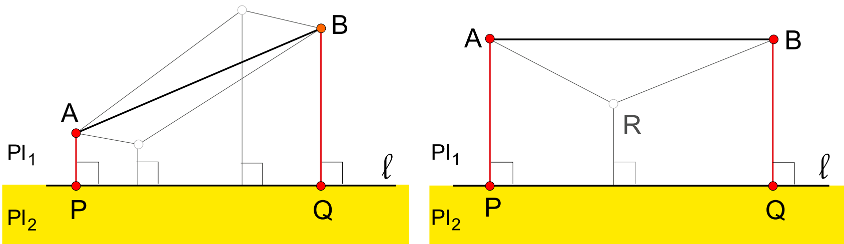


Figure 10.2 – Theorem 30

(Fig. 10.2, left). Let l be a straight line in a plane Pl [Cr. 25], and A and B any two non-common points in the same side, for example* Pl_1, of l in Pl [Cr. 28],

so that A and B are non-equidistant from l [Th. 29].

Draw the perpendiculars AP and BQ to l respectively from A and B [Th. 21],

and assume* $AP < BQ$. Join A and B [Cr. 15].

The points A and B define a distancing direction, from A to B, of the straight line AB [Dfs. 1, 17]

with respect to the straight line l [Df. 16].

All segments of AB must have the same distancing direction with respect to l as AB, otherwise there would be at least one segment whose distancing direction with respect to l would be opposite to that of AB [Dfs. 1, 17].

And then, either the endpoints of that segment are given before drawing AB, which is not the case [Ax. 5, Cr. 15],

or they are unknown before drawing AB, in which case they could only be a consequence of the operation, as such an operation, of

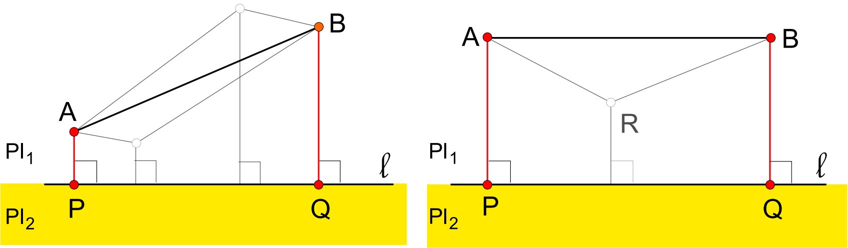


Figure 10.2 – Theorem 30

drawing AB, which is impossible [Df. D, Ax.1, Cr. 15],

or the straight line AB cannot be drawn, which is also impossible [Ax. 5, Cr. 15].

Assume now (Fig. 10.2, right) that A and B are equidistant from l [Th. 29].

Join A with B [Cr. 15].

Let R be any point between A and B [Crs. 5, 4],

and assume its distance to l [Th. 28]

is different from the equidistance of A and B. The segments AR and RB [Cr. 5]

would have different distancing directions with respect to l [Df. 17, Th. 28],

So, either the point R and the distancing directions of AR and of RB with respect to l are given before drawing AB, which is not the case [Ax. 5, Cr. 15],

or they are unknown before drawing AB, in which case they could only be a consequence of the operation, as such an operation, of drawing AB, which is impossible [Df. D, Ax.1, Cr. 15],

or the straight line AB cannot be drawn, which is also impossible [Ax. 5, Cr. 15].

Therefore, R can only be at the same distance from CD as A and B. Consequently, AB is parallel to l [Df. 18].

And being A and B non common points in the same side of l, all points of AB are non common points of the same side of l [Ax. 6, Df. 14]. □

Note.-Though a straight line could be considered parallel to itself by a zero equidistance, hereafter only parallel straight lines whose equidistance is greater than zero will be considered.

Theorem 31 (A variant of Tacquet's Axiom 11) *If a straight line is parallel to another straight line, then the perpendicular from any point of any of the two straight lines to the other straight line is also perpendicular to the first straight line.*

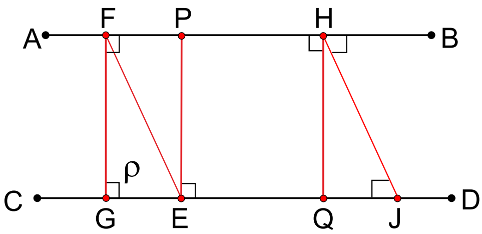


Figure 10.3 – Theorem 31

Let AB be a straight line parallel to another straight line CD [Th. 30].

All points of AB are at the same distance greater than zero from CD [Df. 18].

From a point P of AB draw the perpendicular PE to CD [Th. 21].

Draw the perpendicular from E to AB [Th. 21]

and assume it is not EP but EF. From F draw the perpendicular FG to CD [Th. 21].

It will be different from FE, otherwise there would be two perpendiculars to CD from the same point E, namely PE and FE, which is impossible [Th. 27].

Consider the triangle FGE [Ths. 30, 6].

The right angle ρ [Df. 25]

is the greatest angle of FGE [Th. 26].

So, EF is greater than FG [Th. 17],

and FG is equal to PE because AB is parallel to CD [Df. 18].

In consequence, the shortest distance from E to AB would not be the length of the perpendicular EF, but that of EP [Ps. B],

which is impossible [Th. 28].

So, EP is also perpendicular to AB. Let now Q be any point in CD. Draw the perpendicular QH to AB [Th. 21].

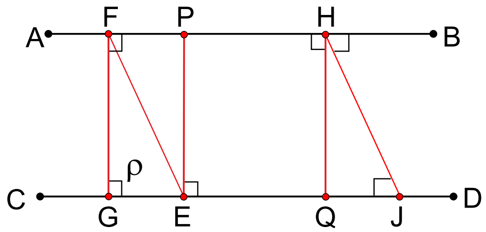


Figure 10.3 – **Theorem 31**

Assume the perpendicular from H to CD is not HQ but HJ. It has just been proved that HJ is also perpendicular to AB. So, there would be two different perpendiculars, HJ and HQ, to AB from the same point H, which is impossible [Th. 27].

Hence, the perpendicular QH is also perpendicular to CD. □

Theorem 32 *A straight line parallel to another given straight line can only be produced as a straight line parallel to the given straight line.*

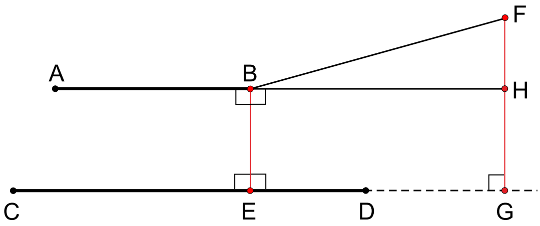


Figure 10.4 – Theorem 32

Let AB be a straight line parallel to another straight line CD [Th. 30],

Draw the perpendicular BE from B to CD [Th. 21].

BE is also perpendicular to AB [Th. 31].

Produce AB, for instance* from B by any given distance to the point F [Cr. 16].

Draw the perpendicular FG from F to CD [Th. 21].

Assume BF is not parallel to CD. We will have $BE \lessgtr FG$ [Th. 30].

Assume* $BE < FG$ [Ps. A].

Take a point H in FG such that $HG = BE$ [Th. 1].

Join B and H [Cr. 15].

BH is parallel to CD [Th. 30]

and EB is perpendicular to BH [Th. 31].

So, AB and BH make at their common endpoint B a straight angle [Cr. 51]

and then AH is straight line [Cr. 42].

In consequence, AH and AF would be two straight lines with a common segment AB, and they would be locally collinear, which is impossible [Df. 11].

Therefore, a straight line parallel to another given straight line can only be produced as a straight line parallel to the given straight line. □

Theorem 33 (Posidonius-Geminus' Axiom) *If two points of a given straight line are equidistant from a second straight line, then the given straight line is parallel to the second straight line.*

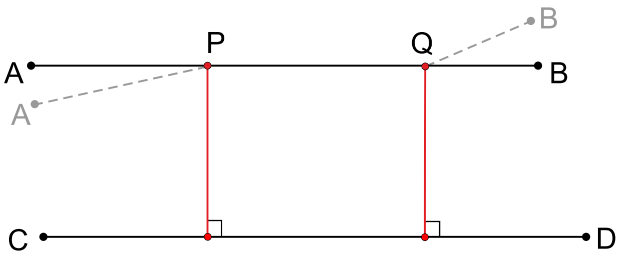


Figure 10.5 – Theorem 33

Let AB and CD be two straight lines such that two points P and Q of AB are equidistant from CD [Th. 30].

The segment PQ, which is the only straight line joining P and Q [Cr. 15],

is parallel to CD [Th. 30].

If PA were not parallel to CD, the straight line PQ [Cr. 14]

could be produced from P by a length PA as a straight line PA [Cr. 16]

non-parallel to CD, which is impossible [Th. 32].

The same applies to QB. AB is then parallel to CD. □

Theorem 34 *If a straight line is parallel to another straight line, this second straight line is also parallel, and by the same equidistance, to the first straight line.*

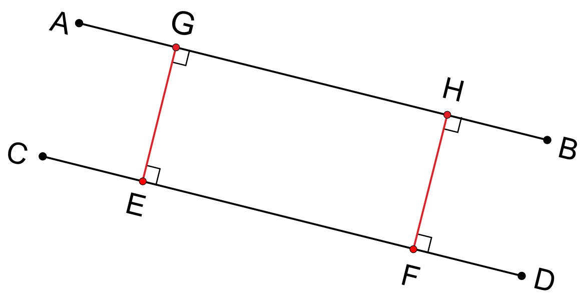


Figure 10.6 – Theorem 34

Let AB be a straight line parallel to another straight line CD [Cr. 33].

Let E and F be any two points of CD [Cr. 1].

From E and from F draw the respective perpendiculars EG and FH to AB [Th. 21].

These perpendiculars are also perpendicular to CD [Th. 31].

So, the distance from E to AB is the same as the distance from G to CD [Th. 28];

and the distance from F to AB is the same as the distance from H to CD [Th. 28].

Since AB is parallel to CD, the distances to CD from G and H are equal to each other [Df. 18].

Hence, the distances to AB from E and F are also equal to each other [Ps. B].

E and F are, then, two points in CD at the same distance from AB. Therefore, CD is parallel to AB [Cr. 33],

and by the same equidistance GE. □

Theorem 35 *To draw a straight line parallel to a given straight line through a given point not in straight line with the given straight line.*

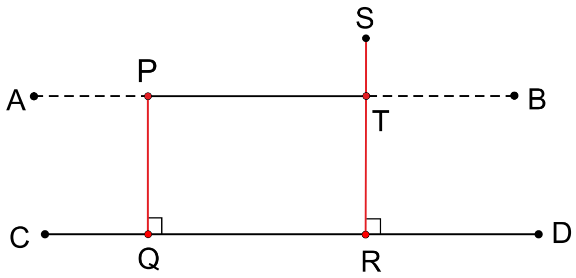


Figure 10.7 – **Theorem 35**

Let CD be a straight line [Cr. 25],

and P a point not in straight line with CD [Cr. 22].

From P draw the perpendicular PQ to CD [Th. 21].

Take a point R in CD different from Q [Cr. 1].

From R draw the perpendicular RS to CD [Th. 20].

And in RS take a point T such that $RT = QP$ [Th. 1].

Join P and T [Cr. 15]

and produce PT respectively from P and from T to any two points A and B [Cr. 16].

The straight line AB has two points, P and T, equidistant from CD. Therefore, AB is a parallel to CD [Cr. 33] through the point P. $\square$

Theorem 36 (Playfair's Axiom 11) *Through a given point not in straight line with a given straight line, one, and only one, parallel to the given straight line can be drawn.*

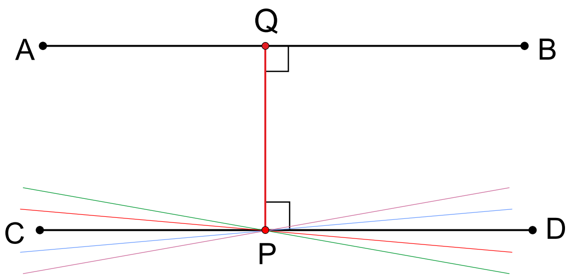


Figure 10.8 – Theorem 36

Let AB be a straight line [Cr. 25]

and P a point not in straight line with AB [Cr. 22].

Through P a parallel CD to AB can be drawn [Th. 35].

Assume that through P more than one parallel to AB can be drawn. From P draw the perpendicular PQ to AB [Th. 21].

PQ is also perpendicular from P to each of the assumed parallels to AB [Th. 31].

And each of these assumed parallels would be a different perpendicular to PQ through the same point P [Cr. 52],

which is impossible [Th. 27].

Therefore, through a given point not in straight line with a given straight line, one [Th. 35],

and only one, parallel to a given straight line can be drawn. □

Theorem 37 *For any given straight line and through different points, a number of parallels to the given straight line greater than any given number can be drawn.*

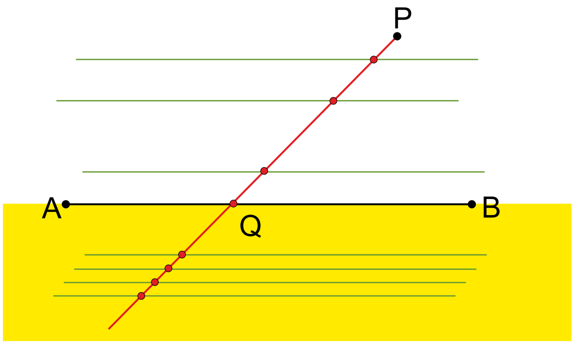


Figure 10.9 – **Theorem 37**

Let AB be a straight line [Cr. 25]

and P a point not in a straight line with AB [Cr. 22].

Join P with any point Q of AB [Cr. 15].

PQ has a number of points greater than any given number n [Cr. 1],

none of which, except Q, is in straight line with AB, even arbitrarily producing AB and PQ [Df. 12, Cr. 21].

Through each of those n points of PQ one, and only one, parallel to AB can be drawn [Ths. 35, 36].

Therefore, it is possible to draw a number greater than any given number of parallels to a given straight line, each one through a different point. □

Theorem 38 *If two straight lines have a common perpendicular, then they are parallel to each other.*

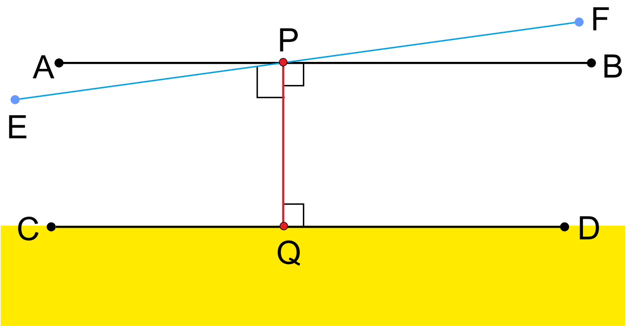


Figure 10.10 – **Theorem 38**

Let AB be a straight line in the same side of another straight line CD [Cr. 29].

From a point P of AB draw the perpendicular PQ to CD [Th. 21].

If PQ is also perpendicular to AB, then AB must also be parallel to CD, otherwise through P a parallel EF to CD could be drawn [Th. 35],

PQ would be perpendicular to EF [Th. 31],

and EF would be perpendicular to PQ [Cr. 52],

and there would be two perpendicular to PQ, namely AB and EF, through the same point P, which is impossible [Th. 27]. □

Theorem 39 *Two parallel straight lines cannot intersect.*

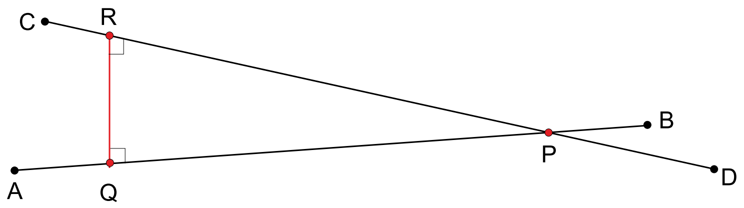


Figure 10.11 – **Theorem 39**

Assume two parallel straight lines AB and CD [Cr. 37]

intersect at a point P. From a point Q of AB different from P [Cr. 1]

draw the perpendicular QR to CD [Th. 21].

QR is also perpendicular to AB [Th. 31].

And PQ and PR would be two perpendiculars to QR [Cr. 52]

through the same point P, which is impossible [Th. 27]. □

Note. The fact that two parallel straight lines cannot intersect with each other, does not imply that non parallel straight lines have to intersect, as Posidonius defended. His pupil Geminus of Rhodes discovered the flaw [9, p. 40, 190], [29, pp. 58-59].

Theorem 40 *If a common transversal cuts two straight lines and makes with them equal the angles of a couple of alternate angles, or of corresponding angles, then the two angles of each couple of alternate angles, and of corresponding angles, are also equal. And the interior angles of the same side of the transversal sum two right angles. If the interior angles of the same side of the transversal sum two right angles, then the two angles of each couple of alternate angles, and of corresponding angles, are equal to each other.*

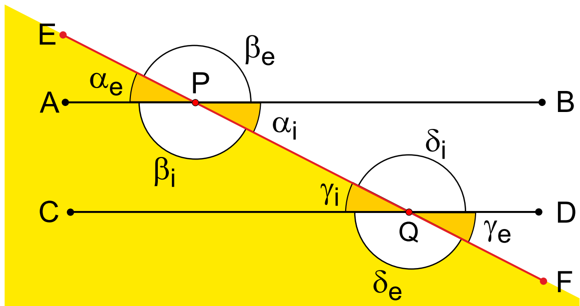


Figure 10.12 – Theorem 40

Let AB and CD be two straight lines that are intersected by a common transversal EF [Df. 26, Cr. 32]

at P and Q respectively. On the one hand we have: $\alpha_i = \alpha_e$; $\beta_i = \beta_e$; $\gamma_i = \gamma_e$; $\delta_i = \delta_e$ [Th. 5].

On the other, and being ρ a right angle: $\rho+\rho = \alpha_e+\beta_e = \alpha_i+\beta_i = \gamma_i + \delta_i = \gamma_e + \delta_e = \alpha_e + \beta_i = \beta_e + \alpha_i = \gamma_i + \delta_e = \delta_i + \gamma_e$ [Th. 4].

So, if $\alpha_i = \gamma_i$, and being $\alpha_i = \alpha_e$ and $\gamma_i = \gamma_e$, we immediately get $\alpha_e = \gamma_e$; $\alpha_i = \gamma_e$; $\alpha_e = \gamma_i$ [Ps. A].

A similar argument proves that the two angles of any other couple of alternate angles, or of corresponding angles [Df. 27],

are equal to each other. In addition, from $\alpha_i + \beta_i = \rho + \rho$ [Th. 4],

$\alpha_i = \gamma_i$ and $\beta_i = \delta_i$, it follows $\gamma_i + \beta_i = \rho + \rho$; $\alpha_i + \delta_i = \rho + \rho$ [Ps. A].

On the other hand, if $\alpha_i + \delta_i = \rho + \rho$, and being $\rho + \rho = \gamma_i + \delta_i$ [Th. 4],

we immediately get $\alpha_i + \delta_i = \gamma_i + \delta_i$ [Ps. B].

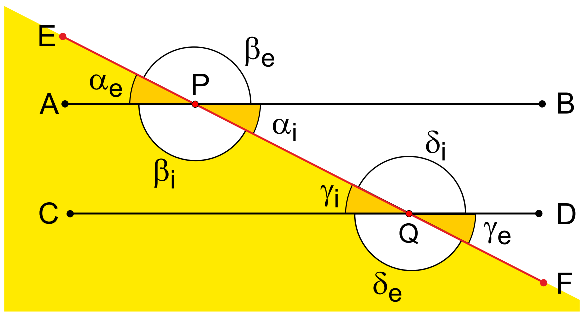


Figure 10.12 – **Theorem 40**

Therefore, $\alpha_i = \gamma_i$ [Ps. B],

and the same argument above proves that the two angles of each couple of alternate angles, and of corresponding angles, are equal. □

Theorem 41 *A common transversal makes with two parallel straight lines equal the two angles of each couple of alternate angles and of corresponding angles.*

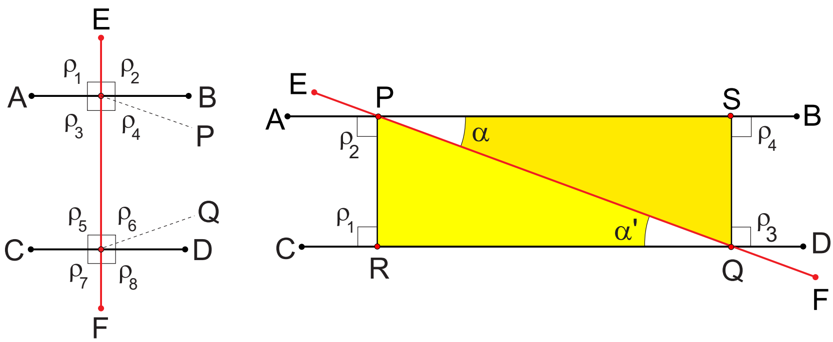


Figure 10.13 – Theorem 41

Let AB and CD be any two parallel straight lines [Cr. 37]. And EF any common transversal [Df. 26, Cr. 32]

that cuts them at P and Q respectively [Cr. 32].

(Fig. 10.13, left). If EF is perpendicular to AB, it is also perpendicular to CD [Th. 31],

and the eight angles it makes with AB and CD at its corresponding intersection points are right angles [Cr. 52],

in which case the two angles of each couple of alternate and of each couple of corresponding angles are equal [Th. 22, Df. 27].

(Fig. 10.13, right). If EF is not perpendicular to AB, draw the perpendicular PR to CD [Th. 21],

which is also perpendicular to AB [Th. 31].

And AB and CD are perpendicular to PR [Cr. 52].

Therefore ρ_1 and ρ_2 are right angles [Df. 25].

Take a point S in PB such that $PS = RQ$ [Th. 1].

Join S and Q [Cr. 15].

SQ and PR are parallel [Cr. 33, Th. 34].

Hence, AB and CD are perpendicular to SQ [Th. 31],

and SQ is perpendicular to AB and to CD [Cr. 52].

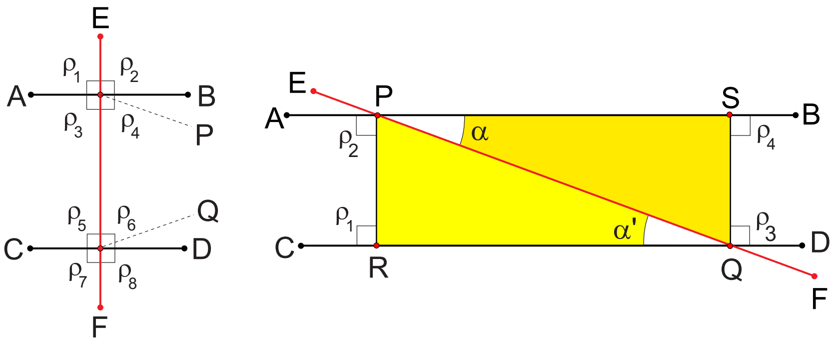


Figure 10.13 – **Theorem 41**

Therefore, ρ_3 and ρ_4 are right angles [Df. 25, Cr. 52].

And being AB parallel to CD, it holds $PR = SQ$ [Df. 18].

Consider the triangles PRQ and PQS [Ths. 30, 6].

The three sides of PRQ are equal to the three sides of PQS. So, $\alpha = \alpha'$ [Th. 14].

Therefore, the two angles of any other couple of alternate angles, and of corresponding angles [Df. 27],

are also equal [Th. 40]. □

Theorem 42 *If a common transversal makes with two straight lines equal the angles of a couple of alternate angles, then both straight lines are parallel to each other.*

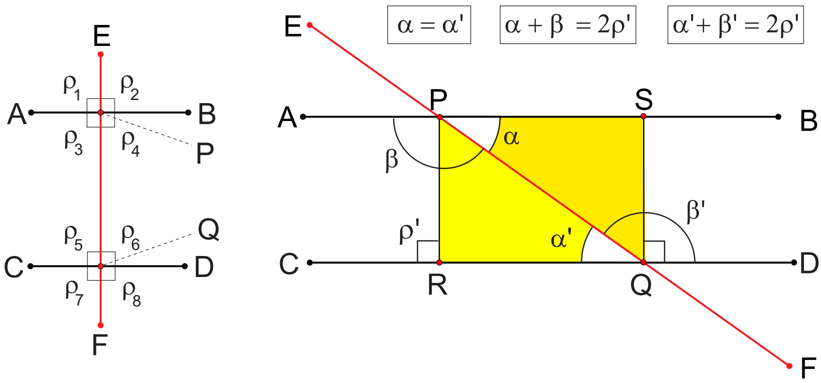


Figure 10.14 – Theorem 42

Let EF be a common transversal [Df. 26, Cr. 32]

that intersects two straight lines AB and CD respectively at P and Q, where EF makes with AB and CD equal the two angles of a couple of alternate angles α and α' [Df. 27].

The angles α and β' sum two right angles; and α' and β also sum two right angles [Th. 40].

Therefore, AB and CD cannot intersect each other, otherwise there would be a triangle with two angles that sum to two right angles, which is impossible [Th. 26].

Therefore, AB and CD are each on the same side of the other [Cr. 31].

(Fig. 10.13, left). If α and α' are right angles, EF will be perpendicular to AB and to CD [Cr. 52],

and AB and CD will be parallel [Th. 38].

(Fig. 10.13, right). If α and α' are not right angles, EF is not perpendicular to AB or to CD [Df. 25].

In this case, draw the perpendicular PR from P to CD [Th. 21].

On PB take a point S such that $PS = RQ$ [Th. 1].

Join S and Q [Cr. 15].

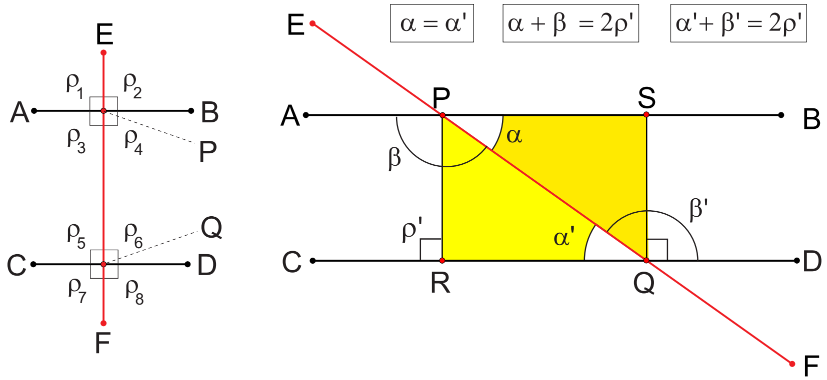


Figure 10.14 – **Theorem 42**

SQ and PR are parallel [Cr. 33, Th. 34],

and then CD is perpendicular to SQ [Th. 31],

and SQ is perpendicular to CD [Cr. 52].

and being SQ and PR parallel, SQP and PQR are triangles [Ths. 30, 6].

They have a a common side PQ, and also $PS = RQ$, and $\alpha = \alpha'$. Therefore $SQ = PR$ [Cr. 48].

Since SQ and PR are perpendicular to CD, S and P are at the same distance from CD [Th. 28].

So, AB and CD are parallel to each other [Cr. 33, Th. 34]. □

Theorem 43 *Two straight lines are parallel to each other if, and only if, a common transversal makes with them two interior angles in the same side of the transversal that sum two right angles.*

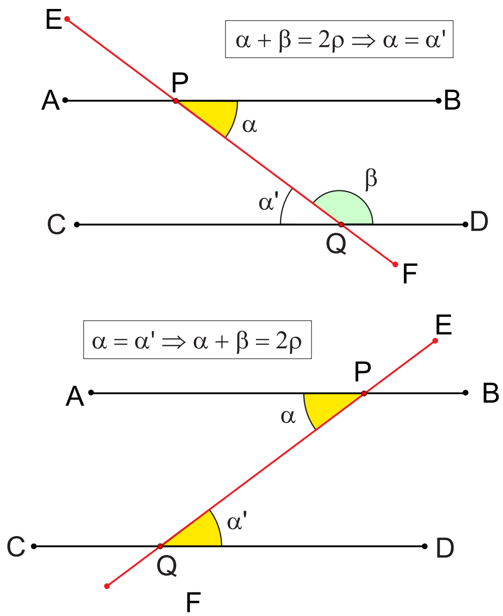


Figure 10.15 – **Theorem 43**

(Fig. 10.15, top) If a common transversal EF makes with two straight lines AB and CD [Df. 26, Cr. 32]

two interior angles α and β [Df. 26]

on the same side of the transversal [Df. 22]

that sum two right angles, then the two angles of any couple of alternate angles α and α' are equal to each other [Th. 40]

and both straight lines are parallel [Th. 42].

(Fig. 10.15, bottom) If a transversal cuts two parallel straight lines [Df. 26, Cr. 32],

it makes with them equal the two angles of each couple alternate angles, for instance α and α' [Th. 41]

and then the two interior angles of the same side of the transversal [Dfs. 22, 26]

sum two right angles [Th. 40]. □

Theorem 44 (Proclus' Axiom) *If a first straight line is parallel to a second straight line and the second straight line is parallel to a third straight line, then the first straight line is also parallel to the third straight line.*

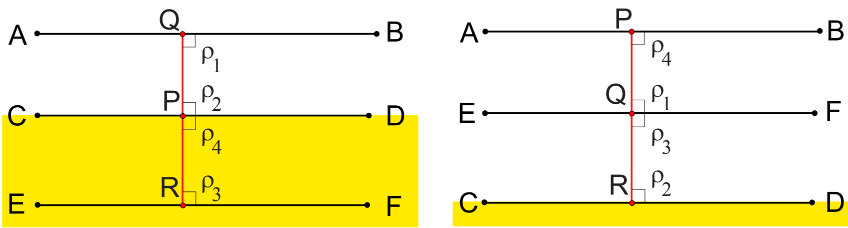


Figure 10.16 – **Theorem 44**

(Fig. 10.16, left) Let AB be a straight line parallel to another straight line CD [Cr. 37],

which is parallel to another straight line EF [Cr. 37].

Assume first that AB and EF are in different sides of CD [Ax. 6] (Fig. 10.16, left).

From any point P of CD draw the perpendicular PQ to AB and the perpendicular PR to EF [Th. 21].

PQ and PR are also perpendicular to CD [Th. 31].

So, ρ_1, ρ_2, ρ_3 and ρ_4 are right angles [Df. 25].

PQ and PR cannot be two different perpendiculars to CD from P [Th. 27].

So, QR is a unique straight line, which is a common transversal of AB and EF, and makes with them in the same side of QR two interior angles ρ_1 and ρ_3 [Df. 26]

that sum two right angles. Therefore AB is parallel to EF [Th. 43].

(Fig. 10.16, right) If AB and EF are in the same side of CD [Cr. 29],

then draw the perpendicular PQ from any point P of AB to EF, and from Q the perpendicular QR to CD [Th. 21].

So, ρ_1 and ρ_2 are right angles [Df. 25].

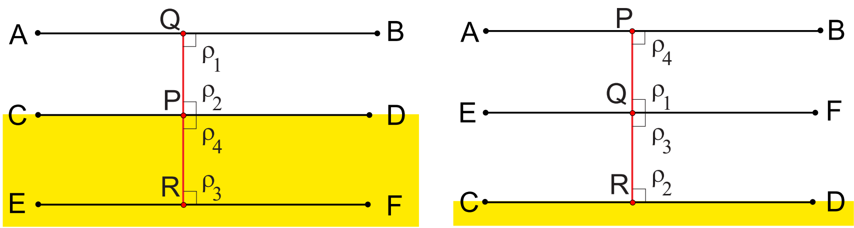


Figure 10.16 – Theorem 44

Since EF is parallel to CD, QR is also perpendicular to EF [Th. 31],

and ρ_3 is a right angle [Df. 25].

And, for the same reasons above, PR is a unique straight line, which is perpendicular to EF through Q. And being perpendicular to CD, PR is also perpendicular to AB [Th. 31],

and then ρ_4 is a right angle [Df. 25].

In consequence, PQ is a transversal of AB and EF that make two interior angles, ρ_1 and ρ_4 [Df. 26],

on the same side of PQ that sum two right angles. So, AB is also parallel to EF [Th. 43]. $\square$

Theorem 45 *If a common transversal makes with two straight lines two interior angles in the same side of the transversal that sum less (more) than two right angles, the interior angles in the other side of the transversal sum more (less) than two right angles.*

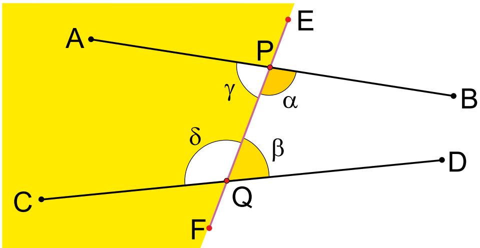


Figure 10.17 – **Theorem 45**

Let EF be a common transversal of two straight lines AB and CD [Df. 26, Cr. 32]

that cuts them respectively at P and Q and makes with them the interior angles α and β [Df. 26]

on the same side of EF [Ax. 6, Df. 22]

so that $\alpha + \beta < \rho + \rho$, where ρ is a right angle [Th. 22].

AB and CD are not parallel [Th. 43].

Let γ and δ be the interior angles that AB and CD make with EF on the other side of EF [Dfs. 22, 26].

On the one hand we have: $\alpha + \gamma = \beta + \delta = \rho + \rho$ [Th. 4],

so that $\alpha + \gamma + \beta + \delta = \rho + \rho + \rho + \rho$ [Pss. B, A].

On the other hand $\gamma + \delta \lesseqgtr \rho + \rho$, otherwise AB and CD would be parallel [Th. 43].

But if $\gamma + \delta < \rho + \rho$, we would have $\gamma + \delta + \rho + \rho < \rho + \rho + \rho + \rho$ [Ps. B];

and being $\alpha + \beta < \rho + \rho$, we also have $\alpha + \beta + \gamma + \delta < \rho + \rho + \gamma + \delta$ [Ps. B].

Therefore $\alpha + \beta + \gamma + \delta < \rho + \rho + \rho + \rho$ [Ps. B], which is not the case.

So, it must be $\gamma + \delta > \rho + \rho$. A similar argument applies to the case $\alpha + \beta > \rho + \rho$. $\square$

Theorem 46 *All segments with the same length of a given straight line have the same distancing direction and the same relative distancing with respect to any other non-parallel straight line in the same side of the given straight line.*

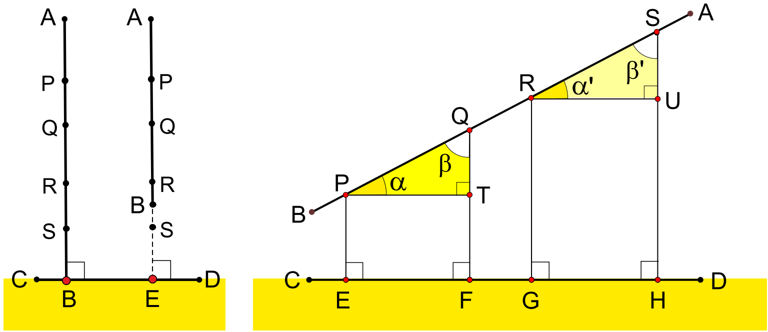


Figure 10.18 – **Theorem 46**

Let AB be a straight line in the same side of another straight line CD [Cr. 29]

to which it is not parallel [Th. 29].

All segments of AB have the same distancing direction, for instance* from B to A with respect to CD [Th. 30].

Let P, Q and R be any three points of AB [Cr. 1].

Assuming* Q is between P and R [Cr. 9],

take in AB a point S at a distance PQ from R in the direction from A to B so that $PQ = RS$ [Th. 1],

If AB were perpendicular (Fig. 10.18, left), or a segment of the perpendicular AE, to CD [Th. 21],

the relative distancing of any segment of AB [Df. 17]

with respect to CD would be the length of the segment [Cr. 13, Th. 28].

So, PQ and RS would have the same relative distancing with respect to CD [Dfs. 17, 9, C].

Assume AB is not perpendicular to CD (Fig. 10.18, right). From P, Q, R and S draw the perpendiculars PE, QF, RG and SH to CD [Th. 21].

And from P and R draw the perpendiculars PT to QF, and RU

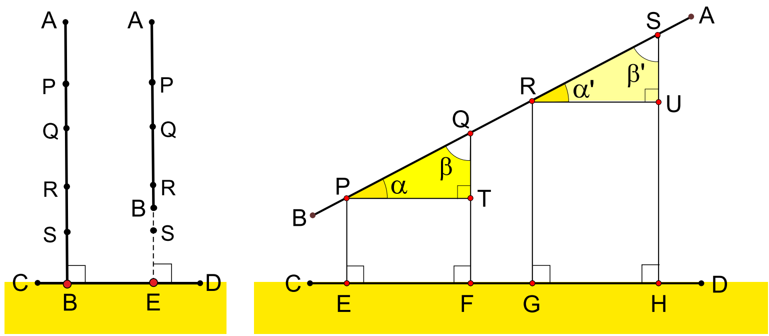


Figure 10.18 – Theorem 46

to SH [Th. 21].

PT is parallel to CD, and PE to QF [Th. 38].

Since right angles are not straight angles and are greater than zero [Cr. 51, Th. 22],

P is not in straight line with QT [Cr. 43].

So, QPT is a triangle [Th. 6].

For the same reason SRU is also a triangle. PT and RU are parallel to CD [Th. 43],

and then they are parallel to each other [Th. 44].

Therefore $\alpha = \alpha'$ [Th. 41].

QF and SH are parallel to each other [Th. 38],

and then $\beta = \beta'$ [Th. 41].

The triangles QPT and SRU verify: $\alpha = \alpha'$; $\beta = \beta'$ [Th. 41],

and $PQ = RS$. Consequently, $QT = SU$ [Th. 11].

Being $PE = TF$ [Df. 18]

and $QT = QF - TF$ [Cr. 13],

it will be $QT = QF - PE$ [Ps. A].

QT is, then, the relative distancing of the segment PQ with respect to CD [Df. 17].

For the same reasons SU is the relative distancing of the segment RS with respect to CD [Df. 17].

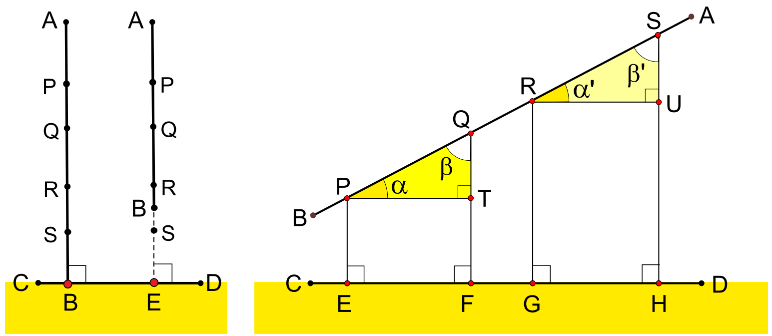


Figure 10.18 – Theorem 46

Since $QT = SU$, and PQ and RS are any two segments of AB with the same length, we conclude that all segments of AB with the same length have the same relative distancing with respect to CD [Df. 17].

in the same distancing direction [Th. 30]. □

Theorem 47 *If a straight line cuts a second straight line, then it can be produced from either endpoint to a new endpoint whose distance to the second straight line is greater than any given distance.*

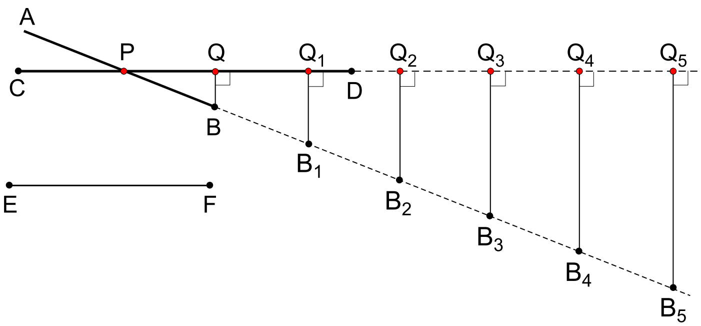


Figure 10.19 – Theorem 47

Let AB be a straight line that cuts another straight line CD at a point P [Cr. 27].

And let EF be any distance, which is the length of the straight line EF [Df. 15].

AB is not parallel to CD [Th. 39].

From the endpoint B of AB draw the perpendicular BQ to CD [Th. 21].

BQ is the relative distancing of the segment PB with respect to the straight line CD [Th. 28, Df. 17].

If $BQ \leq EF$, there will exist a number n such that n times BQ is greater than EF, otherwise there would exist a last natural number, which is impossible according to Peano's Axiom of the Successor [19, p. 1].

In the direction from P to B [Df. 1],

produce n times (five in Figure 10.19) the straight line AB in the same direction and by the same length PB up to the successive extremes B_1, $B_2 \ldots$, B_n [Cr. 16].

The successive distances to the straight line CD from the successive endpoints B_1, $B_2 \ldots$, B_n are always increased by the same distance BQ in each production [Th. 46].

Therefore, the distance B_nQ_n from B_n to CD is equal to n times

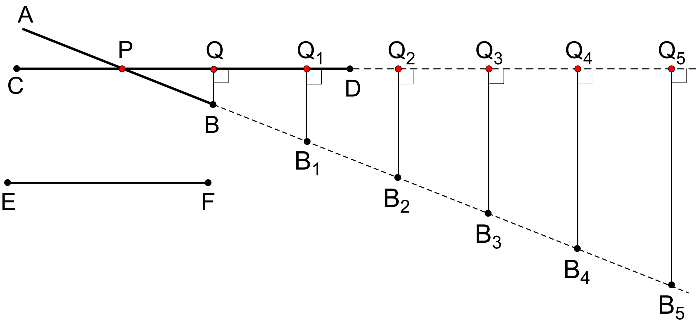


Figure 10.20 – **Theorem 47**

the relative distancing BQ [Th. 46].

And being n times BQ greater than EF, the distance B_nQ_n is greater than the given distance EF. The same argument applies to the other endpoint A of AB. $\square$

Theorem 48 (Khayyam's Axiom) *Two non intersecting straight lines are either parallel, or they can be produced to a unique intersection point.*

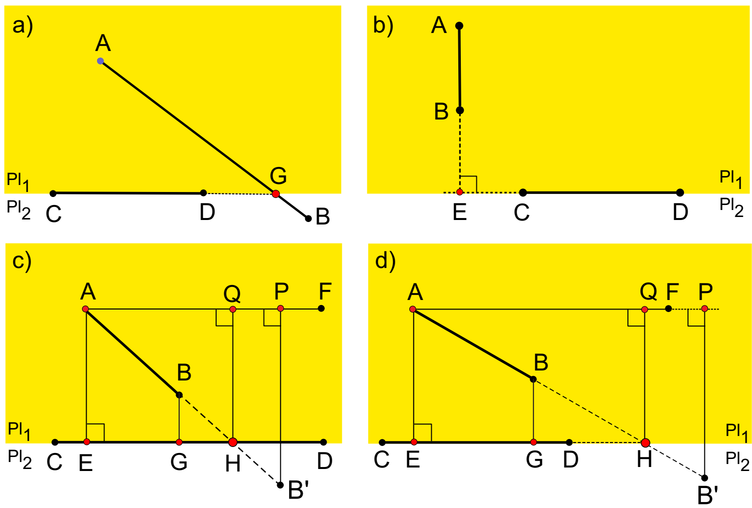


Figure 10.21 – Theorem 48

Let AB and CD be any two non-intersecting straight lines [Cr. 32]

If one of them has its two endpoints on different sides of the other, then it is cut by a finite production of the other at a unique point [Crs. 31, 45] (Fig. 10.21, a).

If not, AB will be in the same side, for instance* Pl_1, of CD [Cr. 30].

In this case, draw the perpendicular AE from A to CD [Th. 21] (Fig. 10.21, b, c, d).

If AB is a segment of AE, then the finite production BE of AB cuts CD at a unique point E [Crs. 16, 45, 21] (Fig. 10.21, b).

If not, draw the parallel AF to CD and the perpendicular BG to CD [Ths. 35, 21] (Fig. 10.21, c, d).

If $AF = BG$, the straight lines AB and CD are parallel to each other [Th. 33].

If not, the length of one of them, for example* of BG, will be less than the length of other.[Ps. A],

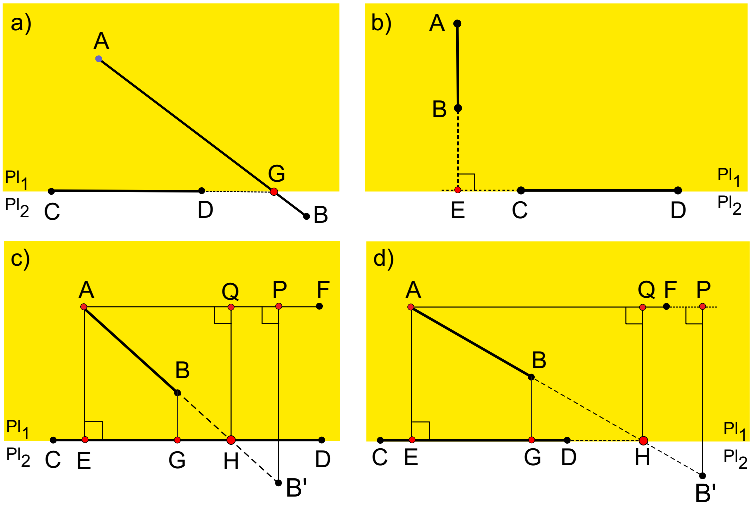


Figure 10.21 – **Theorem 48**

and the distancing direction of AB with respect to CD will be from B to A [Df. 17, Th. 30].

AB can be produced from B to a point B' such that its distance $B'P$ to AF is greater than AE [Th. 47].

Therefore, the distance to AP from the points of AB' vary in a continuous way from zero at A, to $B'P > AE$ at B' [Ax. 7].

And there will exist a point H in AB' such that its distance HQ to AP is the equidistance AE between AP and CD [Df. B, Ax. 7].

If H is in CD, the finite production BH of AB cuts CD at a unique point H [Cr. 21] (Fig. 10.21, c).

If H is not in CD, join D and H [Cr. 15] (Fig. 10.21, d).

DH is parallel to AP [Cr. 33],

and it must be a production of CD, from D to H [Cr. 16],

otherwise there would be two different parallels, CD and DH, to AF through the same point D, which is impossible [Th. 36].

Thus, AB and CD can be produced respectively from B and from D. each by a finite length [Crs. 16, 45],

to the point H, that is their unique intersection point [Cr. 21]. □

Theorem 49 (Euclid's Postulate 5) *If a common transversal makes with two given non-intersecting straight lines two angles in the same side of the transversal that sum less than two right angles, then the given straight lines can be produced in that side of the transversal by a finite length to a unique point where they intersect with each other.*

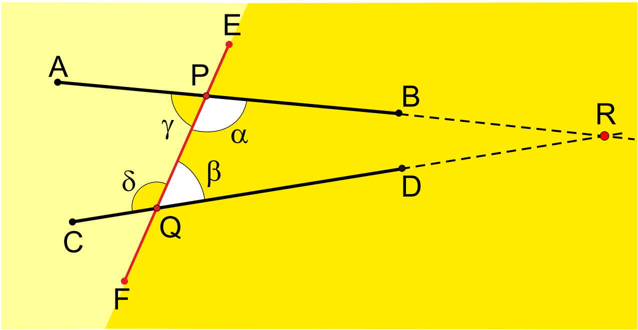


Figure 10.22 – **Theorem 49**

Let AB and CD be any two non-intersecting straight lines [Cr. 32].

Each of them will be in the same side of the other [Cr. 31].

Let EF be a common transversal of AB and CD [Cr. 32]

that makes with AB and CD at its respective and unique intersection points P and Q [Cr. 21]

two interior angles α and β on the same side of EF [Dfs. 22, 26]

whose sum is less than two right angles [Ax. 9].

AB and CD are not parallel to each other [Th. 43].

Therefore, they can be produced [Cr. 16]

by a finite length to a unique intersection point R [Th. 48].

R is not in straight line with P and Q, otherwise AB, EF and CD would belong to the same straight line [Cr. 18],

which is not the case. Therefore, PQR is a triangle [Th. 6].

The vertex R can only be a point on the side of EF where α and β are, because in the other side [Ax. 6]

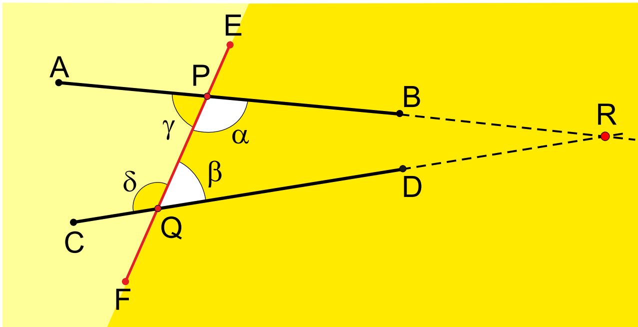


Figure 10.22 – **Theorem 49**

the interior angles sum more than two right angles [Th. 45],

and PRQ would have two angles whose sum is greater than two right angles, which is impossible [Th. 26]. □

11. Variations on Euclid's Fifth Postulate

11.1 Introduction

In this chapter most of the statements that were historically proposed as alternatives to Euclid's Fifth Postulate are demonstrated. As shown in Chapter 10, the famous postulate can be proved (Theorem 49) on the new basis of formal elements introduced in Chapter 7. The demonstrations of the alternatives to the Fifth Postulate do not always need to make use of Theorem 49. One of the most remarkable results whose proof does use Proposition 49 is Hilbert's Axiom I.3: any three points not in a straight line with each other, are contained in a unique plane. Another remarkable result is Theorem 65, whose statement Gauss considered sufficient to demonstrate all geometry: the existence of a triangle whose area is greater than any given area.

Unless otherwise indicated, historical information is taken from [9, p. 153-369]. And as in the preceding chapters, all angles will be greater than zero.

Theorem 50 *The three interior angles of a triangle sum two right angles.*

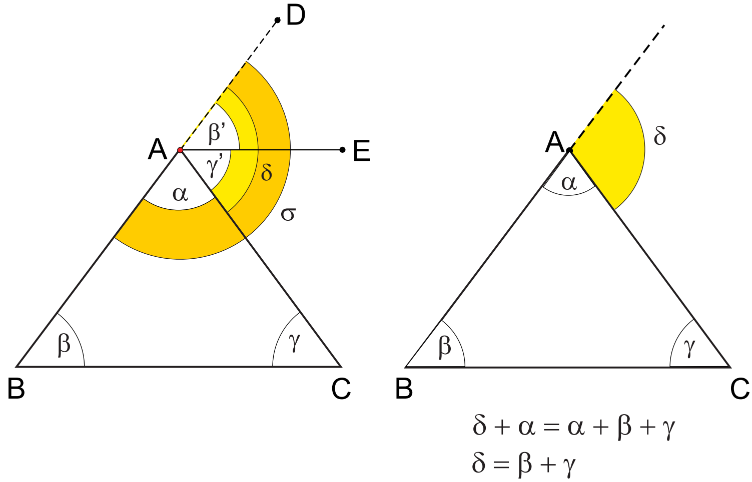


Figure 11.1 – **Theorem 50**

(Figure 11.1, left) Let ABC be any triangle [Ths. 10, 9, 8].

Produce AB from A to any point D [Cr. 16].

AB, AC and AD are adjacent at A [Dfs. 28, 29, Cr. 16].

Therefore, α and δ are adjacent angles whose union angle is the straight angle σ [Ths. 3, 4].

Through A draw the parallel AE to BC [Th. 35].

AE is adjacent at A to AC and to AD, otherwise it would be superposed on one of them [Df. 21]

and it would not be parallel to BC [Dfs. 28, 29]

which is not the case. So, β' and γ' are adjacent angles whose union angle is δ and $\delta = \beta' + \gamma'$ [Ths. 3, 4].

Therefore, $\sigma = \alpha + \delta = \alpha + \beta' + \gamma'$ [Ps. A].

The angles β' and γ' also satisfy $\beta' = \beta$; $\gamma' = \gamma$ [Th. 41].

In consequence, $\sigma = \alpha + \beta + \gamma$ [Ps. A].

Or, what is the same, the angles of a triangle sum two right angles [Cr. 51]. □

Note.-Nasiraddin at-Tutsi (1201-1274 AC) and Legendre took the

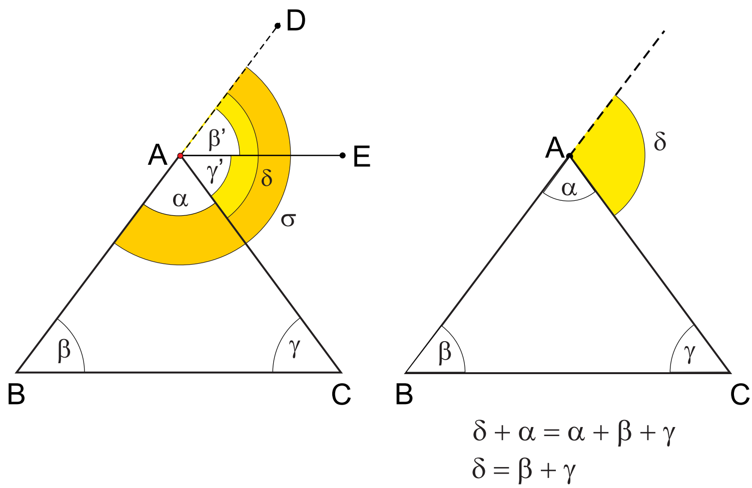


Figure 11.1 – Theorem 50

statement of this theorem as a substitute for Euclid's Postulate 5.

Corollary 54 (Worpitzky's hypothesis) *There exists no triangle in which every angle is as small as we please.*

It is an immediate consequence of [Th. 50]. □

Corollary 55 *Any exterior angle of a triangle is the sum of the two interior and opposite angles.*

(Figure 11.1, right) It is an immediate consequence of [Ths. 50, 4]. □

Theorem 51 *If a straight line cuts one of two parallels and it is coplanar with them, it also cuts the other parallel or a production of it.*

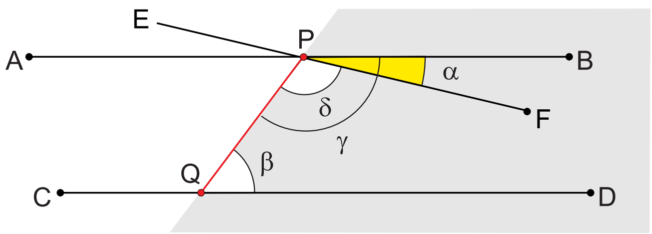


Figure 11.2 – Theorem 51

Let AB and CD be any two parallels [Th. 30],

and EF a straight line coplanar with AB that cuts AB at the unique point P. [Crs. 32, 21].

The angle α that EF makes with AB will be greater than zero, otherwise EF would be superposed on AB in a unique straight line [Df. 21, Cr. 18],

which is not the case. Take at random any point Q on CD and join it with P by a straight line [Cr. 15].

If PQ is not adjacent to PF then they would be superposed on a unique straight line [Dfs. 21, Cr. 18]

and Q would be the intersection point of EF and CD. If PQ and EF are adjacent at P, the sides PB, PF and PQ of α and δ are adjacent at P [Df. 22],

and the angles α and δ are adjacent angles [Cr. 41]

whose union angle is γ [Th. 3].

So, δ is less than γ [Th. 3].

Therefore, $\delta + \beta < \gamma + \beta$ [Ps. B].

Since AB is parallel to CD the angles β and γ sum two right angles $\rho + \rho$ [Th. 43].

And then $\delta + \beta < \rho + \rho$ [Ps. B].

Thus, the interior angles δ and β that PQ makes with CD and EF in the same side of PQ sum less than two right angles [Ps. B].

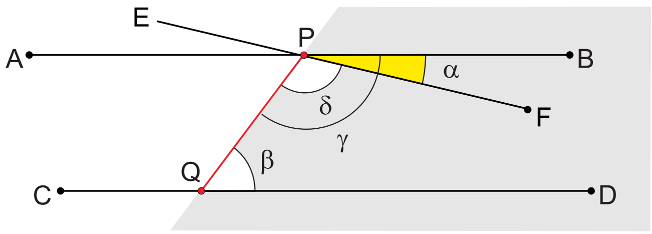


Figure 11.3 – **Theorem 51**

Therefore CD and EF intersect with each other on the same side of those interior angles [Th. 49]. □

Note.-Proclus took the statement of Theorem 51 as a substitute for Euclid's Postulate 5.

Theorem 52 *If three of the four interior angles of a quadrilateral figure are right angles then the fourth one is also a right angle.*

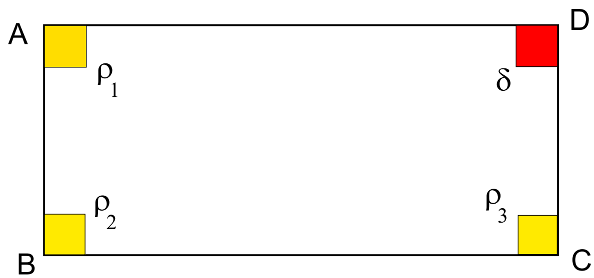


Figure 11.4 – **Theorem 52**

Let $ABCD$ be a quadrilateral such that the angles ρ_1, ρ_2 and ρ_3 are right angles.

Being ρ_1 and ρ_2 right angles, AD and BC will be parallel [Th. 43].

Therefore ρ_3 and δ sum two right angles [Th. 43].

That is to say, $\rho_3 + \delta = \rho_3 + \rho_3$ [Th. 22],

and $\delta = \rho_3$ [Ps. B].

Therefore the fourth angle δ is a right angle. $\square$

Note.-The statement of this theorem was used by Clairaut in the place of Euclid's Postulate 5.

Theorem 53 *To draw a rectangle.*

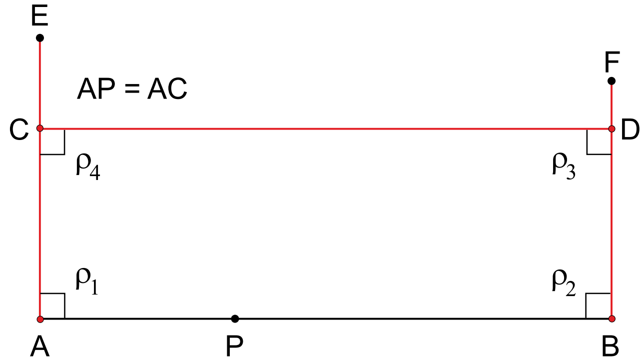


Figure 11.5 – Theorem 53

Let P be any point between the endpoints of any straight line AB of any finite length [Crs. 5, 45].

It holds $AP < AB$ [Cr. 13].

From A draw the perpendicular AE to AB [Th. 20].

From B draw the perpendicular BF to AB [Th. 20].

AE and BF are parallel [Th. 38],

and then equidistant [Df. 18].

On AE take a point C such that $AC = AP$ [Th. 1].

On BF take a point D such that $BD = AP$ [Th. 1].

it holds $AC = BD < AB$ [Ps. B].

Join C and D [Cr. 15].

Since $CA = DB$ [Ps. B],

CD is parallel to AB [Th. 33],

Being ρ_1 and ρ_2 right angles, ρ_3 and ρ_4 are also right angles [Th. 31].

The quadrilateral $CABD$ is a rectangle for it is right angled [Df. 29]

but it is not equilateral because $AC < AB$. $\square$

Note.-The existence of a rectangle was used by G. Saccheri as a substitute for Euclid's Postulate 5.

Theorem 54 *If through each endpoint of a given straight line a perpendicular of the same length is drawn, then by joining the free endpoints of each perpendicular a rectangle is drawn.*

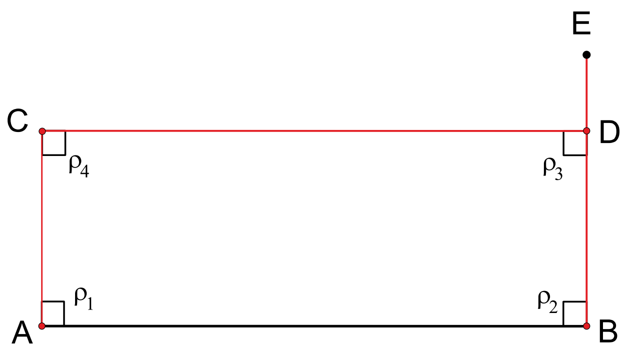


Figure 11.6 – **Theorem 54**

Let AB the given straight line [Cr. 25].

Through A draw the perpendicular AC to AB [Th. 20].

Though B draw the perpendicular BE to AB [Th. 20].

The angles ρ_1 and ρ_2 are right angles [Df. 25].

On BE, produced if necessary, take a point D such that $BD = AC$ [Th. 1].

AC and BD are parallel [Th. 38]

and then equidistant [[Df. 18].

Join C and D [Cr. 15].

Since $AC = BD$, the straight lines AB and CD are parallel [Th. 33],

and $AB = CD$ because AC and BD are parallel [Df. 18].

And being ρ_1 and ρ_2 right angles, ρ_3 and ρ_4 are also right angles [Th. 31; Df. 25].

Thus, $CABD$ is a rectangle because $AB = CD$; $AC = BD$; $AC < AB$; AB and CD are parallel; AC and BD are parallel; and ρ_1, ρ_2, ρ_3 and ρ_4 are right angles [Df. 29]. □

Note.-The statement of this theorem was proposed by Farkas Bolyai as a sort of substitute for Euclid's Fifth Postulate.

Theorem 55 *If two straight lines of the same length and on the same side of a given straight line are adjacent to the given straight line, each at a different endpoint of the given straight line, and make the same non-straight angle with the given straight line, then the straight line joining the non-common endpoints of both straight lines is parallel to the given straight line.*

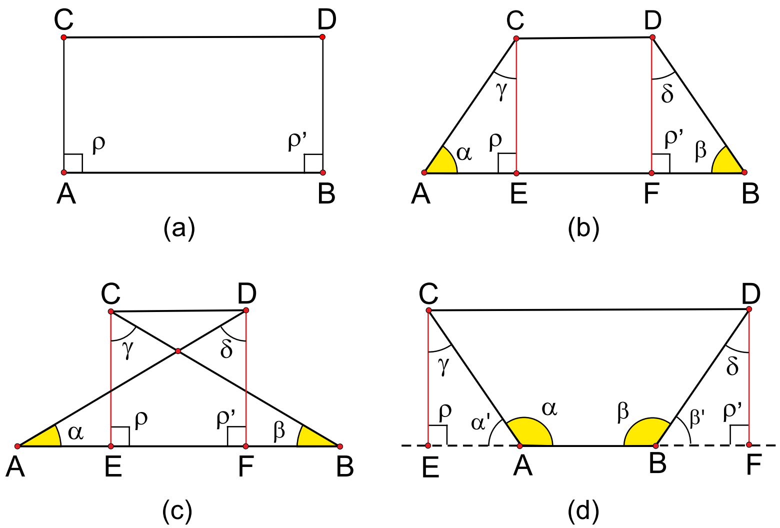


Figure 11.7 – Theorem 55

Let AB the given straight line [Cr. 25],

and CA and DB two straight lines of the same length and adjacent to AB respectively at A and B [Cr. 27],

and so that CA and DB make equal non-straight angles with AB at A and B respectively.

If both angles are right angles (Figure 11.7 a), CA and DB are the respective distances from C and D to AB [Th. 28],

Join C and D [Cr. 15],

Since $CA = DB$, CD is parallel to AB [Th. 30].

If the equal angles that CA and DB make with AB are not right angles (Figure 11.7, b, c, d),

and being equal all right angles [Th. 22],

both angles will be either less or greater than a right angle [Ps. B],

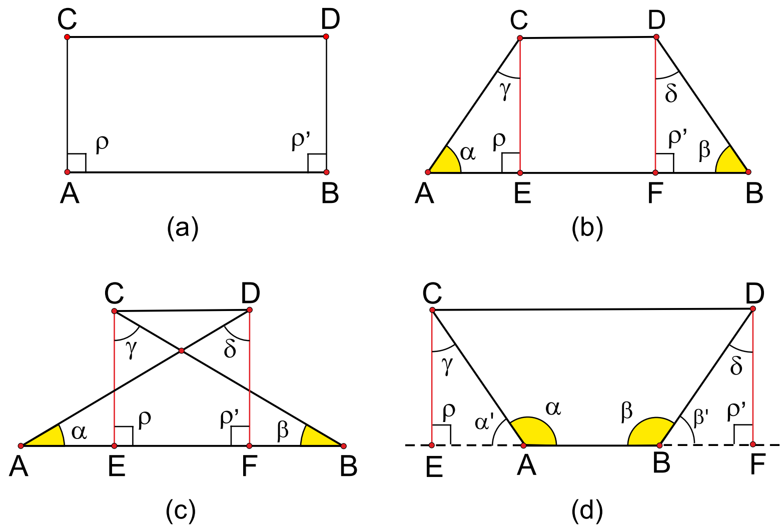


Figure 11.7 – Theorem 55

and then either acute or obtuse [Df. 25, Th. 25].

In this case, from C and D draw the respective perpendiculars CE and DF to AB [Th. 21].

Since neither α nor β are straight angles, neither C nor D are in straight line with AB [Cr. 43].

So, C, A and E define the triangle CAE; and D, B and F define another triangle DBF [Th. 6].

The following argument applies to the cases (b) and (c): since $\gamma + \alpha + \rho = \delta + \rho' + \beta$ [Th. 50]

and taking into account that $\rho = \rho'$ [Th. 22],

it holds $\gamma + \alpha = \delta + \beta$ [Ps. B].

and being $\alpha = \beta$, it holds: $\gamma = \gamma'$ [Ps. B].

Since $AC = DB$ it also holds $CE = DF$ [Pr 11].

Therefore, C and D are at the same distance from AB [Th. 28],

Join C and D [Cr. 15],

Since $CE = DF$, CD is parallel to AB [Th. 30].

For case (d), it is sufficient to take into account that $\alpha + \alpha' = \beta + \beta'$

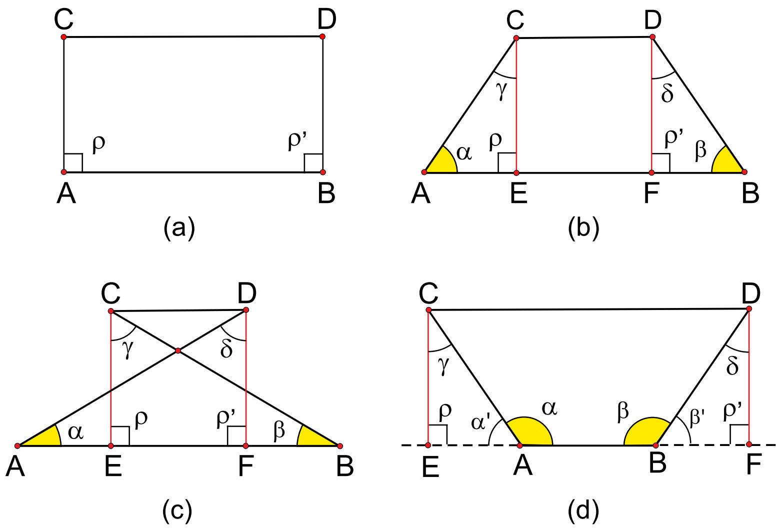


Figure 11.7 – Theorem 55

[Ths. 4, 2].

Therefore, $\alpha' = \beta'$ [Ps. B],

and the same reasoning applies as in cases (b) and (c). □ **Note**.- The statement of this theorem was proposed by Gerolamo Saccheri as a substitute for Euclid's Fifth Postulate.

Theorem 56 *If a common transversal to two straight lines is perpendicular to one of them, then either the perpendicular is common and the lines do not intersect, or they intersect, produced if necessary.*

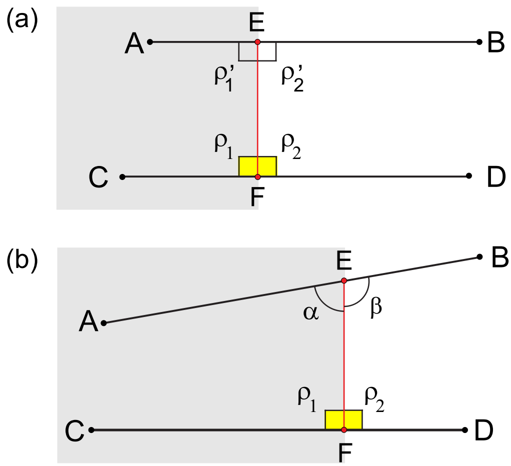


Figure 11.8 – **Theorem 56**

Let AB and CD be any two straight lines with a common transversal EF [Cr. 32]

which is perpendicular to one of them, for example* to CD. EF makes at F with CD two right angles ρ_1 and ρ_2 [Df. 25, Th. 21].

EF makes in E with AB two adjacent angles that can be equal or unequal [Th. 4].

If they are equal (Fig. 11.8, top), they are right angles [Df. 25, Th. 21],

and EF is also perpendicular to AB [Df. 25].

in which case AB and CD are parallel to each other [Th. 43]

and they don cut each other [Th. 39].

If EF makes at E two unequal angles α and β (Fig. 11.8, bottom), one of them, for instance* α, will be less than a right angle [Th. 25, Df. 25],

in which case EF is a common transversal of AB and CD that makes with them two interior angles α and ρ_1 whose sum is less

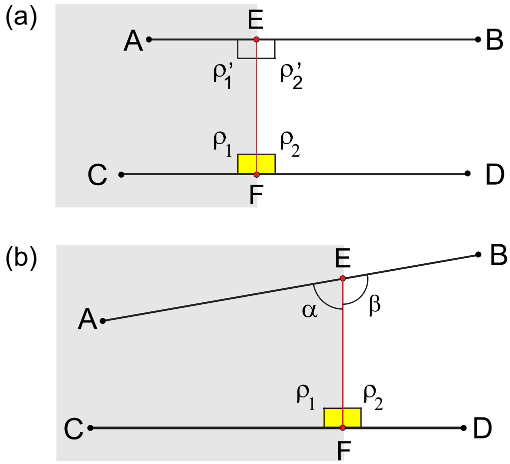


Figure 11.8 – Theorem 56

than two right angles [Df. 25].

And then, AB and CD, produced if necessary by a finite length, cut each other on the side of EF of the acute angle α [Th. 49]. □

Note.-Saccheri, Bolyai and Lobachewsky proved [Th. 56] on the hypothesis of the acute angle [Th. 55]

Theorem 57 *Through a given point not in a straight line with a given straight line, draw a straight line that makes with the given straight line, produced if necessary, an angle greater than zero and less than any given acute angle.*

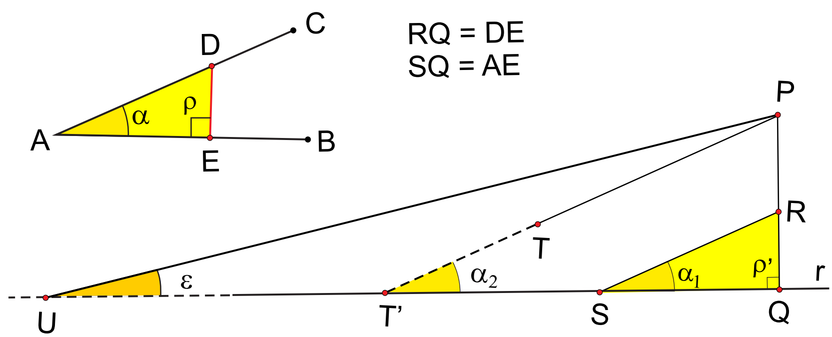


Figure 11.9 – **Theorem 57**

Let AB and AC be two straight lines adjacent at A [Cr. 27] where they make any given acute angle $\alpha > 0$ [Cr. 37].

Let r be the given straight line, and P the given point not in straight line with r [Cr. 22].

Draw the perpendicular PQ from P to r [Th. 21].

Take in AC a point D such that $AD = PQ$ [Th. 1].

Draw the perpendicular DE from D to AB [Th. 21].

Being α an acute angle greater than zero, D is not in straight line with AB [Cr. 43],

and then DAE is a triangle [Th. 6],

of which ρ is its greater angle [Th. 26],

so that and $AD > DE$ [Th. 17],

and being $AD = PQ$, it holds $DE < PQ$ [Ps. B].

Being $DE < PQ$, on PQ take a point R such that $QR = DE$ [Th. 1],

on r take a point S such that $QS = AE$ [Th. 1],

and joint R and S [Cr. 15].

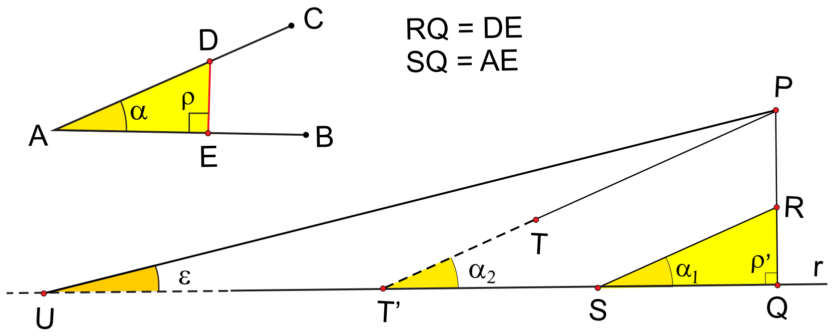


Figure 11.9 – Theorem 57

R is not in straight line with r [Cr. 43].

Therefore, R, S and Q define the triangle RSQ [Cr. 20, Th. 6].

The sides AE and DE and the right angle ρ of DAE are respectively equal to the sides QS, RQ and the right angle ρ' of RSQ [Th. 22],

Therefore $\alpha = \alpha_1$ [Th. 12].

Now, through P draw the parallel PT to RS [Th. 35].

Since r intersects RS at S, it also intersect its parallel PT at T' [Th. 51].

PT' makes at T' an angle α_2 with $T'Q$ such that $\alpha_2 = \alpha_1 > 0$ [Th. 41, Cr. 44].

On r and from T' take any point U on the side of T' of r on which it is not Q [Th. 1].

Join U and P [Cr. 15].

Since P is not in straight line with r, $PT'U$ is a triangle [Th. 6],

of which α_2 is an exterior angle [Df. 28, Cr. 44],

In consequence, the angle ϵ that PU makes at U with r is less than α_2 [Th. 16].

And being $\alpha_2 = \alpha_1 = \alpha$, we conclude that PU is a straight line through the given point P that makes at U with the given straight line r an angle ϵ less than the given angle α [Ps. B]. □

Note.-Legendre proved [Th. 57] on the hypothesis of [Th. 50]

Theorem 58 *The bisected point of a given common transversal of two parallel straight lines is also the bisected point of any other common transversal of both parallels that passes through the bisected point of the given transversal.*

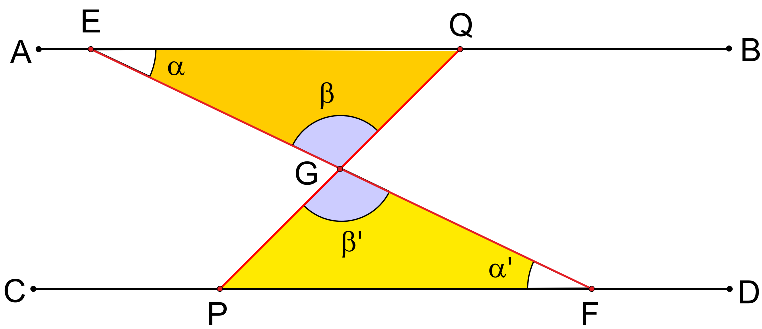


Figure 11.10 – **Theorem 58**

Let AB and CD any two parallel straight lines [Th. 37],

and EF a common transversal of AB and CD [Cr. 32],

which makes with them equal the alternate angles α and α' [Th. 41].

Bisect EF at G [Th. 15].

Join any point P of CD with G [Cr. 15].

PG intersects AB in Q [Th. 51].

Since neither α nor α' are straight angles, G is not in straight line with AB, nor with CD [Cr. 43].

Therefore Q, E and G define the triangle EGQ; and G, P and F define the triangle GPF [Th. 6].

And being $\alpha = \alpha'$ [Th. 41],

$\beta = \beta'$ [Th. 5],

and $EG = GF$, it holds $PG = GQ$ [Th. 11].

Therefore, the point P also bisects the transversal PQ [Df. 9] □

Note.-Veroness and Ingrami deduced from [Th. 58] Playfair's axiom, a substitute for Euclid's postulate 5.

Theorem 59 *There exist two similar triangles which are unequal.*

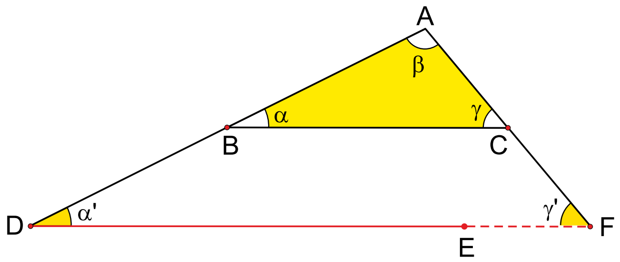


Figure 11.11 – Theorem 59

Let ABC be any triangle [Th. 6].

Produce the side AB from B to any point D [Cr. 16].

It holds $AB < AD$ [Cr. 13].

Through D draw the parallel DE to BC [Th. 35].

AC and DE can be produced respectively from C and from E [Cr. 16]

to its intersection at point F [Th. 51].

AD makes at D with DF an angle $\alpha' = \alpha$ [Th. 41, Cr. 44]

which is not an straight angle [Th. 26, Cr. 51].

Therefore, A is not in a straight line with DF [Cr. 43],

and A is not in straight line with D and F [Cr. 20].

Therefore ADF is a triangle [Th. 6].

The triangles ABC and ADF have a common angle β, and in addition $\alpha = \alpha'$, $\gamma = \gamma'$ [Th. 41].

So, ABC and ADF are two similar triangles [Df. 28].

And being $AB < AD$, ABC and ADF are not equal [Df. 28]. □

Note.-The statement of the above theorem is part of a Wallis, Carnot and Laplace's assumption intended to replace Euclid's Fifth Postulate. Saccheri proved it suffices to replace Euclid's postulate [9].

Theorem 60 *Through any point it is possible to draw a straight line that cuts each side of any non-straight angle at different points, producing or not the sides.*

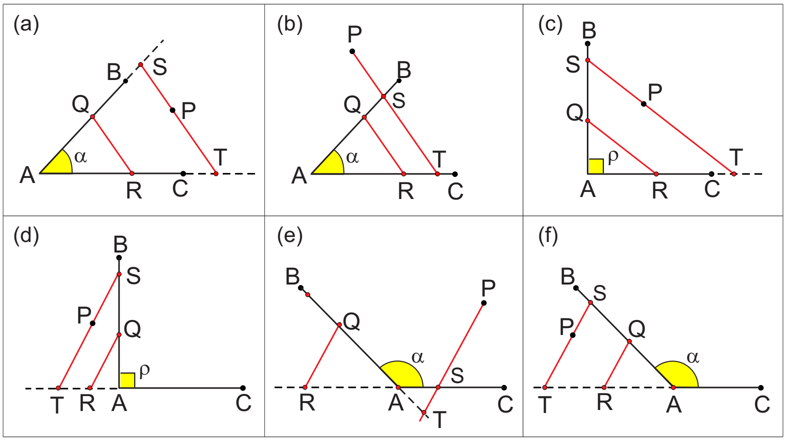


Figure 11.12 – **Theorem 60**

Let P be any point [Cr. 25]

and AB and AC any two adjacent straight lines not in a straight line with each other [Cr. 29]

which make a non-straight angle at their unique common point A [Df. 24, Cr. 37].

If P is in a straight line with any of the sides of the angle (not in Figure 11.12), join it with any point Q of the other side [Cr. 15],

PQ intersects both sides, whether or not produced. If P is not in a straight line with any of the sides of the angle (Fig. 11.12 (a)-(f)), join any point Q on one side with any point R on the other side [Cr. 15].

Through P, draw a parallel to QR [Th. 35].

Since AB and AC cut QR at Q and R, they also cut, whether or not produced, the parallel to QR through P at the points S and T [Th. 51]. □

Note.-The statement of [Th. 60] is a generalized version of Legendre's statement: *through any point within an acute angle a straight line can always be drawn which meets both sides of the angle*, intended to replace Euclid's Postulate 5. Figure 11.12 illustrates the cases of interior and exterior points of an acute, right and obtuse angle, respectively (a), (b); (c), (d); and (e), (f).

Theorem 61 (Hilbert Axiom I.3) *Given three points not on the same straight line, there is one, and only one, plane that contains them.*

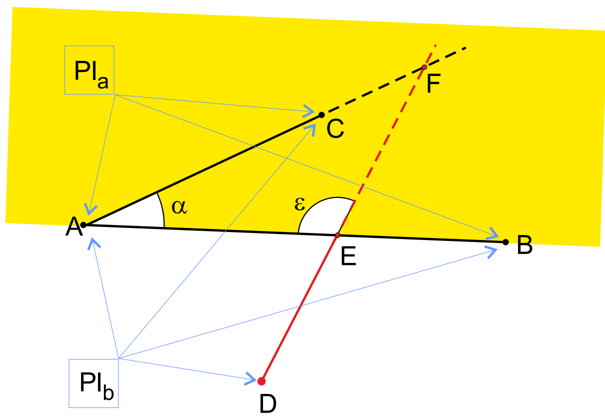


Figure 11.13 – **Theorem 61**

Let A, B and C be any three points not in straight line [Cr. 22]].

There is at least a plane Pl that contains them [Ax. 6].

Assume there is another plane Pl' that contains A, B and C. Join A with B and A with C [Cr. 15].

Both planes contain the straight lines AB and AC [Cr. 25].

Being different Pl and Pl', there will be at least a point D in one of them, for instance in Pl, which is not in the other [Df. 13].

Join D with any point E of AB (the same would apply to AC) [Cr. 15].

If DE were parallel to AC, take another point E' on AB and joint it with D, the straight line DE' will not be parallel to AC [Th. 36].

In any case, there will be a straight line DE (or DE') which is not parallel to AC. And the common transversal AE makes with them two angles α and ϵ on the same side of AE that sum less than two right angles [Th. 43].

Therefore AC and DE, produced or not, intersect each other at a point F [Th. 49].

Since E is on AB, and F is on AC (produced if necessary [Cr. 16]),

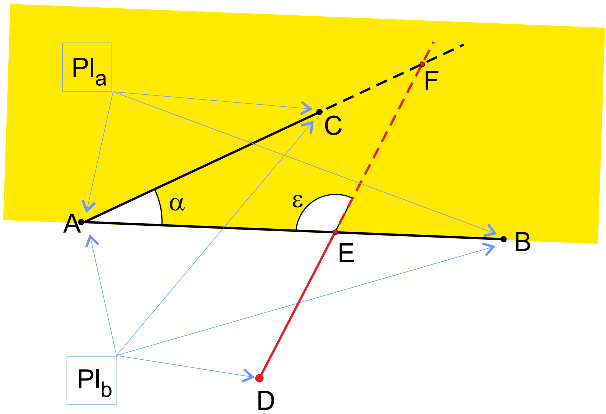


Figure 11.13 – **Theorem 61**

both points also belongs to Pl' [Df. 13].

Therefore, the straight line EF belongs to Pl' [Df. 13],

and the same applies to FD [Df. 13].

So, D also belongs to Pl', which contradicts our initial assumption. It is then impossible that two different planes contain the same three different points not on the same straight line. □

Theorem 62 *Given any three points not in straight line, there is a circle passing through them.*

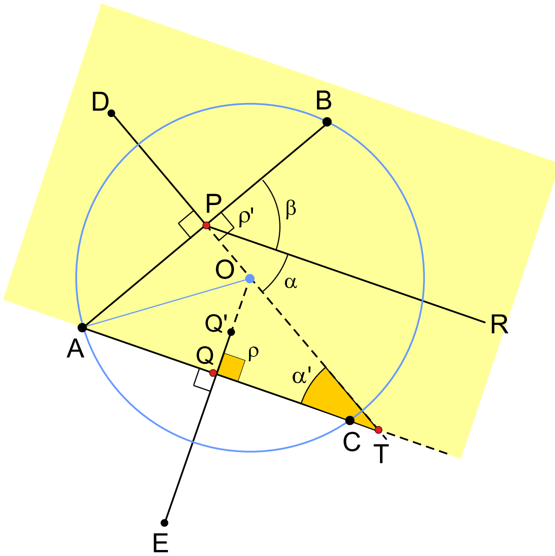


Figure 11.14 – **Theorem 62**

Let A, B and C be any three points not in straight line [Ax. 1, Cr. 22].

Join A with B and with C [Cr. 15].

Bisect AB at P and AC at Q [Th. 15]

and draw the perpendiculars PD to AB through P, and QE to AC through Q [Th. 21].

Produce EQ to any point EQ' [Cr. 16].

From P draw the parallel PR to AC [Th. 35].

AC is intersected by the production PT of DP [Th. 51].

PB and PT are the adjacent sides of the right angle ρ' [Cr. 52, Df. 22].

PR is also adjacent at P with PB, otherwise both straight lines would be superposed in a unique straight line [Df. 21, Cr. 18]

and PR would not be parallel to AC, which is not the case. So, PB, PR and PT are adjacent at T where they make two adjacent angles α and β [Cr. 41]

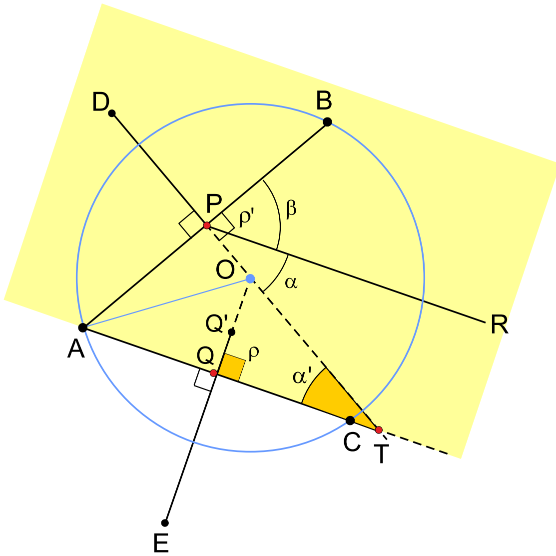


Figure 11.14 – **Theorem 62**

whose sum is their union angle ρ' [Th. 3],

which is a right angle [Cr. 52].

Thus, α is less than a right angle [Th. 3].

Being $\alpha = \alpha'$ [Th. 41]

and ρ a right angle [Cr. 52],

AT is a common transversal of EQ' and DT that makes with them two angles ρ and α' that sum less than two right angles. Therefore, they, or their productions, intersect at a point O in the side of ρ and $alpha'$.[Th. 49].

Since each point of DT is at the same distance from A as from B, and each point of EO is at the same distance from A as from C [Th. 24]

their intersection point O is at the same distance from A, as from B and from C [Ps. B].

In consequence, a circle with centre O and radius OA passes through the three given point A, B and C [Df. 19]. □

Note.-Some mathematicians, as Legendre and Bolyai, took [Th. 62] as a substitute for Euclid's postulate 5.

Theorem 63 *It is impossible to join by means of a straight line a point on one side of a triangle different from its vertices with two different points, one from each of the other two sides; or one of each production of the other two sides.*

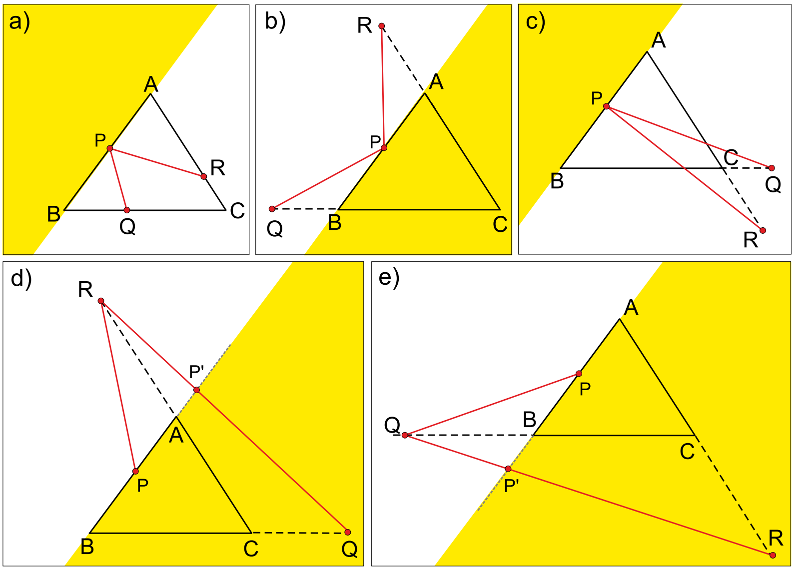


Figure 11.15 – **Theorem 63**

(Fig. 11.15, a)) Let ABC be any triangle in a plane Pl [Ths. 10, 9, 8]

and P any point on one of its sides, for example* of AB [Cr. 1].

Join P with two different points, Q of BC and R of AC [Crs. 1, 15].

PQ and PR cannot be in a straight line, because in that case P would be in a straight line with the points A and R of AC, and with points Q and B of BC. Therefore, P would be in a straight line with AC and with BC [Df. 12],

what is impossible [Cr. 47].

For the same reasons, the straight lines that join P with any point in the production of each side cannot be in a straight line either (Fig. 11.15, b)-e)). □

Theorem 64 (Strong version of Pasch's Axiom) *If a straight line intersects a side of a triangle and does not pass through any of its vertexes then it intersects one, and only one, of the other two sides of the triangle. And if it is not parallel to any of the sides, it also intersects a production of the non-intersected side.*

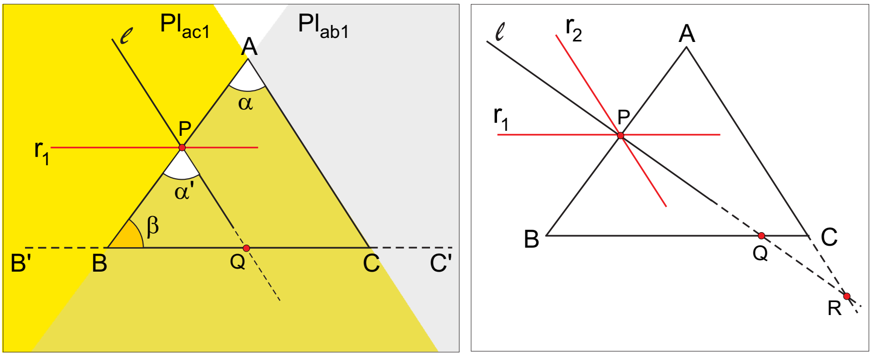


Figure 11.16 – Theorem 64

(Fig. 11.16, left) Let ABC be any triangle in a plane Pl [Ths. 10, 9, 8]

and l any straight line that cuts the side AB at a point P [Cr. 21]

and does not pass through C. If l is parallel to one of the other two sides of ABC, for instance* to AC (Fig. 11.16, left), then draw the parallel r_1 to the other side BC [Th. 35].

It holds $\alpha = \alpha'$ [Th. 41],

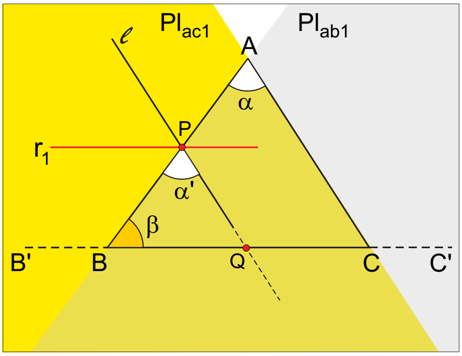
$\alpha' + \beta = \alpha + \beta <$ two right angles [Ps. A, Th. 26].

So, AB is a common transversal of l and BC that makes with them two angles α' and β that sum less than two right angles. Therefore l and BC, whether or not produced intersect at a point Q in the side Pl_{ab1} of AB [Th. 49].

And since all points of l are in the side Pl_{ac1} of AC [Ths. 30, 39],

the intersection point Q can only be between B and C because the points of any production BB' are not in the side Pl_{ab1} of AB, and the point of any production CC' are not in the side Pl_{ac1} of AC [Cr. 30].

On the other hand, if l is not parallel to any side of ABC (Fig. 11.16, right), then draw through P the parallels r_1 and r_2 respec-



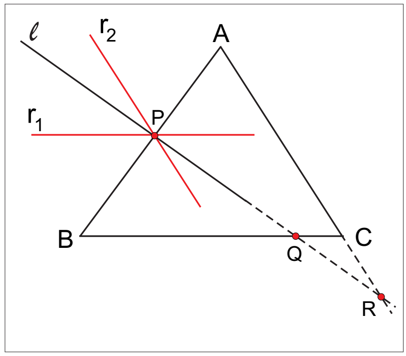


Figure 11.17 – Theorem 64

tively to BC and AC [Th. 35].

The straight line l intersect BC and AC, whether or not produced [Th. 51].

But it can only intersect one side and the production of the other [Th. 63]. □

Theorem 65 (Gauss' Strong Hypothesis) *There is a rectilinear triangle similar to a given triangle whose area is greater than any given finite area.*

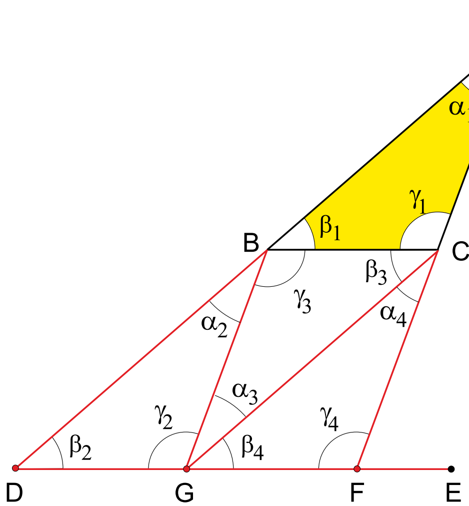


Figure 11.18 – Theorem 65

Let ABC be any triangle of any finite area S_{ABC} [Ths. 10, 9, 8, Ax. 10].

From B produce AB to a point D such that $BD = AB$ [Cr. 16, Th. 1].

Through D draw a parallel DE to BC [Th. 35].

AC can be produced to the intersection point F with DE [Th. 51].

On DF take a point G such that $DG = BC$ [Th. 1],

and join G with B and with C [Cr. 15].

Since BC and DE are parallel, neither B nor C are in straight line with DE [Th. 30, Df. 14].

Since AD makes an angle $\beta_2 = \beta_1$ with DE [Th. 41],

and β_1 is not an straight angle [Th. 26, Cr. 51],

A is not in straight line with DE [Ps. B, Cr. 43].

Hence, BDG, BGC, CGF and ADF are triangles [Th. 6].

Since $BD = AB$; $DG = BC$ and $\beta_1 = \beta_2$ [Th. 41],

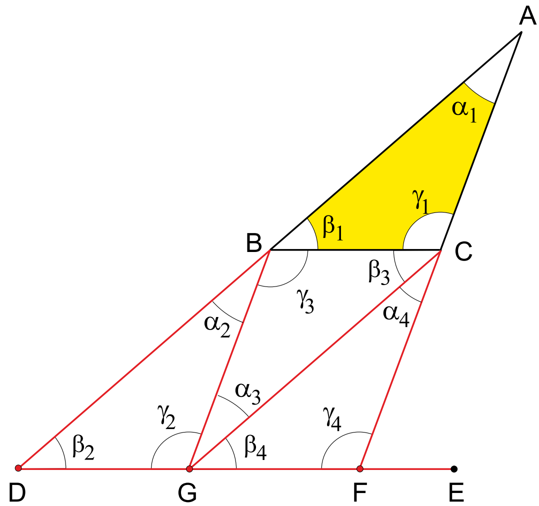


Figure 11.18 – Theorem 65

the triangles ABC and BDG are equal to each other [Cr. 48],

so that $\gamma_1 = \gamma_2$; $\alpha_1 = \alpha_2$; $BG = AC$. It also holds $\gamma_3 = \gamma_2$ [Th. 41],

and the triangles BDG and BGC are also equal because they have a common side BG, $BC = DG$ and $\gamma_3 = \gamma_2$ [Cr. 48].

On the other hand, $\beta_3 = \beta_4$ and $\gamma_1 = \gamma_4$ [Th. 41],

and taking into account that $\gamma_1 = \gamma_2 = \gamma_3$, it follows $\gamma_3 = \gamma_4$ [Ps. B].

So, it must be $\alpha_3 = \alpha_4$ because $\alpha_3 + \beta_3 + \gamma_3 = \alpha_4 + \beta_4 + \gamma_4$ [Th. 50].

Consequently, the triangles BGC and CGF, which have a common side CG, are also equal [Th. 11, Cr. 48].

It has been proved that the four triangles ABC, BDG, BGC and CGF are equal to one another [Ps. B],

and that the triangles ABC and ADF are similar [Df. 28],

being the area of ADF four times the area S_{ABC} of ABC [Ax. 10].

Therefore, and taking into account that for any finite are S there is a natural number n such that $4^n S_{ABC} > S$, by repeating the above construction n successive times a triangle similar to ABC

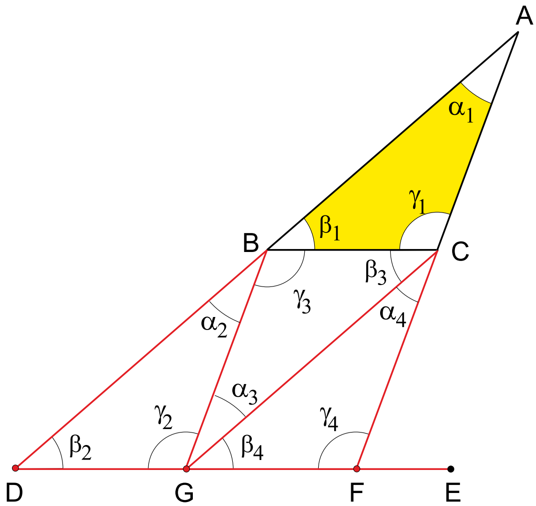


Figure 11.18 – **Theorem 65**

with an area greater than S will have been constructed. □

In 1791 and in a letter to János Bolyai, F. Gauss wrote (quoted in [9, p. 220]):

> If I could prove that a rectilinear triangle is possible the content of which is greater than any given area, I am in a position to prove perfectly rigorously the whole of geometry.

The new foundation of Euclidean geometry makes it possible the above solution to Gauss' hypothesis.

12. Pythagorean Variations

12.1 Introduction

Pythagoras' Theorem is surely the most popular theorem in geometry. There must be few people who have not learned it. And many who remember it for a lifetime. Less well known is the inverse of the Pythagorean Theorem: if in a triangle the square of one of its sides is equal to the sum of the squares of the other two sides, then the angle contained by these two sides is a right angle. With these two theorems ends the Book I of Euclid's Elements. And so ends this introductory book to the new Euclidean geometry built on the formal basis introduced in Chapter 7. This last chapter includes ten theorems, seven of which are also propositions of the Book I of Euclid's Elements, although only the statements and, partially, the strategy to reach the Pythagorean Theorem and its converse coincide.

Theorem 66 *With three given points as vertices not in straight line, draw a parallelogram.*

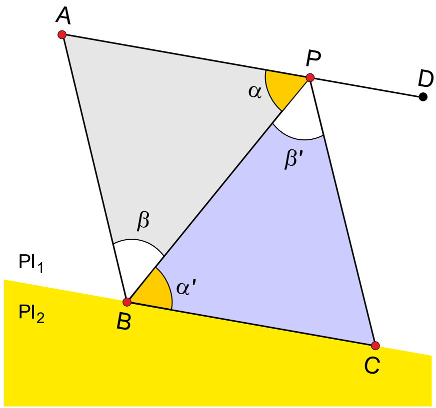


Figure 12.1 – Theorem 66

Let A, B and C be any three given points not in straight line [Cr. 22].

There is a plane Pl that contains them [Ax. 6].

Join A with B and B with C [Cr. 15].

From A draw a parallel AD to BC [Th. 35].

In AD take a point P such that $AP = BC$ [Th. 1].

Join P with B and with C [Cr. 15].

All points of AD are non-common points of one of the sides, Pl_1, of BC [Ax. 6, Th. 30] and then points that are not in straight line with BC [Df. 14].

Therefore ABP is a triangle [Th. 6].

For the same reasons PBC is a triangle. ABP and PBC have a common side PB, being also $AP = BC$ and $\alpha = \alpha'$ [Th. 41].

Therefore, $AB = PC$ and $\beta = \beta'$ [Cr. 48].

Consequently, AB is parallel to PC [Th. 42],

and $ABCP$ is a parallelogram in the plane Pl [Df. 29]. □

Theorem 67 *To draw two triangles and two parallelograms with an equal side and between the same two parallels.*

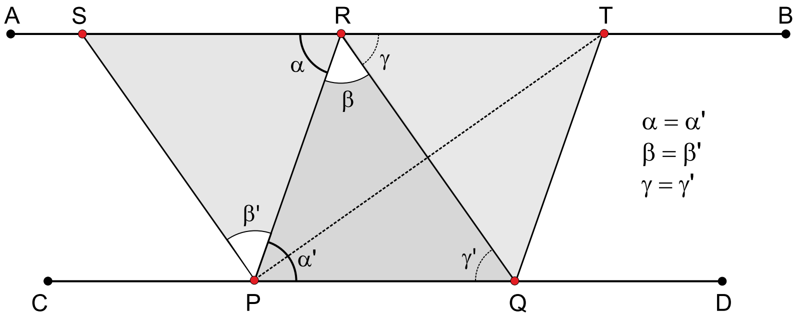


Figure 12.2 – Theorem 67

Let AB and CD be any two parallels [Cr. 37],

P and Q any two points of AB, and R any point of CD [Cr. 1].

In AB and in the direction from R to A take a point S such that $RS = PQ$ [Th. 1].

Join R with P and with Q; and S with P [Cr. 15].

Since the points of AB are not in a straight line with CD, and vice versa [Th. 30, Df. 14],

RPQ and SPR are triangles [Th. 6],

they have a common side RP, and also $RS = PQ$ and $\alpha = \alpha'$ [Th. 41].

Therefore $SP = RQ$ and $\beta = \beta'$ [Cr. 48].

Since $\beta = \beta'$, SP is parallel to RQ [Th. 42],

and $SPQR$ is a parallelogram [Df. 29].

In AB and in the direction from R to B, take a point T such that $RT = PQ$ [Th. 1].

Join T with P and with Q [Cr. 15].

The same above argument proves that TPQ is a triangle and $RPQT$ is a parallelogram. The triangles RPQ and TPQ, and the parallelograms $SPQR$ and $RPQT$ have the same base PQ and are between the same parallels AB and CD [Df. 29]. □

Theorem 68 (Euclid's Proposition 46) *In one of the sides of a given straight line to describe a square one of whose sides is the given straight line.*

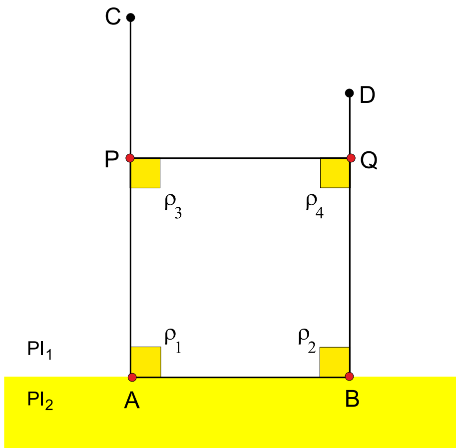


Figure 12.3 – **Theorem 68**

Let AB be the given straight line [Cr. 25].

On A draw the perpendicular AC to AB in its side Pl_1 [Th. 20].

On B draw the perpendicular BD to AB in its side Pl_1 [Th. 20].

Being perpendicular to same line AB, the straight lines AC and BD are parallel to each other [Th. 38].

In AC take a point P such that $AP = AB$ [Th. 1].

In BD take a point Q such that $BQ = AB$ [Th. 1].

Join D with B [Cr. 15].

Since P and Q are equidistant from AB, PQ is parallel to AB [Th. 33].

BD is also in Pl_1 of AB [Ax. 6].

Being AP and BQ perpendicular to AB, and PQ parallel to AB, the angles ρ_1, ρ_2, ρ_3, ρ_4 are right angles [Th. 43].

Thus, the quadrilateral $PABQ$ is equilateral, right angled and all its sides are in Pl_1. So, it is a square in the side Pl_1 of AB [Df. 29]. $\square$

Theorem 69 *To draw a square on each side of a triangle, each square being on the side on which it is not the vertex that is not on that side.*

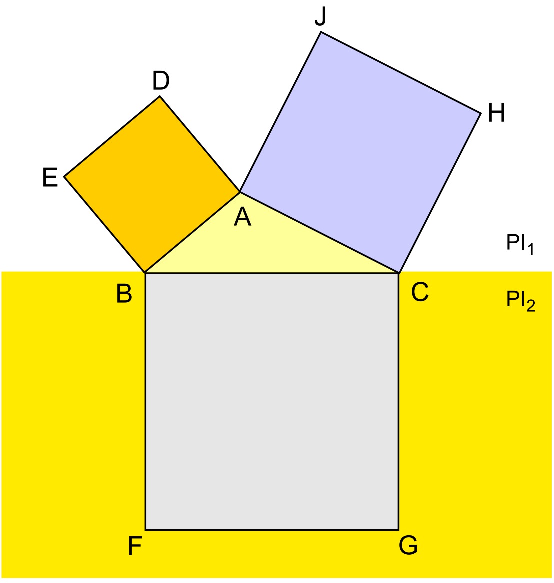


Figure 12.4 – Theorem 69

Let ABC be any triangle [Ths. 10, 9, 8].

Since A, B and C are not in a straight line with each other [Th. 6],

C is not in a straight line with AB [Cr. 20],

and it is a non-common point of one of the sides of AB [Df. 14].

So, it can be drawn a square on AB on the side where it is not C [Th. 68].

For the same reasons it can be drawn a square on BC on the side where A is not, and another square on CA on the side where B is not.[Th. 68]. □

Theorem 70 (Euclid's Proposition 34) *In parallelograms the opposite angles are equal to each other; and the diagonal bisects the area of the parallelogram.*

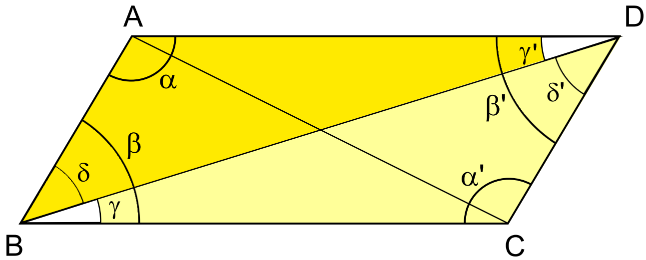


Figure 12.5 – **Theorem 70**

Let $ABCD$ be a parallelogram [Th. 66].

Draw the diagonal BD [Df. 28, Cr. 15].

Since AD and BC are parallel and BD is a common transversal, it holds: $\gamma = \gamma'$ [Th. 41].

Since AB and DC are parallel and BD is a common transversal, we will have: $\delta = \delta'$ [Th. 41].

A, B, C and D are not in straight line [Th. 30, Df. 14].

So, ABD and DBC are triangles [Th. 6]

and BA, BC and BD are adjacent at B [Df. 29].

Therefore, β is the union angle of the adjacent angles δ and γ [Th. 3].

The union angle β is, then, the sum of the adjacent angles δ and γ [Th. 3].

For the same reasons, β' is the sum of the adjacent angles δ' and γ'. Therefore, and being $\gamma = \gamma'$ and $\delta = \delta'$, we will have [Ps. A]:

$$\begin{aligned}\beta &= \gamma + \delta \\ &= \gamma' + \delta' \\ &= \beta'\end{aligned}$$

The angles δ and γ' of ABD are equal to the angles δ' and γ of DBC. And both triangles have a common side BD,

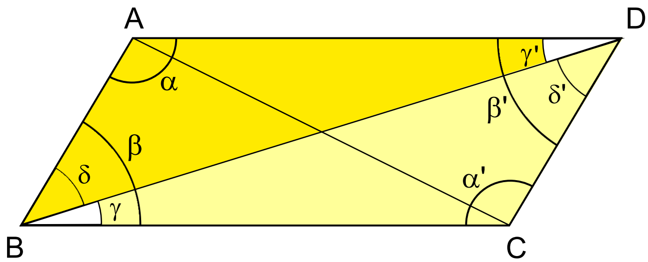


Figure 12.5 – Theorem 70

therefore $AB = DC$, $AD = BC$ [Th. 11],

and also $\alpha = \alpha'$ [Th. 12],

In consequence, ABD and DBC are equal [[Df. 28]]

and they have the same area [Ax. 10].

Hence, in parallelograms, the opposite angles are equal to each other, and the diagonal DB bisects the parallelogram [Df. 28].

The same argument proves that the diagonal AC also bisects the parallelogram. □

Theorem 71 (Euclid's Proposition 35) *Parallelograms which are on the same base and in the same parallels have the same area.*

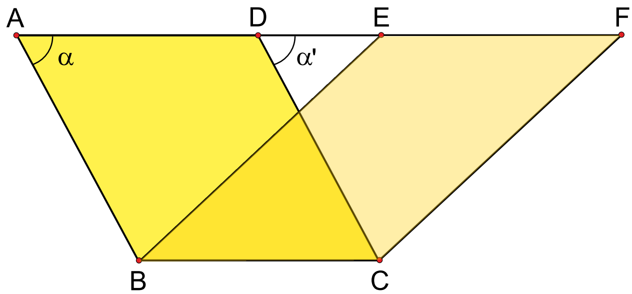


Figure 12.6 – **Theorem 71**

Let $ABCD$ and $EBCF$ be two parallelograms on the same base BC and between the same parallels AF, BC [Th. 67].

Being parallelograms, we will have: $AD = BC$ and $EF = BC$ [Df. 29].

Therefore $AD = EF$ [Ps. B],

$AD + DE = EF + DE$ [Ps. B].

And being $AE = AD + DE$, $DF = DE + EF$ [Cr. 13],

it holds $AE = DF$ [Ps. A].

On the other hand AB and DC are parallel to each other [Df. 29,

and AF a common transversal. Therefore $\alpha = \alpha'$ [Th. 41].

Being AF and BC parallel, A, B and E are not in straight line [Th. 30, Df. 14]

and they define the triangle ABE [Th. 6].

For the same reasons D, C and F define the triangle DCF [Ths. 30, 6].

Being the sides AB and AE of the triangle ABE respectively equal to the sides DC and DF of the triangle DCF. And being also $\alpha = \alpha'$, the other two angles of ABE are equal to the corresponding other two angles of DCF, and $BE = CF$ [Cr. 48].

Therefore, ABE and DCF are two similar triangles [Df. 28],

and the sides of the one are equal to the corresponding sides of the

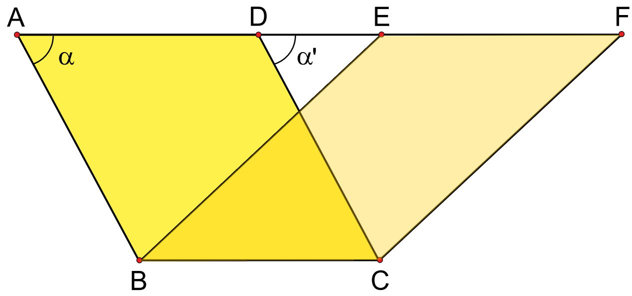


Figure 12.6 – Theorem 71

other. They are then equal [Df. 28],

and they have the same area [Ax. 10].

Since areas, as lengths, are metric properties, they can be arithmetically operated [Df. C], and we will have:

$$\begin{aligned} ABCF &= ABCD + DCF && \text{[Ax. 10]} \\ ABCF &= EBCF + ABE && \text{[Ax. 10]} \\ ABCD + DCF &= EBCF + ABE && \text{[Ps. A]} \\ ABCD &= EBCF && \text{[Ps. B]} \end{aligned}$$

So then, the parallelograms $ABCD$ and $EBCF$ have the same area. □

Theorem 72 (Euclid's Proposition 37) *Triangles which are on the same base and in the same parallel have the same area.*

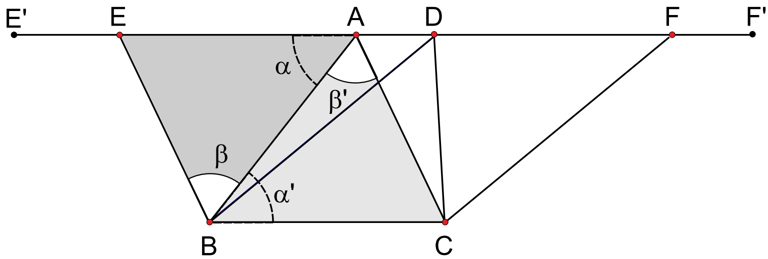


Figure 12.7 – **Theorem 72**

Let ABC and DBC be two triangles on the same base BC and between the same parallels BC, $E'F'$ [Th. 67].

On AE' take a point E such that $AE = BC$ [Th. 1].

On DF' take a point F such that $DF = BC$ [Th. 1].

Join B with E, and C with F [Cr. 15].

Since BC and $E'F'$ are parallel, the points of the one are not in a straight line with the points of the other [Th. 30, Df. 14].

Hence, EBA is a triangle [Th. 6],

ABC and EBA have a common side AB, and also $AE = BC$ and $\alpha = \alpha'$ [Th. 41].

Therefore $EB = AC$ and $\beta = \beta'$ [Cr. 48].

In consequence EB is parallel to AC [Th. 42].

so that $EBCA$ is a parallelogram [Df. 29].

And for the same reasons, $DBCF$ is also a parallelogram. Therefore $EBCA$ and $DBCF$ are parallelograms on the same base and in the same parallels. So, they have the same area [Th. 71].

The diagonal AB bisects the area of $EBCA$ [Th. 70].

And the diagonal DC bisects the area of $DBCF$ [Th. 70].

Consequently the area of the triangle ABC is half the area of the parallelogram $EBCA$ [Df. 28],

and the area of the triangle BDC is half the area of the parallelogram $DBCF$ [Df. 28].

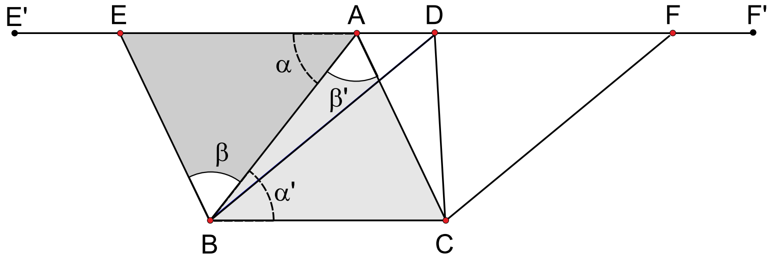


Figure 12.7 – Theorem 72

And being $EBCA = DBCF$, the areas of both triangles, ABC and BDC, will be equal to each other [Ps. B]. □

Theorem 73 (Euclid's Proposition 41) *The area of a parallelogram with the same base and in the same parallels than a triangle is double than the area of the triangle.*

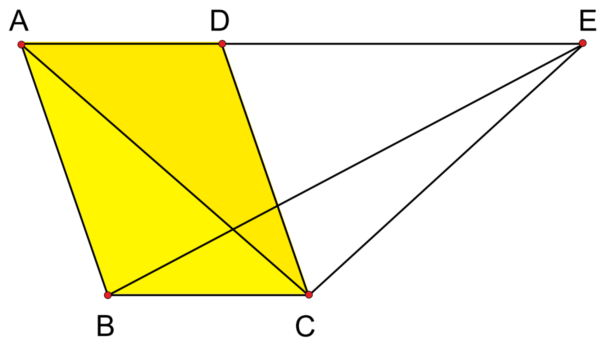


Figure 12.8 – Theorem 73

Let $ABCD$ be a parallelogram with the same base BC and in the same parallels, BC, AE, as the triangle EBC [Th. 67].

Join A with C [Cr. 15].

The points of AE are not in a straight line with those of BC [Th. 30, Df. 14].

Therefore, ABC is a triangle [Ths. 30, 6].

The diagonal AC bisects the parallelogram $ABCD$ [Th. 70].

Therefore, the area of the triangle ABC is half the area of the parallelogram $ABCD$. On the other hand, the area of the triangle ABC is equal to the area of the triangle EBC for they are on the same base BC and in the same parallels BC, AE [Th. 72].

Hence, the area of the parallelogram $ABCD$ doubles the area of the triangle BCE [Ps. B]. □

Theorem 74 (Euclid's Proposition 47. Pythagoras Theorem) *In right triangles the area of the square on the side subtending the right angle is equal to the sum of the areas of the squares on the sides containing the right angle.*

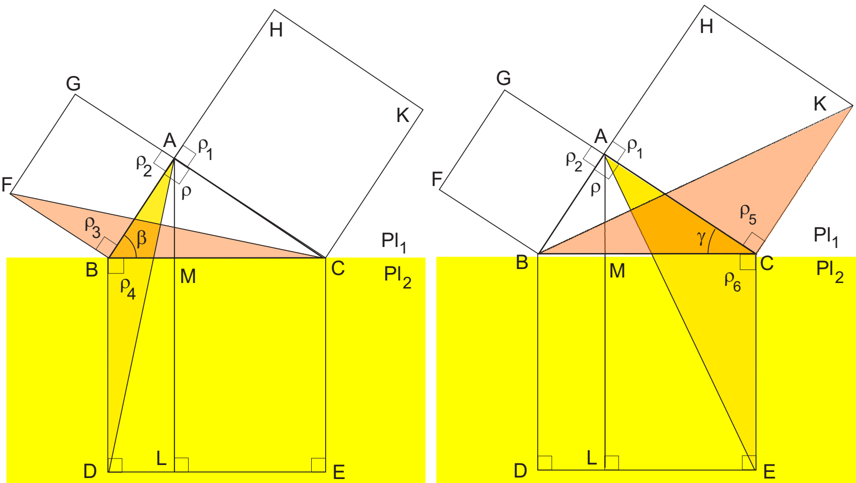


Figure 12.9 – **Theorem 74**

Let ABC be a right triangle whose right angle is ρ [Th. 23].

Describe the square $GFBA$ on AB; the square $HACK$ on AC; and the square $BDEC$ on BC, each in the side where it is not the vertex which is not in that side [Th. 69].

GA and AC are in straight line with each other because ρ and ρ_2 are right angles [Crs. 51, 42].

For the same reason, BA and AH are also in straight line with each other. From A draw the perpendicular AL to DE [Th. 21].

Since A and L are in different sides of BC there will be an intersection point M between AL and BC [Cr. 31].

Being BD, AL, and CE perpendicular to DE, they are parallel to one another [Th. 38].

Join A with D, and F with C by straight lines [Cr. 15].

GC is parallel to FB, and BD to AL [Th. 38],

and since the points of a straight line are not in a straight line with their parallels [Th. 30, Df. 14],

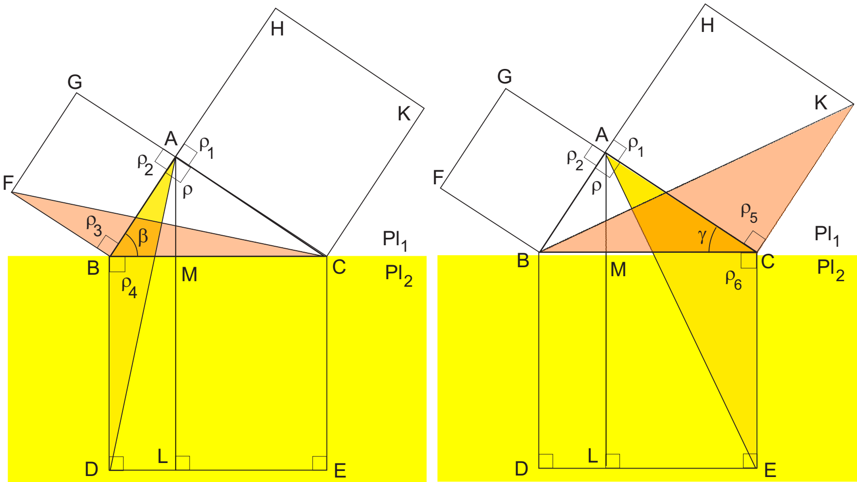


Figure 12.9 – **Theorem 74**

FBC and ABD are triangles [Ths. 30, 6].
(Figure 12.9, left) FBC and ABD have two sides of the one equal to two sides of the other, namely $FB = BA$ and $DB = BC$. And the angles $\rho_3 + \beta$ and $\rho_4 + \beta$ contained by the equal sides are also equal to each other [Cr. 48; Ps. B].

Consequently the triangles FBC and ABD are equal to each other [Cr. 48].

The area of the triangle FBC is half the area of the square $GFBA$ because they are on the same base (FB) and in the same parallels (FB, GC) [Th. 73].

And the area of the triangle ABD is half the area of the square $BDLM$ because they are on the same base (BD) and in the same parallels (BD, AL) [Th. 73].

Consequently, and being equal the triangles FBC and ABD, half of the area of $GFBA$ is equal to half of the area of $BDLM$. Therefore the whole area of $GFBA$ is equal to the whole area of $BDLM$ [Ps. B].

For the same reasons as in the case of triangle FBC, KBC and AEC are triangles (Figure 12.9, right). They have two sides of the one equal to two sides of the other, namely $CK = AC$ and $BC = CE$. And the angles $\rho_5 + \gamma$ and $\rho_6 + \gamma$ contained by the equal sides are also equal to each other [Th. 22; Ps. B].

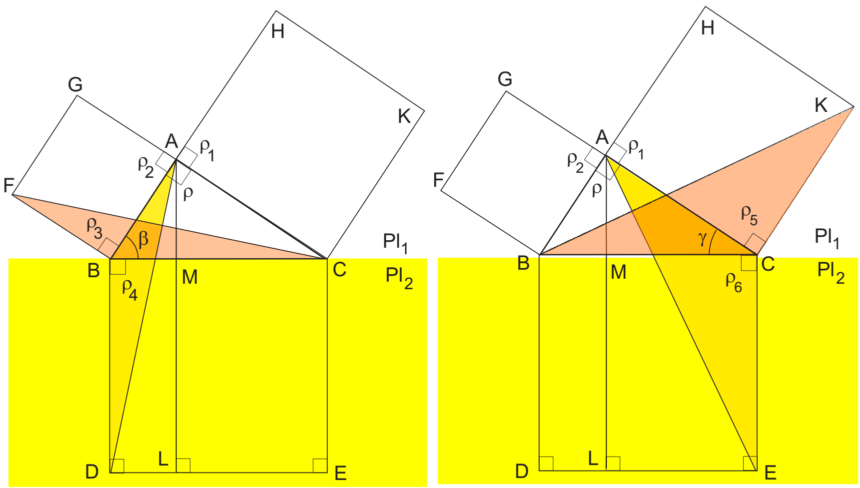


Figure 12.9 – Theorem 74

Consequently both triangles are equal to each other [Cr. 48].

The area of the triangle KBC is half the area of the square $HACK$ because they are on the same base (CK) and in the same parallels (CK, BH) [Th. 73].

And the area of the triangle ACE is half the area of the square $MLEC$ because they are on the same base (CE) and in the same parallels (CE, AL) [Th. 73].

Consequently, and being equal the triangles KBC and ECA, half of the area of $HACK$ is equal to half of the area of $MLEC$. Therefore the whole area of $HACK$ is equal to the whole area of $MLEC$ [Ps. B].

And since the area of $BDEC$ is the sum of the areas of $BDLM$ and $MLEC$, the area of square on the side BC subtending the right angle ρ of ABC is equal to the sum of the areas of the squares on the sides AB and AC containing the right angle ρ of ABC. $\square$

Theorem 75 (Euclid's Proposition 48) *If in a triangle the area of the square built on one of the sides is equal to the sum of the areas of the two squares built on the remaining sides, the angle contained by those remaining sides is a right angle.*

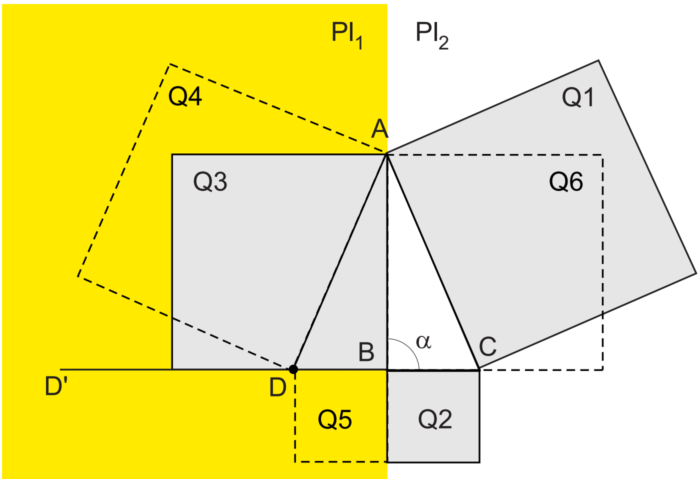


Figure 12.10 – Theorem 75

Let ABC be a triangle [Ths. 10, 9, 8]

and suppose the area of the square $Q1$ is equal to the sum of the areas of the squares $Q2$ and $Q3$. From B draw the perpendicular BD' to AB [Th. 20].

In BD' take a point D such that $BD = BC$ [Th. 1].

Join D and A [Cr. 15].

The sides BD' and BC of the point B of the straight line $D'C$ are on opposite sides, PL_1 and Pl_2, of AB in the common plane [Cr. 30].

Therefore D is a non-common point of one of the sides, Pl_1, of AB and it is not in straight line with A and B [Df. 14],

and ADB is a triangle [Th. 6].

Draw the squares $Q4$ on AD, $Q5$ on BC, and $Q6$ on AB[Th. 68].

Since ADB is a right triangle [Df. 29],

the area of $Q4$ is the sum of the areas of $Q5$ and $Q6$. [Th. 74].

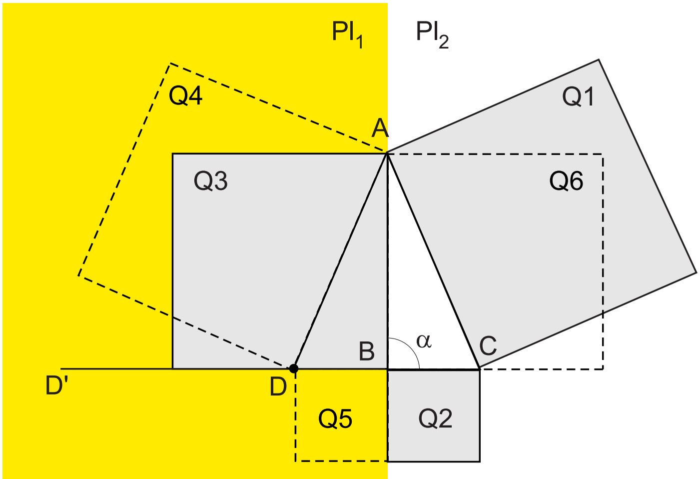


Figure 12.10 – **Theorem 75**

On the other hand the squares $Q2$ and $Q5$ are equal because they have been drawn on equal sides. And for the same reason the squares $Q3$ and $Q6$ are also equal. Consequently, the sum of the areas of $Q2$ and $Q3$ is equal to the sum of the areas of $Q5$ and $Q6$ [Ps. B].

And being the first sum equal to the area of $Q1$, and the second sum equal to the area of $Q4$, we conclude that the area of $Q1$ is equal to the area of $Q4$ [Ps. B].

Therefore the squares $Q1$ and $Q4$ are equal, and the side AC of $Q1$ is equal to the side AD of $Q4$. Therefore, the three sides of triangle ABC are equal to the corresponding three sides of triangle ADB. And both triangles are equal [Th. 14, Df. 28].

And the angle α is equal to the angle ρ, which is a right angle. □

Appendix A.
Classical Foundations of Euclidean Geometry

A.1 Introduction

By way of historical reference, this appendix contains three of the most significant foundational bases of Euclidean geometry. In the case of Euclid's the text is taken from [9, p. 153-155], in that of J. Playfair's from [21, p. 8-11], and in that of Hilbert's from [10, p. 2-16]. Because of its self-evidence, Playfair gave no demonstrations of the corollaries included in its foundational basis. On the other hand, Hilbert includes 20 theorems, of which only 4 include a proof proper.

A.2 Euclid's Foundation (around 300 BC)

DEFINITIONS

1. A point is that of which there is no part.
2. A line is a length without breadth.
3. The extremities of a line are points.
4. A straight-line is a line which lies evenly with the points on itself.
5. A surface is that which has length and breadth only.
6. The extremities of a surface are lines.
7. A plane surface is a surface which lies evenly with the straight

lines on itself.

8. A plane angle is the inclination to one another of two lines in a plane which meet one another, and do not lie in a straight line.
9. And when the lines containing the angle are straight then the angle is called rectilineal.
10. When a straight line set up on a straight line makes adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.
11. An obtuse angle is an angle greater than a right-angle.
12. And an acute angle is an angle less than a right-angle.
13. A boundary is that which is an extremity of anything.
14. A figure is that which is contained by any boundary or boundaries.
15. A circle is a plane figure contained by one line such that all of the straight lines falling upon it from one point among those lying within the figure are equal to one another.
16. And the point is called the center of the circle.
17. And a diameter of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such straight line also also bisects the circle.
18. And a semicircle is the figure contained by the diameter and the circumference cuts off by it. And the center of the semicircle is the same as that of the circle.
19. Rectilineal figures are those which are contained by straight lines: trilateral figures being those contained by three, quadrilateral by four, and multilateral those contained by more than four straight lines.
20. Of trilateral figures: an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.
21. Further, of trilateral figures, a right-angled triangle is that which has a right-angle, an obtuse-angled triangle that which

has an obtuse angle, and an acute-angled triangle that which has its three angles acute.

22. Of the quadrilateral figures, a square is that which is both equilateral and right-angled, an oblong that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.
23. Parallel straight lines are straight-lines which, being in the same plane, and being produced indefinitely in both directions, do not meet one another in either direction.

Postulates

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To draw a circle with any center and distance.
4. That all right-angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right-angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

Common Notions

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equal be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

A.3 John Playfair's Foundation (1813)

Definitions and Corollaries

1. A Point is that which has position, but not magnitude.
2. A line is length without breadth.
   Corollary. The extremities of a line are points; and the intersections of one line with another are also points.
3. If two lines are such that they cannot coincide in any two points, without coinciding altogether, each of them is called a straight line.
   Corollary. Hence two straight lines cannot enclose a space. Neither can two straight lines have a common segment; for they cannot coincide in part, without coinciding altogether.
4. A superficies is that which has only length and breadth.
   Corollary. The extremities of a superficies are lines; and the intersections of one superficies with another are also lines.
5. A plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies.
6. A plane rectilineal angle is the inclination of two straight lines to one another, which meet together, but are not in the same straight line.
7. When a straight line standing on another straight line makes adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which stands on the other is called a perpendicular to it.
8. An obtuse angle is that which is greater than a right angle.
9. An acute angle is that which is less than a right angle.
10. A figure is that which is inclosed by one or more boundaries. The space contained within a figure is called the Area of the Figure.
11. A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another.
12. This point is called the centre of the circle.
13. A diameter of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

14. A semicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.
15. Rectilineal figures are those which are contained by straight lines.
16. Trilateral figures, or triangles, by three straight lines.
17. Quadrilateral, by four straight lines.
18. Multilateral figures, or polygons, by more than four straight lines.
19. Of three-sided figures, an equilateral triangle is that which has three equal sides.
20. An isosceles triangle is that which has only two sides equal.
21. A scalene triangle, is that which has three unequal sides.
22. A right angled triangle, is that which has a right angle.
23. An obtuse angled triangle, is that which has an obtuse angle.
24. An acute angled triangle, is that which has three acute angles.
25. Of four sided figures, a square is that which has all its sides equal, and all its angles right angles.
26. An oblong, is that which has all its angles right angles, but has not all its sides equal.
27. A rhombus, is that which has all its sides equal, but its angles are not right angles.
28. A rhomboid, is that which has its opposite sides equal to one another, but all its sides are not equal, nor its angles right angles.
29. All other four-sided figures besides these, are called Trapeziums.
30. Straight lines, which are in the same plane, and being produced ever so far both ways, do not meet, are called Parallel Lines.

Postulates

1. Let it be granted that a straight line may be drawn from any one point to any other point.
2. That a terminated straight line may be produced to any length in a straight line.

3. And that a circle may be described from any centre, at any distance from that centre.

AXIOMS

1. Things which are equal to the same thing are equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be taken from equals, the remainders are equal.
4. If equals be added to unequals, the wholes are unequal.
5. If equals be taken from unequals, the remainders are unequal.
6. Things which are doubles of the same thing, are equal to one another.
7. Things which are halves of the same thing, are equal to one another.
8. Magnitudes which coincide with one another; that is, which exactly fill the same space, are equal to one another.
9. The whole is greater than its part.
10. All right angles are equal to one another.
11. Two straight lines which intersect one another, cannot be both parallel to the same straight line.

A.4 David Hilbert's Foundation (1899)

AXIOMS OF CONNECTION

I,1. Two distinct points A and B always completely determine a straight line a. We write $AB = a$ or $BA = a$.

I,2. Any two distinct points of a straight line completely determine that line; that is, if $AB = a$ and $AC = a$, where $B \neq C$, then is also $BC = a$.

I,3. Three points A, B, C not situated in the same straight line always completely determine a plane α. We write $ABC = a$.

I,4. Any three points A, B, C of a plane α, which do not lie in the same straight line, completely determine that plane.

I,5. If two points A, B of a straight line a lie in a plane α, then every point of a lies in α.

I,6. If two planes α and β have a point A in common, then they have at least a second point B in common.

I,7. Upon every straight line there exist at least two points, in every plane at least three points not lying in the same straight line, and in space there exist at least four points not lying in a plane.

Axioms of Order

II,1. If A, B, C are points of a straight line and B lies between A and C, then B lies also between C and A.

II,2. If A and C are two points of a straight line, then there exists at least one point B lying between A and C and at least one point D so situated that C lies between A and D.

II,3. Of any three points situated on a straight line, there is always one and only one which lies between the other two.

II,4. Any four points A, B, C, D of a straight line can always be so arranged that B shall lie between A and C and also between A and D, and, furthermore, that C shall lie between A and D and also between B and D.

Definition. We will call the system of two points A and B, lying upon a straight line, a segment and denote it by AB or BA. The points lying between A and B are called the points of the segment AB or the points lying within the segment AB. All other points of the straight line are referred to as the points lying outside the segment AB. The points A and B are called the extremities of the segment AB.

II,5. Let A, B, C be three points not lying in the same straight line and let a be a straight line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the straight line a passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.

Axiom of Parallels (Euclid's Axiom)

III,1. In a plane α there can be drawn through any point A, lying outside of a straight line a, one and only one straight line which does not intersect the line a. This straight line is called the parallel to a through the given point A.

AXIOMS OF CONGRUENCE

IV,1. If A, B are two points on a straight line a, and if A' is a point upon the same or another straight line a', then, upon a given side of A' on the straight line a', we can always find one and only one point B' so that the segment AB (or BA) is congruent to the segment $A'B'$. We indicate this relation by writing $AB \equiv A'B'$. Every segment is congruent to itself; that is, we always have $AB \equiv AB$.

We can state the above axiom briefly by saying that every segment can be laid off upon a given side of a given point of a given straight line in one and and only one way.

IV,2. If a segment AB is congruent to the segment $A'B'$ and also to the segment $A''B''$, then the segment $A'B'$ is congruent to the segment $A''B''$; that is, if $AB \equiv A'B'$ and $AB \equiv A''B''$, then $A'B' \equiv A''B''$.

IV,3. Let AB and BC be two segments of a straight line a which have no points in common aside from the point B, and, furthermore, let $A'B'$ and $B'C'$ be two segments of the same or of another straight line a' having, likewise, no point other than B' in common. Then, if $AB \equiv A'B'$ and $BC \equiv B'C'$, we have $AC \equiv A'C'$.

Definitions. Let α be any arbitrary plane and h, k any two distinct half-rays lying in α and emanating from the point O so as to form a part of two different straight lines. We call the system formed by these two half-rays h, k an angle and represent it by the symbol $\angle(h, k)$ or $\angle(k, h)$. From axioms II, 1.5, it follows readily that the half-rays h and k, taken together with the point O, divide the remaining points of the plane a into two regions having the following property: If A is a point of one region and B a point of the other, then every broken line joining A and B either passes through O or has a point in common with one of the half-rays h, k. If, however, $A, A0$ both lie within the same region, then it is always possible to join these two points by a broken line which neither passes through O nor has a point in common with either of the half-rays h, k. One of these two regions is distinguished from the other in that the segment joining any two points of this region lies entirely within the region. The region so characterised is called the interior of the angle (h, k). To distinguish the other region from this, we call it the exterior of the angle (h, k). The half rays h and k are called the sides of the angle, and the point O is called the vertex of the angle.

IV,4. Let an angle (h, k) be given in the plane α and let a straight line a' be given in a plane α'. Suppose also that, in the plane α, a definite side of the straight line a' be assigned. Denote by h' a half-ray of the straight line a' emanating from a point O' of this line. Then in the plane α' there is one and only one half-ray k' such that the angle (h, k), or (k, h), is congruent to the angle (h', k') and at the same time all interior points of the angle (h', k') lie upon the given side of a'. We express this relation by means of the notation $\angle(h, k) \equiv \angle(h', k')$. Every angle is congruent to itself; that is,$\angle(h, k) \equiv \angle(h, k)$, or $\angle(h, k) \equiv \angle(k, h)$.

We say, briefly, that every angle in a given plane can be laid off upon a given side of a given half-ray in one and only one way.

IV,5. If the angle (h, k) is congruent to the angle (h', k') and to the angle (h", k"), then the angle (h', k') is congruent to the angle (h", k"); that is to say, if $\angle(h, k) \equiv \angle(h', k')$ and $\angle(h, k) \equiv (h'', k'')$, then $\angle(h', k') \equiv \angle(h'', k'')$.

Suppose we have given a triangle ABC. Denote by h, k the two half-rays emanating from A and passing respectively through B and C. The angle (h, k) is then said to be the angle included by the sides AB and AC, or the one opposite to the side BC in the triangle ABC. It contains all of the interior points of the triangle ABC and is represented by the symbol $\angle BAC$, or by $\angle A$.

IV,6. If, in the two triangles ABC and $A'B'C'$ the congruences $AB \equiv A'B'$, $AC \equiv A'C'$, $\angle BAC \equiv \angle B'A'C'$ hold, then the congruences $\angle ABC \equiv \angle A'B'C'$ and $\angle ACB \equiv \angle A'C'B'$ also hold.

Axiom of Continuity (Archimedean Axiom)

V,1. Let A_1 be any point upon a straight line between the arbitrarily chosen points A and B. Take the points $A_2, A_3, A_4, \ldots$. so that A_1 lies between A and A_2, A_2 between A_1 and A_3, A_3 between A_2 and A_4 etc. Moreover, let the segments AA_1, A_1A_2, A_2A_3, $A_3A_4, \ldots$ be equal to one another. Then, among this series of points, there always exists a certain point A_n such that B lies between A and A_n.

Remark. To the preceeding five groups of axioms, we may add the following one, which, although not of a purely geometrical nature, merits particular attention from a theoretical point of view. It may be expressed in the following form:

Axiom of Completeness (Vollständigkeit): To a system of points, straight lines, and planes, it is impossible to add other elements in such a manner that the system thus generalized shall form a new geometry obeying all of the five groups of axioms. In other words, the elements of geometry form a system which is not susceptible of extension, if we regard the five groups of axioms as valid.

Bibliographic references

[1] Aristotle. *Prior and Posterior Analytics.* Clarendon Press, Oxford, 1949.

[2] E. Beltrami. Note fisico-matematiche. *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, 3(1):67–79, 1889.

[3] Eugenio Beltrami. Saggio di interpretazione della geometria non-euclidea. *Giornale di Mathematiche*, pages 285–315, 1868.

[4] Charles L. Dogson. *Euclid and His Modern Rivals.* Forgotten Books, London, 2013.

[5] Ana Millán Gasca. *Euclides. La fuerza del razonamiento matemático.* Nivola, 2004.

[6] Jeremy Gray. *Worlds Out of Nothing. A Course in the History of Geometry in the 19th Century.* Spriger Verlag, 2010.

[7] Marvi Jay Greenberg. *Euclidean and Non-Euclidean Geometries. Development and History.* W.H. Freeman, 2008.

[8] Marvin Jay Greenberg. *Euclidian and Non-Euclidian Geometries: History and Development.* W. H. Freeman, San Francisco, CA, 1994.

[9] Thomas Heath. *The Thirteen Books of Euclid's Elements*, volume I. Dover Publications Inc, New York, second edition, 1956.

[10] David Hilbert. *The Foundations of Geometry.* The Open Court Publishing Company, La Salle, Illinois, 1950.

[11] A. León. Infinity, Language and non-Euclidean Geometries. *The General Science Journal*, 2023.

[12] A. León. *Infinity put to the test.* Self edition in KDP. Printed at amazon.com. Free pdf, 2023 (2021).

[13] John Losee. *Introducción histórica a la filosofía de la ciencia.* Alianza, Madrid, 1987.

[14] Nelo Alberto Maestre Blanco. *Este no es el título de este libro.* EMSE EDAPP S.L., 2019.

[15] Leónard Mlodinow. *Euclid's Windows. The Story of Geometry from Parallel Lines to Hyperspace.* Penguin Books, 2002.

[16] José Luis Muñoz. *Riemann. Una nueva visión de la geometría.* Nivola, 2009.

[17] J.J. O'Connor and E.F. Robertson. Non-Euclidean Geometry History, 1996. McTutor History of Mathematics. Accesed: 2016-02-16.

[18] Pappus of Alexandria. *Mathematicae Collectiones.* Apud Francifcum de Francifcis Senenfem, 1589. Translated to Latin by Federico Commandino. Digitalized by Google.

[19] Giuseppe Peano. *Arithmetices Principia. Nova Methodo Exposita.* Libreria Bocca, Roma, 1889.

[20] Josep Pla i Carrera. *Las matemáticas presumen de figura.* Ediciones RBA, 2012.

[21] J. Playfair. *Elements of Geometry.* W.E. Dean Printer and Publisher, New York, 1846.

[22] John Playfair. *Elements of Geometry.* Number London. Forgotten Books, 2015.

[23] Proclus and Gottfried Friedlein. *Procli Diadochi in primum Euclidis Elementorum librum commentarii.* in aedibus B. G . Teubneri, 1873. Digitalized by Google.

[24] Thomas Reid. *Inquiry Into the Human Mind: on the Principle of Common Sense.* Edinburgh University Press, 1997 (1764).

[25] Pietro Riccardi. *Saggio di una bibliografia euclidea.* Tipografia Gamberini e Parmeggiani, 1887. Digitalized by Google.

[26] Vicenzo De Risi. *Leibniz on the Parallel Postulate and the Foundations of Geometry. The unpublished manuscripts*, volume Science Networks Histoical Studies. Birkhäuser, 2016.

[27] Boris Abramovich Rosenfeld. *A History of Non-Euclidian Geometry. Evolution of the Concept of a Geometric Space.* Spriger Verlag, New York, 1988.

[28] Bertrand Russell. Sobre la inducción. In Richard Swinburne, editor, *La justificación del razonamiento inductivo.* Alianza, Madrid, 1976.

[29] Bardi. J. S. *The Fifth Postulate: How Unraveling a Two Thousand Year Old Mystery Unraveled the Universe.* John Wiley & Sons, Hoboken, New Jersey, 2009.

[30] Mark J. Schiefsky. New technologies for the study of Euclid's Elements. `http://www.archimedes.fas.harvard.edu`, February 2007. Accessed: 2016-02-13.

[31] Lee Smolin. *The Life of the Cosmos.* Phoenix, London, 1998.

[32] Roberto Torretti. Nineteenth Century Geometry. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy.* Stanford University, 2014.

[33] Gideon Yaffe. Reconsidering Reid's geometry of visibles. *The Philosophical Quarterly*, 52(209):602–620, 2009.

Alphabetical index